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# EIGENFUNCTIONS IN THE THREE-BODY PROBLEM

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## § 1. Introduction

The study of a three-body system in nuclear physics led to the construction of basis functions in the form of so-called $K$ -polynomials, i.e. harmonic polynomials in the six- dimensional space.[1] In order to obtain these functions, which look like a generalization of the usual spherical functions, it is necessary to find the complete set of solutions of the Laplace equation on the five dimensional sphere. For this purpose we worked out a method for calculating the generators of the group of motion on the five dimensional sphere, and constructed the explicit form of the corresponding complete set of commuting operators[2,3].

The salient feature of the problem which makes it somewhat difficult is that the required functions have to be the eigenfunctions of the angular momentum operator and a representation of the three-particle permutation group simultaneously. It is, of course, easy to find the solution of the problem, if we do not require the permutation symmetry of the eigenfunctions. The simplest way is to obtain such a solution directly by using the method of graphs /4/. In this case the eigenfunctions of the Laplacean (i.e. the basis functions of the problem) are characterized by the following five quantum numbers:

$K$ – the degree of the polynomial

$j_1 M_1, j_2 M_2$ – angular momenta and their projections conjugated to $\vec{\xi}$ and $\vec{\eta}$ . (1)

Instead of $j_1$ and $j_2$ we may, of course, introduce the total momentum $J$ .

In our previous papers /2,3/ the aim of the work was the construction of a set of functions possessing certain symmetry properties with respect to the permutations. This set of functions was characterized by another five quantum numbers

$$K, J, M, \nu, \Omega . \qquad (2)$$

The meaning of these quantum numbers was considered in detail in /3/. It was shown, that the general form of the required harmonic function is

$$\Phi^{J}_{M,\nu} = \sum_{\Lambda, M'} a_{\nu}(\Lambda, M') D^{\Lambda}_{\nu, M'}(\lambda, a, 0)\, D^{J}_{2M', M}(\phi_1 \theta \phi_2) . \qquad (3)$$

The coefficients $a_{\nu}(\Lambda, M')$ are determined by the condition that the function $\Phi^{J}_{M,\nu}$ has to fulfill the Laplace equation on the five-dimensional sphere and the eigenvalue equation of operator $\Omega$ ; the meaning of the variables will be discussed later.

In the present paper we choose another way: namely, we try to find a transformation from the complete set of "tree" – functions, i.e. functions constructed with the help of the method of graphs, to the $K$-harmonics. In this case we first have to transform the initial "tree" - functions to a set with a given total angular momentum, that is, to a set characterized by the quantum numbers

$$K, J, M, j_1, j_2 . \tag{4}$$

In the next step we pass over to the quantum numbers

$$K, J, M, \nu, (j_1 j_2). \tag{5}$$

In order to do this it is necessary to carry out a simple Fourier transform. To be correct, $(j_1 j_2)$ is not a real quantum number (in the sense, that functions corresponding to different pairs $(j_1 j_2)$ do not form an orthonormal set), but this notation demonstrates the parentage of the functions (i.e. it shows where we got them from). Let's point out, that $j_1$ and $j_2$ cease to be eigenvalues any more after performing the Fourier transform.

Finally, we have to take the sum

$$C_{(j_1 j_2)} \Psi^{(j_1 j_2)}_{KJM} , \tag{6}$$

where $(j_1 j_2)$ will run over each pair of values which can give the total angular momentum $J$ such that

$$J \leq j_1 + j_2 \leq K .$$

In the present paper we will show, how to construct the set of functions (5). So far we couldn't find a set of eigenfunctions of $\Omega$ in a closed form. Although a method for calculating such functions expressed in a power series is given in /5/, it is very difficult to get a general solution because the increase of the eigenvalue makes the corresponding equation too complicated. As soon as our final purpose is the construction of a complete orthonormal set of functions, we have to come back to the properties of $\Omega$ once more in the next paper.

In the following we will obtain the "tree" functions in such a form, for which the Fourier transform becomes almost trivial. Thus, from a formal point of view, the present work demonstrates how to calculate the Fourier transform of the "tree" functions.

## §2. Coordinates and Parametrization

Dealing with a three-particle system, we will use the coordinates introduced in /3/. Let $\vec{x}_i$ $(i = 1,2,3)$ be the radius-vectors of the three particles, and fix

$$\vec{x}_1 + \vec{x}_2 + \vec{x}_3 = 0. \tag{7}$$

The Jacobi coordinates for equal masses will be defined as

$$\begin{aligned} \vec{\xi} &= -\sqrt{\frac{3}{2}}(\vec{x}_1 + \vec{x}_2) \\ \vec{\eta} &= \sqrt{\frac{1}{2}}(\vec{x}_1 - \vec{x}_2) \\ \xi^2 + \eta^2 &= \rho^2, \end{aligned} \tag{8}$$

where $\rho$ is the radius of the five-dimensional sphere. (We will take for simplicity $\rho = 1$ ). Further, following /3/ we introduce the complex vector

$$\begin{aligned} \vec{z} &= \vec{\xi} + i\,\vec{\eta} \\ \vec{z}^{\,*} &= \vec{\xi} - i\,\vec{\eta} . \end{aligned} \tag{9}$$

Let us consider now a triangle, the vertices of which are determined by three particles. The situation of this triangle in the space will be characterized by the complex vectors $\vec{\ell}_+$ and $\vec{\ell}_-$ which form together with $\vec{\ell}_0 = \vec{\ell}_+ \times \vec{\ell}_-$ the moving coordinate system. Vectors $\vec{\ell}_+$ and $\vec{\ell}_-$ fulfill the usual conditions

$$\ell_+^2 = \ell_-^2 = 0 \qquad \vec{\ell}_+ \vec{\ell}_- = 1 . \tag{10}$$

They are connected with the vectors $\vec{z}$ and $\vec{z}^*$ in the following way

$$\begin{aligned} \vec{z} &= e^{-i\frac{\lambda}{2}} \left( \cos\frac{a}{2}\, \vec{\ell}_+ + i \sin\frac{a}{2}\, \vec{\ell}_- \right) \\ \vec{z}^* &= e^{i\frac{\lambda}{2}} \left( \cos\frac{a}{2}\, \vec{\ell}_- - i \sin\frac{a}{2}\, \vec{\ell}_+ \right). \end{aligned} \tag{11}$$

Variables $\lambda$ and $a$ determine the form of the triangle (we excluded similarity transformations, taking $\rho$ =const). Note, that the components of the moment of inertia can be expressed as

$$\sin^2\left(\frac{a}{2} - \frac{\pi}{4}\right), \quad \cos^2\left(\frac{a}{2} - \frac{\pi}{4}\right), \quad 1 . \tag{12}$$

