ОБЪЕДИНЕННЫЙ
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Дубна

E2 - 4733

V.A.Meshcheryakov, K.V.Rerikh

A METHOD OF THE LOCAL CONSTRUCTION OF INVARIANT SUBSPACES IN THE SPACE OF SOLUTIONS OF THE CHEW-LOW TYPE EQUATIONS

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1969

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Submitted to "Annals of Phys."

and "Теоретическая и математическая физика".

Abstract

In the introduction the nonlinear system of functional equations for the matrix elements of the S-matrix is formulated, which we call the Chew-Low type equations. A short review of some papers devoted to equations of this type is given, and the advantages of the approach of the present paper are discussed.

In part II, §1 the transition to the projective coordinates in the space of the matrix elements of the S-matrix and the linearization of the unitarity conditions are performed.

An interpretation of the system of functional equations as a transformation in the $(n-1)$ dimensional real space is given. It is shown that some of the solutions of the initial system of equations are contained on the invariant hypersurfaces of this space (§2).

In §3 a method of the local construction of the invariant subspaces is proposed. In part III the method suggested is applied to the Chew-Low equations with the 3x3 crossing matrix. It is established that, if the Chew-Low equations possess a solution, then the arbitrariness of the solutions of the class (2.12), being the generalization of the familiar β-arbitrariness, is not exhaustive.

I. Introduction

The problem of P-wave scattering of π-mesons by a static nucleon with the two-particle approximation to the unitarity condition is described by the Chew-Low equations[1], which are of the form:

$$h_i(\omega) = \frac{\lambda_i}{\omega} + \frac{1}{\pi}\int_1^{\infty}\left[\frac{\operatorname{Im} h_i(\omega')}{\omega' - \omega} + \frac{A_{ij}\operatorname{Im} h_j(\omega')}{\omega' + \omega}\right] d\omega'. \quad (1.1)$$

Here $h_i(\omega) = \dfrac{e^{i\delta_i(\omega)}\sin\delta_i(\omega)}{q^3 u^2(q^2)}$, $\delta_i(\omega)$ is the scattering phase shift in the state with definite values of total isospin T and total angular momentum $J\,(i=(2T,2J))$, ω is the meson total c.m. energy, $\omega = \sqrt{q^2+\mu^2}$ and μ (the meson mass) $=1$, $u(q^2)$ is the Fourier transform of the source function, and A_{ij} are the elements of the crossing matrix. The numbers λ_i are proportional to the square of meson-nucleon coupling constant and satisfy the relations

$$A_{ij}\lambda_j = -\lambda_i .$$

The Chew-Low equations were first established within the framework of the special model of πN scattering characterized by a static source interaction[2]. However, they can be obtained as the static limit of the rigorously proved relativistic dispersion relations. It is well known[4] that the analytic properties of the partial waves $f_i(\omega) = \frac{e^{i\delta_i(\omega)}\sin\delta_i(\omega)}{q^{2\ell+1}}$ are not as simple as those of the functions $h_i(\omega)$. In the static limit the latter have, in addition to the two cuts $(-\infty,-1]$, $[+1,+\infty)$ from the u and s channels, also the cut $(-i\infty,+i\infty)$ from the t-channel $(\pi\bar{\pi}\to N\bar{N})$. In ref.[5] the method of obtaining the equations (1.1) from the Mandelstam representation has been proposed and the interpretation of the cut-off function $u^2(q^2)$ as a model of the t channel cut has been given. Following ref.[5] we shall suppose that the cut $(-i\infty,+i\infty)$ can be represented by the system of poles corresponding to the resonances in the t channel and the function $\frac{1}{u^2(q^2)}$ is an entire function of q^2. It is also known[6] that the crossing properties of the functions $f_\ell(\omega)$ are essentially more complex before the transition to the static limit than is assumed in (1.1). The function $f_\ell(-\omega)$ is expressed in terms of the infinite system of partial-wave amplitudes and their derivatives. In the static limit this relation is significantly simplified and is of the form

$$h_{\ell,i}(-\omega) = \sum_j A_{ij} h_{\ell,j}(\omega),$$

where i,j assume the values (1.1, 1.3, 3.1 and 3.3).