In the following it will be useful to return to $\vec{\xi}$ and $\vec{\eta}$ and connect them with the coordinates $\vec{\ell}_+$ and $\vec{\ell}_-$ :

$$\begin{aligned} \vec{\xi} &= \frac{1}{2}\left( u\, \vec{\ell}_+ + u^* \vec{\ell}_- \right) \\ \vec{\eta} &= -\frac{i}{2}\left( v\, \vec{\ell}_+ - v^* \vec{\ell}_- \right). \end{aligned} \tag{13}$$

The introduced u , v , u* and v* are, as follows

$$
\begin{aligned}
u &= e^{-i\frac{\lambda}{2}} \cos\frac{a}{2} - i e^{i\frac{\lambda}{2}} \sin\frac{a}{2} \\
v &= e^{-i\frac{\lambda}{2}} \cos\frac{a}{2} + i e^{i\frac{\lambda}{2}} \sin\frac{a}{2} \\
u^* &= e^{i\frac{\lambda}{2}} \cos\frac{a}{2} + i e^{-i\frac{\lambda}{2}} \sin\frac{a}{2} \\
v^* &= e^{i\frac{\lambda}{2}} \cos\frac{a}{2} - i e^{-i\frac{\lambda}{2}} \sin\frac{a}{2} .
\end{aligned}
\tag{14}
$$

In formulae (13) the Euler angles, describing the orientation of the triangle, and the coordinates, characterizing its deformations, are separated explicitly x/. These expressions can not be obtained as products of funct ons of the Euler angles and functions of the coordinates related tc the deformations (they are, in fact, sums of such functions).This feature of (13) corresponds to the connection of rotations and defor nations. The meaning of the coordinates introduced becomes clear if we rewrite the expreessions for $\vec{\xi}$ and $\vec{\eta}$ in the form

x/ Let's turn our attention to a happy correspondence between spinors and the variables u , v , u* , v* . Consider

$$\xi^2 + \eta^2 = 1 = \frac{1}{2}(uu^* + vv^*) .$$

Introducing two spinors

$$\alpha = \begin{pmatrix} u^* \\ v \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} -v^* \\ u \end{pmatrix}$$

we can rewrite the above expression as an invariant product of them:

$$\xi^2 + \eta^2 = \alpha_1 \beta_2 - \alpha_2 \beta_1$$

$$\vec{\xi} = \left(\frac{1}{2} u u^*\right)^{1/2} \frac{1}{\sqrt{2}} \left[ e^{i\psi_1} \vec{\ell}_+ + e^{-i\psi_1} \vec{\ell}_- \right] \tag{15}$$

$$\vec{\eta} = \left(\frac{1}{2} v v^*\right)^{1/2} \frac{1}{\sqrt{2}} \left[ e^{i\psi_2} \vec{\ell}_+ + e^{-i\psi_2} \vec{\ell}_- \right] \quad . \tag{16}$$

Here we have introduced the phases $\psi_1$ and $\psi$

$$u = \rho_1 e^{i\psi_1} \qquad v = \rho_2 e^{i\psi} \tag{17}$$

from where, obviously,

$$\frac{u}{u^*} = e^{2i\psi_1} \qquad \frac{v}{v^*} = e^{2i\psi} \quad . \tag{18}$$

Finally,

$$\psi_2 = \psi - \frac{\pi}{2}$$

and the angle

$$\Theta = \psi_1 - \psi_2 \tag{19}$$

is the angle between the vectors $\vec{\xi}$ and $\vec{\eta}$ :

$$\vec{\xi} \cdot \vec{\eta} = |\xi| |\eta| \cos\Theta . \tag{20}$$

Making use of the equations

$$\xi^2 = \frac{1}{2} u u^* \qquad \eta^2 = \frac{1}{2} v v^* \tag{21}$$

we can express the angle $\Theta$ in terms of our variables

$$\cos\Theta = \frac{\cos\lambda \sin a}{\sqrt{1 - \sin^2\lambda \sin^2 a}} \quad . \tag{22}$$

This irrational connection between $\Theta$ and the angles $\alpha$ and $\lambda$ makes it necessary to find another way for the Fourier transform.

Let's introduce the unit vectors $\vec{n}$ and $\vec{m}$ defined by

$$\vec{n} = \frac{\vec{\xi}}{|\xi|} \qquad \vec{m} = \frac{\vec{\eta}}{|\eta|}. \tag{23}$$

From (15) and (16) it is clear, that

$$\vec{n} = \frac{1}{\sqrt{2}}\left(e^{i\psi_1}\vec{\ell}_+ + e^{-i\psi_1}\vec{\ell}_-\right) \tag{24}$$

$$\vec{m} = \frac{1}{\sqrt{2}}\left(e^{i\psi_2}\vec{\ell}_+ + e^{-i\psi_2}\vec{\ell}_-\right) \tag{25}$$

It will be useful to rewrite these expressions in the form

$$\vec{n} = D^1_{01}(0, \tfrac{\pi}{2}, 0)\left[e^{i\psi_1}\vec{\ell}_+ + e^{-i\psi_1}\vec{\ell}_-\right] \tag{26}$$

$$\vec{m} = D^1_{01}(0, \tfrac{\pi}{2}, 0)\left[e^{i\psi_2}\vec{\ell}_+ + e^{-i\psi_2}\vec{\ell}_-\right] \tag{27}$$

or, for the components of $\vec{n}$ and $\vec{m}$

$$n^{(M_1)} = \sum_{\mu_1} D^1_{0\mu}\left(0, -\frac{\pi}{2}, \psi_1\right) D^1_{\mu_1 M_1}(\vec{\ell}_+ \vec{\ell}_-) \tag{28}$$

$$m^{(M_2)} = \sum_{\mu_2} D^1_{0\mu_2}\left(0, -\frac{\pi}{2}, \psi_2\right) D^1_{\mu_2 M_2}(\vec{\ell}_+ \vec{\ell}_-) \cdot \tag{29}$$

Remember, that the components of $\vec{\ell}_+$ and $\vec{\ell}_-$ can be expressed in terms of the Wigner $D$-functions

$$D^1_{mn}(\phi_1 \theta \phi_2) = \ell^{(n)}_m . \tag{30}$$

In fact we have $D(\phi_1 \theta \phi_2) \equiv D(\vec{\ell}_+ \vec{\ell}_-)$, and we introduced this somewhat unusual notation just to show, that the Euler angles determine the position of a trihedron defined by the unit vectors $\vec{\ell}_1, \vec{\ell}_2, \vec{\ell}_3$. The expression (28) describes in fact the rotation, in consequence of which the trihedron $\vec{\ell}_1, \vec{\ell}_2, \vec{\ell}_3$ transforms to a trihedron determined by the unit vector $\vec{n}$ and two other orthogonal vectors $\vec{n}_1, \vec{n}_2$. (The precise definition of them is of no interest to us). Similarly, the formula (29) describes the rotation to a trihedron defined by the unit vectors $\vec{m}, \vec{m}_1, \vec{m}_2$. Thus, expressions (28) and (29) might be considered as transformations of Legendre polynomials of first degree (namely, vectors $\vec{n}$ and $\vec{m}$ ) and they can be generalized easily to the case of an arbitrary degree. For example, for a polynomial constructed from the unit vector $\vec{n}$ we can obtain the following expression

$$D^{j_1}_{0M_1}(\vec{n}) = \sum_{\mu_1=-j_1}^{j_1} D^{j_1}_{0\mu_1}(0, \frac{\pi}{2}, -\psi_1) D^{j_1}_{\mu_1 M_1}(\phi_1 \theta \phi_2) \qquad (31)$$

and similarly, for a polynomial built up from $\vec{m}$ :

$$D^{j_2}_{0M_2}(\vec{m}) = \sum_{\mu_2=-j_2}^{j_2} D^{j_2}_{0\mu_2}(0, \frac{\pi}{2}, -\psi_2) D^{j_2}_{\mu_2 M_2}(\phi_1 \theta \phi_2) \qquad (32)$$

It is easy to rewrite these formulae to such a form, in which $D^{j_1}_{0\mu_1}$ and $D^{j_2}_{0\mu_2}$ are functions of the same arguments:

$$D^{j_1}_{0M_1}(\vec{n}) = \sum_{\mu_1} e^{i\mu_1 \frac{\psi_1-\psi_2}{2}} D^{j_1}_{0\mu_1}(0, \frac{\pi}{2}, -(\frac{\psi_1+\psi_2}{2})) D^{j_1}_{\mu_1 M_1}(\phi_1 \theta \phi_2) \qquad (33)$$

$$D^{j_2}_{0M_2}(\vec{m}) = \sum_{\mu_2} e^{-i\mu_2 \frac{\psi_1-\psi_2}{2}} D^{j_2}_{0\mu_2}(0, \frac{\pi}{2}, -(\frac{\psi_1+\psi_2}{2}))) D^{j_2}_{\mu_2 M_2}(\phi_1 \theta \phi_2). \qquad (34)$$

For the three-body problem we have to introduce one more restriction. Namely, as soon as the problem possesses certain symmetry properties with respect to the reflection in the plane defined by the moving vectors $\vec{\ell}_+$ and $\vec{\ell}_-$, the values of $\mu_1$ and $\mu_2$ have to be either only even, or only odd (e. g. in the sums (28) and (29) only $\mu_{1,2}=\pm 1$ is possible).

The variety of the introduced parameters and coordinates, and the performed transformations might seem so far to be superfluous; among others, $u$, $v$, $u^*$ and $v^*$ are constructed in a rather artificial way. Still, in the following we will see, that just they will make easier the calculation of the Fourier coefficients of our polynomials.

## § 3. The Case $J=0$

For the states with zero total angular momentum the solution might be easily obtained. From the corresponding equation of [3] we get for this case the following result:

$$D^{\frac{K}{4}}_{\frac{\nu}{2},-\frac{\nu}{2}}(2\lambda, 2a, 0). \tag{35}$$

The degree $\frac{K}{4}$ of the $D$-function corresponds to the degree of the polynomial, since the latter is defined by the trigonometrical functions of argument $\frac{a}{2}$ so that every trigonometrical function of $2a$ increases the degree by 4 units.

The structure of (35) is easy to understand even without solving the equation. Note, that at $J=0$ the polynomial can not involve the vectors $\vec{\ell}_+$ and $\vec{\ell}_-$ i.e. it has to be a function of $z^2$ and $z^{*2}$. We know, that

$$\begin{aligned} z^2 &= i \sin a\, e^{-i\lambda} \\ z^{*2} &= -i \sin a\, e^{i\lambda} \end{aligned} \tag{36}$$

Looking at the table of $D$-functions we notice, that

$$z^2 = D^{1/2}_{-1/2,\,1/2}(2\lambda,\, 2a,\, 0)$$

$$z^{*2} = -\, D^{1/2}_{1/2,\,-1/2}(2\lambda,\, 2a,\, 0) \qquad (37)$$

and that the sum of the lower indices is equal to zero. This feature remains valid, naturally, for $D$-functions of higher order also if we construct harmonic polynomials from the functions (37). Thus it is clear, that we come to the expression (35). The specific feature of our problem turns out to be the absence of diagonal elements

$$D^{1/2}_{1/2,\,1/2} \quad \text{and} \quad D^{1/2}_{-1/2,\,-1/2} \qquad (38)$$

in the basis.

It seems to be interesting to study the expansion of (35) in terms of the "tree" functions, i.e. in functions with given pairing angular momenta. The $J=0$ state of the system can be described as the rotation of the vectors $\vec{\xi}$ and $\vec{\eta}$ in opposite directions with equal momenta. This problem leads among others to questions connected with the theory of Clebsch-Gordan coefficients, which will be discussed in another paper.

Presenting the calculations in the Appendix, here we remark only, that the amplitudes of states with $j_1 = j_2 = j$ are proportional to the Wigner coefficient[x/]

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x/ This formula was obtained by us together with V. Efros. After this work has been finished , we received the paper of R.C. Whitten[/8/], in which the expansion of two-body potentials in terms of the functions corresponding to $J=0$ is given.

$$\begin{pmatrix} \frac{K}{4} & \frac{K}{4} & j \\ \frac{\nu}{2} & -\frac{\nu}{2} & 0 \end{pmatrix} . \tag{39}$$

The expansion in functions with definite pairwise momenta leads presumably to different coefficients of vector sums in the case of $J \neq 0$ also. We will consider this case separately.

## §4. The "Tree" Functions

The solution of the Laplace equation on the five dimensional sphere can be given in a coordinate system corresponding to the "tree" on fig.1

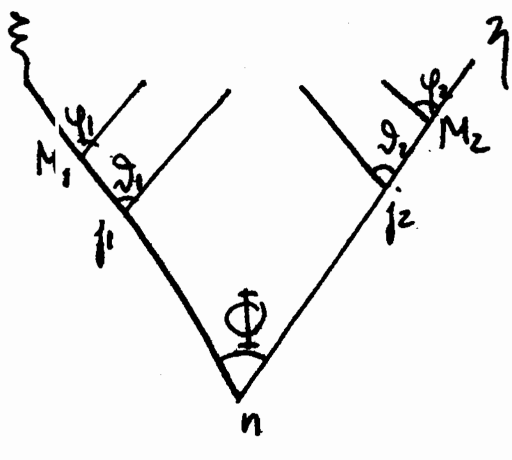


Fig. 1.