Recently some inportant results proving the existence of a solution of equations (1.1) have been obtained in ref.[19,20] assuming the smallness of the coupling constant and also some restrictions on the asymptotic behaviour of the functions $h_l(\omega)$ as $\omega\to\infty$ x/ xx/.

The equations analogous to (1.1) can be established for the higher waves in πN scattering and for other two-particle processes. The specific properties for any particular problem are as follows:

1) the assignment of the meson and source quantum numbers and also the group symmetry under which the interaction is invariant;

2) the assumption about the mass spectrum of the meson plus source system;

3) the assumption about the asymptotic behaviour of the functions $h_l(\omega)$.

Condition 1) allows us to establish the type of crossing matrix A and conditions (2) and 3) fix the poles and the rate of growth of the functions $h_l(\omega)$.

However all these equations have certain common properties[8], which are conveniently formulated in terms of the functions $S_l(\omega)$, where

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x/ For the validity of the existence theorems in ref.[19,20] the admissible upper bound $\sup_l \{\lambda_l\}$ turns out to be an order of magnitude less than the experimental value of the coupling constant. It is obvious that the existence of a solution of (1.1) is in no way related to the smallness of the coupling constant and this limitation is simply a consequence of the method employed in ref.[19,20].

xx/ The authors thank Dr. R.Denchev for a discussion of ref.[19,20].

$$S_l(\omega) = e^{2i\delta_l(\omega)} = 1 + 2iq^{2\ell+1}u^2(q^2)h_l(\omega):$$

1. $S_l(\omega)$ are analytic functions in the complex ω-plane with the cuts $(-\infty,-1],[+1,+\infty)$;
2. $S^*_l(\omega)=S_l(\omega^*)$ is the reality condition;
3. $|S_l(\omega+i0)|^2=1$ for $\omega>1$ is the unitarity condition; (1.2)
4. $S_l(-\omega)=\sum_j A_{lj}S_j(\omega)$ is the crossing relation,

where A is an $(n\times n)$ matrix with the properties[13]

$$A^2 = E \quad \text{and} \quad \sum_j A_{lj} = 1. \tag{1.3}$$

We shall call these equations (1.2) the Chew-Low type equations[13].

The advantages of this approach to the problem (1.1) are the fact that the cut-off function $u^2(q^2)$ is not contained in the equations (1.2), that a definite asymptotic behaviour of the functions $S_l(\omega)$ at infinity is not assumed, and that the coupling constant is not explicitly present.

One must satisfy additional local conditions on the functions $S_l(\omega)$ in the construction of the concrete physical solutions of the problem (1.2):

$$\begin{aligned} &1. \quad S_l(\omega)\underset{\omega\to 1}{=} 1 + O(q^{2\ell+1}) && \text{is the threshold behaviour,} \\ &2. \quad S_l(\omega)\underset{\omega\to 0}{\sim} \frac{\lambda_l}{\omega} && \text{is the Born behaviour.} \end{aligned} \tag{1.4}$$

Many articles[7-21] have been devoted to the investigation of the Chew-Low type equations. However, the complete solution of these equations has so far been obtained only for the 2x2 crossing matrix[9-11].

In ref.[13] it has been demonstrated that the Riemann surface of the functions $S_i(\omega)$ has infinitely many sheets because of the logarithmic branch point at infinity. To take into account this branch point and also the first order branch points in $S_i(\omega)$ at $\omega = \pm 1$, we shall make the conformal transformation[8–10]

$$w = \frac{1}{\pi} \arcsin \omega ,$$

which maps the physical sheet into the strip $|\mathrm{Re}\, w| < \frac{1}{2}$ and leads to the uniformization[23] of the functions $S_1(\omega)$.

Conditions (1.2) are of the following form in the variable w :

1) $S(w)$ is a column matrix of meromorphic functions in the w–-plane;

2) $S^*(w) = S(w^*)$ is the reality condition;

3) $S(1-w) = IS(w)$ is the unitarity condition; (1.5)

4) $S(-w) = AS(w)$ is the crossing relation,

where by I we shall denote the nonlinear operation of the type

$$IS(w) = \begin{pmatrix} 1/S_1 \\ 1/S_2 \\ \vdots \\ 1/S_n \end{pmatrix} .$$

If we obtain all solutions of the system of nonlinear functional equations 3), 4) of (1.5) in the class of functions defined by 1) and 2) of (1.5), satisfying conditions (1.4), then we can construct the solution of the equations (1.1).