In this coordinate system we assume

$$\vec{\xi} = \cos\Phi\,\vec{n}$$
$$\vec{\eta} = \sin\Phi\,\vec{m} \tag{40}$$

taking into account that $\xi^2 + \eta^2 = 1$. The eigenfunction corresponding to this "tree" is build up in the following way/4/. The function

$$(\cos\Phi)^{j_1} (\sin\Phi)^{j_2} P_n^{(j_1+\frac{1}{2},\, j_2+\frac{1}{2})}(\cos 2\Phi) \tag{41}$$

will correspond to the intersection $n$ (where $2n = K - j_1 - j_2$ ). From (40) it follows, that

$$\cos^2\Phi = \xi^2 , \qquad \sin^2\Phi = \eta^2 .$$

So we can rewrite the expression (41) in the form

$$\xi^{j_1} \eta^{j_2} P_{\frac{K-j_1-j_2}{2}}^{(j_1+\frac{1}{2},\, j_2+\frac{1}{2})}(\vec{\xi}, \vec{\eta}), \tag{42}$$

where

$$P_{\frac{K-j_1-j_2}{2}}^{(j_1+\frac{1}{2},\, j_2+\frac{1}{2})}(\vec{\xi},\vec{\eta}) = \sum_{m=0}^{n} \begin{pmatrix} n+j_1+\frac{1}{2} \\ m \end{pmatrix} \begin{pmatrix} n+j_2+\frac{1}{2} \\ n-m \end{pmatrix} (\xi^2)^m (\eta^2)^{n-m} (-1)^{n-m} . \tag{43}$$

The functions corresponding to the "branch" characterized by $j_1 M_1$ and $j_2 M_2$ are

$$P_{j_1 M_1}(\vec{n}) = P_{j_1}^{M_1}(\theta_1)\, e^{-iM_1\phi_1} \tag{44}$$

and

$$P_{j_2 M_2}(\vec{m}) = P_{j_2}^{M_2}(\theta_2)\, e^{-iM_2\phi_2} ,$$

respectively. Multiplying the functions (43) and (44) we obtain the eigenfunction of the whole "tree":

$$\Phi(\vec{\xi},\vec{\eta}) = P_{JM}(\vec{\xi},\vec{\eta})\, P^{(j_1+\frac{1}{2},\, j_2+\frac{1}{2})}_{\frac{K-j_1-j_2}{2}}(\vec{\xi},\vec{\eta}). \tag{45}$$

Here we used the notations

$$P_{JM}(\vec{\xi},\vec{\eta}) = P_{j_1M_1}(\vec{\xi})\, P_{j_2M_2}(\vec{\eta}) \tag{46}$$

$$P_{j_1M_1}(\vec{\xi}) = \xi^{j_1} P_{j_1M_1}(\vec{n}) \tag{47}$$

$$P_{j_2M_2}(\vec{\eta}) = \eta^{j_2} P_{j_2M_2}(\vec{m}). \tag{48}$$

Making use of the equations (18) and (21) we can pass over to the variables $u$, $v$, $u^*$ and $v^*$ and rewrite the expressions (47) and (48) in the form

$$P_{j_1M_1}(\vec{\xi}) = i^{M_1}\left[\frac{(j_1+M_1)!}{(j_1-M_1)!}\right]^{\frac{1}{2}} \sum_{\mu_1} \frac{1}{2^{j_1}}\, u^{\frac{j_1+\mu_1}{2}}\, u^{*\frac{j_1-\mu_1}{2}}\, \Delta^{(j_1)}_{0\mu_1} D^{j_1}_{\mu_1 M_1}(\phi_1\theta\phi_2) \tag{49}$$

and

$$P_{j_2M_2}(\vec{\eta}) = i^{M_2}\left[\frac{(j_2+M_2)!}{(j_2-M_2)!}\right]^{\frac{1}{2}} \sum_{\mu_2} \frac{(-i)^{\mu_2}}{2^{j_2}}\, v^{\frac{j_2+\mu_2}{2}}\, v^{*\frac{j_2-\mu_2}{2}}\, \Delta^{(j_2)}_{0\mu_2} D^{j_2}_{\mu_2 M_2}(\phi,\theta\phi_2). \tag{50}$$

Here we applied the notation of /6/

$$\Delta^{(m)}_{k\ell} = D^{m}_{k\ell}(0,\tfrac{\pi}{2},0) = P^{m}_{k\ell}(\cos\tfrac{\pi}{2}). \tag{51}$$

Expanding the product $D^{j_1}_{\mu_1 M_1}(\phi_1\theta\phi_2)D^{j_2}_{\mu_2 M_2}(\phi_1\theta\varphi_2)$ in terms of the functions $D^{J}_{\mu M}(\phi_1\theta\phi_2)$ (where $M = M_1 + M_2$, $\mu = \mu_1 + \mu_2$ ) and considering only one term J of the sum (where $|j_1 - j_2| \leq J \leq j_1 + j_2$), we get

$$P_{JM}(\vec{\xi},\vec{\eta}) = (i)^{M}\left[\frac{(j_1+M_1)!(j_2+M_2)!}{(j_1-M_1)!(j_2-M_2)!}\right]^{1/2}(j_1 0, j_2 0|J0)(j_1M_1, j_2M_2|JM)\times \tag{52}$$

$$\times\sum_{\mu_1\mu_2}\frac{(-i)^{\mu_2}}{2^{j_1+j_2}}\, u^{*\frac{j_1-\mu_1}{2}}\, v^{\frac{j_2+\mu_2}{2}}\, u^{\frac{j_1+\mu_1}{2}}\, v^{*\frac{j_2-\mu_2}{2}}(j_1\mu_1, j_2\mu_2|J\mu)^2\Delta^{(J)}_{\mu M}D^{(J)}_{\mu M}(\phi_1\theta\phi_2).$$

With the help of (18) and (21) we can rewrite the expression (43)

$$P^{(j_1+\frac{1}{2}, j_2+\frac{1}{2})}_{\frac{K-j_1-j_2}{2}}(\vec{\xi},\vec{\eta}) = \sum_m \frac{1}{2^n}\binom{n+j_1+\frac{1}{2}}{m}\binom{n+j_2+\frac{1}{2}}{n-m}(-1)^{n-m}(uu^*)^{m}(vv^*)^{n-m}. \tag{53}$$