It is well known[13] that conditions (1–4) of (1.5) do not uniquely determine the functions $S_1(w)$. If one has found the functions $S_1(w)$ satisfying (1–4), then the functions $S_1(w+\beta(w))\,D(w)$, where

$$D^*(w) = D(w^*), \quad D(-w) = D(w), \quad D(w)D(1-w) = 1,$$
$$\beta^*(w) = \beta(w^*), \quad \beta(-w) = -\beta(w), \quad \beta(w+1) = \beta(w), \tag{1.6}$$

also satisfy (1-4).

Moreover, if the matrix A can be decomposed into the direct product of two matrices A_1 and A_2 with the same properties (1.3), then at least some of the solutions (1.5) can be represented in the form of the direct product of solutions $S^{(1)}(w)$ and $S^{(2)}(w)$, which individually depend on arbitrary functions $\beta_1(w)$ and $\beta_2(w)$, respectively[13]. In addition to this continuous family of solutions (1.5), a discrete ambiguity is also possible. An example of this is contained in ref.[13,16].

Digressing from the arbitrariness in (1.5) one can speak about the "skeleton" solutions[16] and their classification. The class of solutions (1.5), for which any ratio $S_i(w)/S_j(w)$ of "skeleton" solutions has a limited number of poles, has been obtained for 3x3 and 4x4 crossing matrices in ref.[10-13]. In ref.[13,15] a method of constructing the class of solutions for arbitrary orders of the crossing matrix has been developed.

However, in ref.[16] it has been demonstrated that a solution of equations 3), 4) of (1.5) exists, for which one of the ratios S_i/S_j is a transcendental meromorphic function of w. The difficulty of solving the system 3), 4) of (1.5) is explained by the absence of general methods of solving nonlinear functional equations. The success in finding the complete solution for the 2×2 matrix A is easily understood. In this case the ratio $x(w) = S_1(w)/S_2(w)$ satisfies the simplest nonlinear functional equation

$$x(w+1) = \frac{\alpha x(w) + \beta}{x(w) + \gamma},$$

whose general solution can easily be obtained on the basis of its geometrical interpretation[16]. Moreover, in ref.[16] it has been demonstrated that it is convenient to solve equations 3), 4) of (1.5) on the basis of the preliminary treatment of the functional relations among $S_i(w)$ which satisfy (1.5). The first step in this treatment has been made in ref.[17], in which the authors proposed an algorithm for obtaining the polynomial functional relations among the functions $S_i(w)$ by means of the heuristic use of a computer. In ref.[18] a more direct method of investigating this class of functional relations has been proposed. In the present article the general method of finding solutions of equations 3), 4) of (1.5), contained in the invariant subspaces, is developed.

II. The Method of Finding Solutions of the System (1.5) Contained in the Invariant Subspaces

§1. The transition to the projective coordinates. The linearization of the unitarity condition.

Let S be the n -dimensional complex space of meromorphic real functions of the complex variable w. Then the n -dimensional vector $S(w)$ in this space satisfies the equations

$$\begin{aligned} &1.\ S(1-w) = IS(w) \\ &2.\ S(w+1) = IAS(w), \end{aligned} \tag{2.1}$$

which follow from (1.5).

As is easily seen the operators I and IA of (2.1) are real operators satisfying the following conditions:

$$I^2 = E$$

$$(IA)^{-1} = AI. \tag{2.2}$$

Thus the inverse operator to (IA) exists and is unique. Clearly, from (2.1) it follows that the powers of the operators $(IA)^n$ and $(AI)^n$ are the operators of the continuation of the values of the functions $S_i(w)$ from the strip $|\mathrm{Re}\, w| \leq \frac{1}{2}$ into the strips $|\mathrm{Re}(w \pm n)| \leq \frac{1}{2}$. It is seen from 1) of (2.1) that the system of poles and zeros of the functions $S_i(w)$ is symmetrical about the line $\mathrm{Re}\, w = \frac{1}{2}$.

It has been mentioned above that the solutions of equations (2.1) possess the arbitrariness (1.6). To take into account the existence of the D-arbitrariness in the functions $S_i(w)$ it is convenient to make the transition to the projective coordinates $x_i(w) = S_i(w)/S_k(w)$, $i=1,2,\ldots,n$, $i \neq k$, where we can choose $k = n$ for definiteness, although in concrete cases it may be more convenient to make another choice. The transition to the projective coordinates $x_i(w)$ is also convenient because the functions $x_i(w)$ will not contain the common, possibly infinite, system of poles and zeros in the functions $S_i(w)$.

Then according to this definition and (2.1) the solutions of equations 1), 2) of (2.1) contained in the $(n-1)$ dimensional space of the real meromorphic functions $x_i(w)$ $(i = 1, 2, \ldots, n-1)$ must satisfy the equations[x/]:

---

x/ Here in 2) of (2.2) the summation over j is assumed.

$$1.\ x_i(w)\,x_i(1-w)=1$$

$$2.\ x_i(w+1)=\frac{A_{nj}\,x_j(w)+A_{nn}}{A_{ij}\,x_j(w)+A_{in}}. \tag{2.2}$$

The function $S_n(w)$ is found from the equations

$$S_n(w)\,S_n(1-w)=1$$

$$S_n(w+1)\,S_n(w)=\frac{1}{A_{nj}\,x_j(w)+A_{nn}}. \tag{2.3}$$

The first stage of constructing the functions $S_i(w)$ is to find all the solutions of the system (2.2).

Equations (2.2) are nonlinear. We linearize the unitarity conditions 1) by means of the substitution

$$x_i(w)=\frac{1-a_i(w-\frac{1}{2})}{1+a_i(w-\frac{1}{2})}, \tag{2.4}$$

which transforms them to the form

$$a_i(w-\tfrac{1}{2})+a_i(-(w-\tfrac{1}{2}))=0.$$

Thus the functions $a_i(w)$ are odd meromorphic real functions of w

$$a_i(-w)=-a_i(w). \tag{2.5}$$

We remark that the substitution (2.4) is analogous to the transition from the S-matrix to the K-matrix

$$S=\frac{1-iK}{1+iK}.$$

In the variables $a_i(w)$ equations (2.2) will be of the form

$$a_i(w+1) = \frac{\sum_{j=1}^{n-1}(A_{ij}-A_{nj})(1-a_j)\prod_{k\neq j}^{n-1}(1+a_k)+(A_{in}-A_{nn})\prod_{k=1}^{n-1}(1+a_k)}{\sum_{j=1}^{n-1}(A_{ij}+A_{nj})(1-a_j)\prod_{k\neq j}^{n-1}(1+a_k)+(A_{in}+A_{nn})\prod_{k=1}^{n-1}(1+a_k)}. \quad (2.6)$$

Performing some algebraic manipulations of (2.6) (see Appendix) we obtain the more convenient expression

$$a_i(w+1) = G_i(a_j(w)), \quad (2.7)$$

where $G_i(a_j(w))$ is defined by the following formula

$$G_i(a_j(w)) = \frac{-\sum_{q=1}^{n-1}\sum_{\{k_q\}}^{n-1} a_{k_1}a_{k_2}a_{k_3}\ldots a_{k_q}\left[\sum_{j=\{k_q\}}(A_{ij}-A_{nj})\right]}{1+\sum_{q=1}^{n-1}\sum_{\{k_q\}}^{n-1} a_{k_1}a_{k_2}a_{k_3}\ldots a_{k_q}\left[1-\sum_{j=\{k_q\}}(A_{ij}+A_{nj})\right]}. \quad (2.8)$$

Here the symbol $\sum_{\{k_q\}}^{n-1}$ denotes the summation over the ordered system of q elements a_k with the indices $k_1<k_2<\ldots<k_{q-1}<k_q, 1\le k_q\le n-1$ and the summation index j in the coefficients of the products $a_{k_1}a_{k_2}\ldots a_{k_q}$ with definite indices $k_1, k_2, \ldots, k_q$ takes the values $k_1, k_2, \ldots, k_q$.