Substituting into (45) formulae (52) and (53), we obtain the eigen-function $\Phi(\vec{\xi},\vec{\eta})$ in the following form:

$$\Phi(\vec{\xi},\vec{\eta}) = A'_{JM}\sum_m\sum_{\mu_1\mu_2}\frac{(-i)^{\mu_2}}{2^{j_1+j_2+n}}(-1)^{n-m}(j_1\mu_1; j_2\mu_2|J\mu)^2\,\Delta^{(J)}_{0\mu}\binom{n+j_1+\frac{1}{2}}{m}\binom{n+j_2+\frac{1}{2}}{n-m}\times$$

$$\times\, u^{*\frac{j_1-\mu_1}{2}+m}\, v^{\frac{j_2+\mu_2}{2}+n-m}\, u^{\frac{j_1+\mu_1}{2}+m}\, v^{*\frac{j_2-\mu_2}{2}+n-m}\, D^{J}_{\mu M}(\phi_1\theta\phi_2),$$

where

$$A'_{JM} = (-1)^{\frac{M}{2}} \left[ \frac{(j_1+M_1)!(j_2+M_2)!}{(j_1-M_1)!(j_2-M_2)!} \right]^{\frac{1}{2}} (j_1 0, j_2 0 | J 0)( j_1 M_1, j_2 M_2 | JM).$$

We have pointed out it before, that all the following calculations are needed just to get $\Phi(\vec{\xi}, \vec{\eta})$ in a form for which the Fourier transform becomes relatively easy. We will use now the equation given in /7/

$$\frac{1}{\sqrt{(j-k)!(j+k)!}} \left(\cos\frac{\alpha}{2} e^{i\frac{\gamma}{2}} + i\sin\frac{\alpha}{2} e^{-i\frac{\gamma}{2}}\right)^{j-k} \left(i\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2} e^{-i\frac{\gamma}{2}}\right)^{j+k} =$$

$$= \sum_{\ell=-j}^{j} \frac{P^{j}_{\ell k}(\cos\alpha)}{\sqrt{(j-\ell)!\,(j+\ell)!}} e^{-i\ell\gamma} \tag{55}$$

remembering

$$\bar{P}^{j}_{\ell k}(\cos\alpha) = (-1)^{\ell-k} P^{j}_{-\ell-k}(\cos\alpha). \tag{56}$$

Indeed, if we compare (55) and its conjugated with the expressions (14) for $u$, $v$, $u^*$ and $v^*$, we see that

$$u^{*\,\frac{j_1-\mu_1}{2}+m}\; v^{\frac{j_2+\mu_2}{2}+n-m} = \left[\left(\frac{j_1-\mu_1}{2}+m\right)!\left(\frac{j_2+\mu_2}{2}+n-m\right)!\right]^{\frac{1}{2}} \times$$

$$\times \sum_{\nu_1=-(\frac{K-\delta}{4})}^{\frac{K-\delta}{4}} P^{\frac{K-\delta}{4}}_{\nu_1, W+\frac{\mu}{4}}(\cos\alpha) \frac{e^{-i\nu_1\lambda}}{\left[\left(\frac{K-\delta}{4}-\nu_1\right)!\left(\frac{K-\delta}{4}+\nu_1\right)!\right]^{\frac{1}{2}}} \tag{57}$$

and

$$u^{\frac{j_1+\mu_1}{2}+m} v^{*\,\frac{j_2-\mu_2}{2}+n-m} = \left[\left(\frac{j_1+\mu_1}{2}+m\right)!\left(\frac{j_2-\mu_2}{2}+n-m\right)!\right]^{1/2} \times \tag{58}$$

$$\times \sum_{\nu_2=-\left(\frac{K+\delta}{4}\right)}^{\frac{K+\delta}{4}} P^{\frac{K+\delta}{4}}_{-\nu_2,-W+\frac{\mu}{4}}(\cos a)(-1)^{\nu_2-W+\frac{\mu}{4}} \frac{e^{i\nu_2\lambda}}{\left[\left(\frac{K+\delta}{4}-\nu_2\right)!\left(\frac{K+\delta}{4}+\nu_2\right)!\right]^{1/2}},$$

where

$$\delta = \mu_1 - \mu_2$$

$$W = \frac{j_2 - j_1}{4} + \frac{n}{2} - m \quad .$$

Thus, carrying out the expansion of $P^{\frac{K-\delta}{4}}_{\nu_1,W+\frac{\mu}{4}}(\cos a)\, P^{\frac{K+\delta}{4}}_{-\nu_2,-W+\frac{\mu}{4}}(\cos a)$ in terms of the functions $P^{\frac{K}{2}-\kappa}_{\nu,\frac{\mu}{2}}(\cos a)$ (where $\nu = \nu_1 - \nu_2$ ) the eigenfunction $\Phi(\vec{\xi},\vec{\eta})$ might be obtained in the form

$$\Phi(\vec{\xi},\vec{\eta}) = A'_{JM} \sum_{m} \sum_{\mu\,\delta} \sum_{\nu\,\epsilon} \sum_{\kappa} \frac{1}{2^{j_1+j_2+n}} (-1)^{\frac{K-\delta}{4}+\frac{\epsilon-\nu}{2}+\frac{\mu}{2}-\frac{j_2}{2}} \times$$

$$\times \left(j_1 \frac{\mu+\delta}{2}; j_2 \frac{\mu-\delta}{2} \middle| J\mu\right)^2 \left(\frac{K-\delta}{4}, \frac{\nu+\epsilon}{2}; \frac{K+\delta}{4}, \frac{\nu-\epsilon}{2} \middle| \frac{K}{2}-\kappa, \nu\right)\left(\frac{K-\delta}{4}, W+\frac{\mu}{4}; \frac{K+\delta}{4}, -W+\frac{\mu}{4} \middle| \frac{K}{2}-\kappa, \frac{\mu}{2}\right) \times$$

$$\times \Delta^{(J)}_{0\mu} \begin{pmatrix} n+j_1+\frac{1}{2} \\ m \end{pmatrix} \begin{pmatrix} n+j_2+\frac{1}{2} \\ n-m \end{pmatrix} \left[\frac{\left(\frac{j_1}{2}-\frac{\mu+\delta}{4}+m\right)!\left(\frac{j_1}{2}+\frac{\mu+\delta}{4}+m\right)!\left(\frac{j_2}{2}+\frac{\mu-\delta}{4}+n-m\right)!\left(\frac{j_2}{2}-\frac{\mu-\delta}{4}+n-m\right)!}{\left(\frac{K-\delta}{4}-\frac{\nu+\epsilon}{2}\right)!\left(\frac{K-\delta}{4}+\frac{\nu+\epsilon}{2}\right)!\left(\frac{K+\delta}{4}-\frac{\nu-\epsilon}{2}\right)!\left(\frac{K+\delta}{4}+\frac{\nu-\epsilon}{2}\right)!}\right]^{1/2} \times$$

$$\times P^{\frac{K}{2}-\kappa}_{\nu,\mu/2}(\cos a)\, D^{J}_{\mu M}(\phi_1 \theta \phi_2)\, e^{-i\nu\lambda} \quad .$$

Making use of different relations for the Clebsch-Gordan coefficients, we can transform (59) in such a way that it becomes easy to take the sum over $\epsilon = \nu_1 + \nu_2$ by introducing

$$P_k^{(\alpha,\beta)}(0) = \frac{1}{2^k} \sum_{m=0}^{k} \binom{k+\alpha}{m}\binom{k+\beta}{n-m}(-1)^{k-m} . \tag{60}$$

Concerning the sum over $\nu$, in the following only one definite term of it will be considered. Then, after tedious calculations we get the final expression

$$\Phi(\vec{\xi},\vec{\eta}) = A_{JM} \sum_m \sum_{\mu,\delta} \sum_{\kappa} \left(j_1 \frac{\mu+\delta}{2}, j_2 \frac{\mu-\delta}{2} \middle| J\mu\right)^2 \frac{\left(\frac{K-\delta}{4}, W+\frac{\mu}{4}; \frac{K+\delta}{4}, -W+\frac{\mu}{4} \middle| \frac{K}{2}-\kappa ; \frac{\mu}{2}\right)}{\left(\frac{j_1}{2}+m, \frac{\mu+\delta}{4}; \frac{j_2}{2}+n-m, \frac{\mu-\delta}{4} \middle| \frac{K}{2}; \frac{\mu}{2}\right)} \times$$

$$\times \frac{(-1)^{\frac{K+\mu-\delta}{4} - \frac{\nu}{2}+\kappa}}{2^{K/4}} \frac{\Delta^{(J)}_{0\mu} \Delta^{(\frac{K}{2}-\kappa)}_{\frac{\delta}{2},\nu}}{\Delta^{(\frac{K}{2})}_{\frac{K}{2},\frac{\mu}{2}}} \left[\frac{K-2\kappa+1}{(K+\kappa+1)!\,\kappa!}\right]^{\frac{1}{2}} \times \tag{61}$$

$$\times \binom{n+j_1+\frac{1}{2}}{m}\binom{n+j_2+\frac{1}{2}}{n-m} \sqrt{(j_1+2m)!\,(j_2+2n-2m)!} \times$$

$$\times D^{\frac{K}{2}-\kappa}_{\nu,\frac{\mu}{2}}(\lambda, a, 0)\, D^{J}_{\mu,M}(\phi_1\ \theta\phi_2),$$

where

$$A_{JM} = \frac{(-1)^{-\frac{j_2}{2}}}{2^{\frac{j_1+j_2}{2}}} A'_{JM} .$$

Thus, the general solution of the problem is obtained in the form (3). (In our notation there will be $M' = \frac{\mu}{2}$ , $\Lambda = \frac{K}{2} - \kappa$ ). Looking at the structure of the coefficient of $D^{\frac{K}{2}-\kappa}_{\nu,\frac{\mu}{2}}(\lambda, a, 0) D^{J}_{\mu,M}(\phi_1 \theta \phi_2)$ it is easy to understand, that our attempts /3/ to determine $a_\nu(\lambda, M')$ explicitly couldn't be successful.

## §5. Another Way of Calculating the Eigenfunction $\Phi(\vec{\xi}, \vec{\eta})$

While calculating the explicit form of the eigenfunction, we noticed, that besides the expression for it in the form of the product of two D-functions there is another possible solution, which might be more convenient in the following. We begin with the expression (54). Remembering the explicit form of $u$ , $v$ , $u^*$ and $v^*$ (14), we expand them in a power series of $\sin\frac{a}{2}$ and $\cos\frac{a}{2}$ :

$$u^A = \sum_{s=-\frac{A}{2}}^{\frac{A}{2}} \begin{pmatrix} A \\ \frac{A+s}{2} \end{pmatrix} \left(\cos\frac{a}{2}\right)^{\frac{A-s}{2}} \left(\sin\frac{a}{2}\right)^{\frac{A+s}{2}} (-i)^{\frac{A+s}{2}} e^{i\lambda\frac{s}{2}}$$

$$v^B = \sum_{t=-\frac{B}{2}}^{\frac{B}{2}} \begin{pmatrix} B \\ \frac{B+t}{2} \end{pmatrix} \left(\cos\frac{a}{2}\right)^{\frac{B-t}{2}} \left(\sin\frac{a}{2}\right)^{\frac{B+t}{2}} (i)^{\frac{B+t}{2}} e^{i\lambda\frac{t}{2}} \qquad (62)$$

$$u^{*C} = \sum_{u=-\frac{C}{2}}^{\frac{C}{2}} \binom{C}{\frac{C+u}{2}} \left(\cos\frac{a}{2}\right)^{\frac{C+u}{2}} \left(\sin\frac{a}{2}\right)^{\frac{C-u}{2}} (i)^{\frac{C-u}{2}} e^{i\lambda\frac{u}{2}}$$

$$v^{*D} = \sum_{v=-\frac{D}{2}}^{\frac{D}{2}} \binom{D}{\frac{D+v}{2}} \left(\cos\frac{a}{2}\right)^{\frac{D+v}{2}} \left(\sin\frac{a}{2}\right)^{\frac{D-v}{2}} (-i)^{\frac{D-v}{2}} e^{i\lambda\frac{v}{2}} .$$

Using these expressions we can write

$$u^{*\,\frac{j_1-\mu_1}{2}+m}\; v^{\frac{j_2+\mu_2}{2}+n-m}\; u^{\frac{j_1+\mu_1}{2}+m}\; v^{*\,\frac{j_2-\mu_2}{2}+n-m} =$$

$$= \sum_{\substack{s,t,u,v\\ s+t+u+v=-2\nu}} \binom{\frac{j_1+\mu_1}{2}+m}{\frac{\frac{j_1+\mu_1}{2}+m+s}{2}} \binom{\frac{j_2+\mu_2}{2}+n-m}{\frac{\frac{j_2+\mu_2}{2}+n-m+t}{2}} \binom{\frac{j_1-\mu_1}{2}+m}{\frac{\frac{j_1-\mu_1}{2}+m+u}{2}} \binom{\frac{j_2-\mu_2}{2}+n-m}{\frac{\frac{j_2-\mu_2}{2}+n-m+v}{2}} \times$$

$$\times \left(\cos\frac{a}{2}\right)^{\frac{K-s-t+u-v}{2}} \left(\sin\frac{a}{2}\right)^{\frac{K+s+t-u-v}{2}} (i)^{\frac{-\delta-s+t-u+v}{2}} e^{i\left(\frac{s+t+u+v}{2}\right)\lambda}$$

Two of these sums can be easily taken if we introduce $P_k^{(\alpha,\beta)}(0)$ in the form (60). Thus we have

$$\Phi(\vec{\xi},\vec{\eta}) = A'_{JM} \sum_m \sum_{\mu,\delta} \frac{(-i)^{\frac{\mu-\delta}{2}}}{2^{K-n}} (-1)^{n-m} \left(j_1\,\frac{\mu+\delta}{2},\, j_2\,\frac{\mu-\delta}{2} \,\middle|\, J\mu\right)^2 \Delta^{(J)}_{0\mu} \times$$

$$\times \begin{pmatrix} n+j_1+\frac{1}{2} \\ n \end{pmatrix} \begin{pmatrix} n+j_2+\frac{1}{2} \\ n-m \end{pmatrix} \sum_{\substack{t,u \\ s+t+u+v=-2\nu}} (i)^{-\frac{K}{2}-\frac{\delta}{2}} 2^{\frac{K}{2}} e^{-i\nu\lambda} \times$$

$$\times \left[ \frac{(\frac{j_1}{2}+\frac{\mu+\delta}{4}+m)!(\frac{j_1}{2}-\frac{\mu+\delta}{4}+m)!\,(\frac{j_2}{2}+\frac{\mu-\delta}{4}+n-m)!(\frac{j_2}{2}-\frac{\mu-\delta}{4}+n-m)}{(\frac{K}{4}+\frac{\mu}{4}+\frac{s+t}{2})!(\frac{K}{4}+\frac{\mu}{4}-\frac{s+t}{2})!(\frac{K}{4}-\frac{\mu}{4}+\frac{u+v}{2})!(\frac{K}{4}-\frac{\mu}{4}-\frac{u+v}{2})!} \right]^{1/2} \times \tag{64}$$

$$\times \Delta^{(\frac{K}{4}+\frac{\mu}{4})}_{-\frac{s+t}{2},\, W-\frac{\delta}{4}} \Delta^{(\frac{K}{4}-\frac{\mu}{4})}_{-\frac{u+v}{2},\, -W-\frac{\delta}{4}} \times$$

$$\times (\cos\frac{a}{2})^{\frac{K}{2}-\frac{(s+t)-(u+v)}{2}} (\sin\frac{a}{2})^{\frac{K}{2}+\frac{(s+t)-(u+v)}{2}} D^{J}_{\mu,M}(\phi_1\,\theta\phi_2).$$

This formula can be rewritten in the following form

$$\Phi(\vec{\xi},\vec{\eta}) = \sum_{\mu,\delta,\sigma,W} N(K,\nu,j_1,j_2 \mid \mu,\delta,\sigma,W) \times$$

$$\times (\cos a+1)^{\frac{K}{4}-\frac{\sigma}{4}} (\cos a-1)^{\frac{K}{4}+\frac{\sigma}{4}} D^{J}_{\mu M}(\phi_1\,\theta\phi_2). \tag{65}$$

where $\sigma = s + t - (u + v)$ , and

$$N(K,\nu,j_1,j_2\,|\,\mu,\delta,\sigma,W) = A_{JM}(-1)^{n-m}\left(j_1\frac{\mu+\delta}{2},j_2\frac{\mu-\delta}{2}\Big|J\mu\right)^2\times$$

$$\times\,\Delta^{(J)}_{0\mu}\begin{pmatrix} n+j_1+\frac{1}{2} \\ \\ m\end{pmatrix}\begin{pmatrix} n+j_2+\frac{1}{2} \\ \\ n-m\end{pmatrix}\cdot\tilde{\tilde{\Delta}}^{\left(\frac{K}{4}+\frac{\mu}{4}\right)}_{\frac{\nu}{2}-\frac{\sigma}{4},\,W-\frac{\delta}{4}}\;\tilde{\tilde{\Delta}}^{\left(\frac{K}{4}-\frac{\mu}{4}\right)}_{\frac{\nu}{2}+\frac{\sigma}{4},\,-W-\frac{\delta}{4}}\,. \tag{66}$$

We have introduced here the notation

$$\sqrt{\frac{(\ell-n)!\,(\ell+n)!}{(\ell-m)!\,(\ell+m)!}}\;\Delta^{(\ell)}_{mn} = \tilde{\tilde{\Delta}}^{(\ell)}_{mn} \tag{67}$$

$A_{JM}$ is the normalization constant, the explicit form of which is now out of interest for us. The boundaries of summations in (65) are

$$-K \le \sigma \le K\;,\quad -\frac{K-2j_2}{4} \le W \le \frac{K-2j_1}{4}\,. \tag{68}$$

The boundaries in $\mu$ and $\delta$ with given $\sigma$ and $m$ are defined by the factorial in the denominators of the functions $\tilde{\tilde{\Delta}}$ .

## §6. Conclusions

The problem of constructing a basis for a system of three free particles, realizing representations of the three-dimensional rotation group and of the permutation group, turned out to be, quite unexpectedly to the authors, rather complicated. We couldn't solve directly the obtained set of equations for determining the eigenfunctions [x/], and had to construct the solution in another way. Functions were constructed, which fulfill four of the given five equations, so that the final solution (with the quantum number $\Omega$ ) is to be found by substituting a linear combination of solutions with several $(j_1 j_2)$ in the equation of $\Omega$ obtained in [/3/]. It is not difficult to get the solution of the eigenvalue equation for $\Omega$ in every particular case. (For polynomials it was already done in [/5/]). However, if seems to us, that there must be some more effective means for the orthogonalization of the polynomials, may be even without using the operator $\Omega$ , by some symmetrization procedure. In any case this question will be discussed separately.

Besides that, there are several problems arising from our work. It is necessary to generalize the method of expansion of functions into series in terms of functions with given partial momenta in the case of the total angular momentum $J \neq 0$ . It would be very interesting to know how to construct an approximatively orthonormal set of functions with a cut off at a certain given value $j_1 = j_2 = j$ ; such a set may be convenient for the calculation of matrix elements.

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[x/] The obtained formulae are complicated, because we got polynomials which are not classical and their theory is not worked out yet. If our method will lead to useful results, it will not be difficult (in principle, at least) to study the properties of these new polynomials and tabulate them.

Note, that the transition to a larger number of particles makes the formulae still more complicated; in a certain sense this is similar to the transition from the hypergeometrical function of one variable to hypergeometrical functions of few variables, the theory of which is also almost not known.

After the orthogonalization of the wave functions obtained in the present paper it is reasonable to study the problem of the origin of the rotational spectrum of the system. For that purpose we have to construct a wave packet as a superposition of "spherical harmonics" with different $\Omega$ and $K$, which have approximately conserved projections of the angular momentum to the normal of the plane of the triangle. The detailed discussion of this problem will follow in our next paper.

It would be also of interest to see, whether it is possible to make use of an expansion of this kind for the motion of a massive top. Especially interesting (and so far not well understood) is the case of the Kovalevskaya top, the quantum analogue of which is not known yet[x/].

A possible application of the technics worked out in the present paper could be the classification of Dalitz plots of the three-particle decays. It seems to us now, that the generalization of our results to the relativistic case cannot be too difficult.

Finally, from a practical point of view, it will be important to work out the method for the calculation of matrix elements of pairwise interactions.

## APPENDIX

To obtain the contribution of the partial momenta in the $J=0$ state, it is necessary to calculate the Fourier coefficient of the function

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x/ In this case we must start with a more general expansion.

$$\Phi_0(\xi,\eta) = (\cos\Phi)^j (\sin\Phi)^j P_{\frac{K}{2}-j}^{(j+\frac{1}{2},j+\frac{1}{2})}(\cos 2\Phi) P_j(n,m). \qquad (A.1)$$

From the condition $\xi^2+\eta^2=1$ it follows, that we can take $\cos^2\Phi=\xi^2$ and $\sin^2\Phi=\eta^2$ and rewrite (A.1) in the form

$$\Phi_0(\xi,\eta) = (\xi^2)^{j/2} (\eta^2)^{j/2} P_{\frac{K}{2}-j}^{(j+\frac{1}{2},\, j+\frac{1}{2})}(\xi^2-\eta^2) P_j(n.m). \qquad (A.2)$$

The zero total angular momentum state is composed from two equal partial mometa $j_1=j_2=j$ . As soon as in this case the eigenvalue of $\Omega$ plays no role, it is obvious, that the $J=0$ state can be obtained from the states described in the present paper by replacing the quantum number $\nu$ by $j$ . Thus, the function (A.2) has to be the superposition

$$\sum_{\nu} C(j,\nu) D^{1/2}_{\frac{\nu}{2},-\frac{\nu}{2}}(2\lambda,2a,0) \qquad (A.3)$$

and we have to calculate the coefficient $C(j,\nu)$ . The Fourier coefficient of (A.2) will be obtained by the use of the condition $n\cdot m=1$, which means, as it can be seen from (21), $a=\frac{\pi}{2}$ . On the other hand,

$$\cos 2\Phi = \sin a \ \sin\lambda$$

which leads in the case $a = \frac{\pi}{2}$ to

$$\cos 2\Phi = \sin\lambda$$

$$\sin 2\Phi = \cos\lambda \;. \qquad \text{(A.4)}$$

To get the standard formulae, we put $\sin\lambda = \cos\Lambda$ and use the Gegenbauer polynomial instead of the Jacobi polynomial:

$$P^{(j+\frac{1}{2},j+\frac{1}{2})}_{\frac{K}{2}-j}(\cos 2\Phi) = \frac{\Gamma(2j+2)\,\Gamma(K+\frac{3}{2})}{\Gamma(\frac{K}{2}+j+2)\,\Gamma(j+\frac{3}{2})}\, C^{j+1}_{\frac{K}{2}-j}(\cos 2\Phi). \qquad \text{(A.5)}$$

Doing so, we obtain from (A.1)

$$(\cos\Phi)^j (\sin\Phi)^j \frac{\Gamma(2j+2)\,\Gamma(K+\frac{3}{2})}{\Gamma(\frac{K}{2}+j+2)\,\Gamma(j+\frac{3}{2})}\, C^{j+1}_{\frac{K}{2}-j}(\cos 2\Phi). \qquad \text{(A. 6)}$$

Making use of the integral representation of the Gegenbauer polynomial /7/, one can write

$$\frac{1}{2^j}(\sin\Lambda)^j\, C^{j+}_{\frac{K}{2}-j}(\cos\Lambda) = \frac{i^j}{2^{2j+1}}\,\frac{\Gamma(2+j+\frac{K}{2})}{\frac{K}{2}!\,\Gamma(j+1)} \times$$

$$\times \int_0^\pi (\cos\Lambda - i\sin\Lambda\cos\theta)^{\frac{K}{2}}\, C^{\frac{1}{2}}_j(\cos\theta)\sin\theta\, d\theta \,. \qquad \text{(A.7)}$$

Remembering $C^{\frac{1}{2}}_j(\cos\theta) = P_j(\cos\theta)$, we have

$$\Phi_0(\vec{\xi},\vec{\eta}) = \frac{i^j}{2^{2j+1}}\,\frac{\Gamma(2j+2)\,\Gamma(K+\frac{3}{2})}{\frac{K}{2}!\,\Gamma(j+1)\,\Gamma(j+\frac{3}{2})} \times$$

$$\times \int_0^\pi (\cos\Lambda - i\sin\Lambda\cos\theta)^{\frac{K}{2}}\, P_j(\cos\theta)\sin\theta\, d\theta \,. \qquad \text{(A.8)}$$

It is easy to obtain the Fourier coefficient directly, expanding $\cos\Lambda$ and $\sin\Lambda$ in terms of the exponents and opening the parentheses. For the terms with $e^{-i\nu\Lambda}$ in the integral we can write then

$$\begin{pmatrix} \frac{K}{2} \\ \\ \frac{K}{4}-\frac{\nu}{2} \end{pmatrix} \int_0^{\pi} (\sin\frac{\theta}{2})^{\frac{K}{2}-\nu} (\cos\frac{\theta}{2})^{\frac{K}{2}+\nu} P_j(\cos\theta)\, e^{-i\nu\Lambda} \sin\theta\, d\theta \qquad (A.9)$$

or, in another form:

$$i^{\nu-\frac{K}{2}} \int_0^{\pi} P^{\frac{K}{4}}_{\frac{K}{4},\frac{\nu}{2}}(\cos\theta)\, P^{\frac{K}{4}}_{-\frac{K}{4},-\frac{\nu}{2}}(\cos\theta)\, P^{j}_{00}(\cos\theta)\, e^{-i\nu\Lambda} \sin\theta\, d\theta . \qquad (A.10)$$

Finally, we obtain

$$\Phi_0(\vec{\xi},\vec{\eta}) = \frac{i^{j-\frac{K}{2}}}{2^{2j}\sqrt{2j+1}} \frac{\Gamma(2j+2)\,\Gamma(K+\frac{3}{2})}{\Gamma(j+1)\,\Gamma(j+\frac{3}{2})} \times \qquad (A.11)$$

$$\times \left(\frac{K}{4},\frac{\nu}{2};\frac{K}{4},-\frac{\nu}{2}\Big| j\,0\right) \frac{e^{i\nu\lambda}}{\sqrt{(\frac{K}{2}-j)!\,(\frac{K}{2}+j+1)!}} .$$

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