The inverse equation to (2.7), expressing $a_i(w)$ in terms of $a_j(w+1)$, is determined by the inverse operator $G_i^{-1}(a_j(w+1))$, which exists and is unique. Using the oddness (2.5) of the functions $a_i(w)$ it is easy to obtain the inverse equation to (2.7), which is of the form

$$a_i(w) = -G_i(-a_j(w+1)). \quad (2.9)$$

For the physical solutions of (2.7) we shall assume the local conditions (1.4), which in terms of the variables $a_1(w)$ take the form:

$$1)\ a_1(w)\underset{w\to 0}{=} 0(w^{2\ell+1}) \quad \text{is the threshold behaviour}$$

$$2)\ a_1(w)\underset{w=-\frac{1}{2}}{=}\frac{1-\frac{\lambda_1}{\lambda_n}}{1+\frac{\lambda_1}{\lambda_n}} \quad \text{is the Born behaviour.} \tag{2.10}$$

Furthermore, using the fact that $a_1(w)$ are real functions of the complex variable w and that the real operators G_1 and G_1^{-1} determined by (2.8) and (2.9) are the unit shift operators of the argument along a direction parallel to the axis $\operatorname{Im} w = 0$, we can consider the functions $a_1(w)$ on the real axis and the $(n-1)$ dimensional complex space a as the real space of dimensionality $(n-1)$. Let us assume that we have constructed the solutions of (2.5), (2.7-8) on the real axis. The problem of the uniqueness of the analytic continuation of these solutions into the complex plane arises. In so far as the functions $a_1(w)$ are meromorphic, we can always find some interval of the real axis $a \leq \operatorname{Re} w \leq b$ on which all the $a_1(w)$ are analytic. Then the positive answer to this problem gives the "interior theorem of uniqueness". (see, for example, Markushevich[22], §6, p.302). Thus equations (2.7-8) allow a one-to-one transition from the complex space a into the real subspace a.

Let us now turn to the geometrical interpretation of the functional equations (2.7-8).

§2. The Geometrical Interpretation of the Functional Equations (2.7-8)

We shall consider the $(n-1)$ equations (2.7-8) as the transformation of the real space a determined by the functions (2.8):

$$a'_i = G_i(a_j). \tag{2.11}$$

We shall consider the properties of the transformation (2.11) in detail. This transformation is a birational transformation of the $(n-1)$-dimensional real space a. It is not difficult to show from (2.8) that the point $P(-1,-1,\ldots,-1)$ is a singular point of the transformation (2.11), in so far as the functions G_i have a singularity of the type $\frac{0}{0}$ at this point. The position of the transform P' of this point in a' depends on the direction from which one tends to the point P in a.

We remark that the point P corresponding to the point at infinity in the space x (see (2.2) and (2.4)) corresponds to the common system of poles of the functions $x_i(w)$.

Analogously the point $Q(1,1,\ldots,1)$ in the space a is the transform of the common system of zeros of the functions $x_i(w)$, as follows from (2.2) and (2.4). The point in the space a with the i coordinate equal to -1 corresponds to a pole in $x_i(w)$, and that with the j coordinate equal to $+1$ to zero in $x_i(w)$.

Let us find the fixed points of the transformation (2.11). They are defined by the $(n-1)$ equations

$$a_i = G(a_j).$$

In so far as the direct (2.11) and inverse (2.9) transformations are one-to-one, the fixed points of (2.11) coincide with the fixed points of the inverse transformation.

It follows from equation (2.9) that the set of fixed points of the transformation (2.11) is symmetrical with respect to the inversion of all the axes. We remark that in general case we already know one fixed point of (2.11), namely the origin of coordinates in the space a. This point is a physically interesting one, in so far as the threshold behaviour 1) of (2.10) tells us that the physical solution must pass through the origin of coordinates in the space a.

We shall now discuss the question of how the solution of the system (2.5) (2.7-8) will look in the space a. We assume that a solution (one or many) exists and is given by the $(n-1)$ functions $a_i(w)$. Before discussing this question we give a definition of a class of solutions of the system (2.5),(2.7-8), which we shall consider below.

The Definition of the Class of Solutions of the System (2.5); (2.7-8)

Among the set of solutions of the system (2.5),(2.7-8) for the arbitrary $n\times n$ crossing matrix A, there are contained the solutions which are the unique functions of the k variables:

$$a_i(w) = a_i(w+\beta_1(w), w+\beta_2(w), \dots, w+\beta_k(w)), \tag{2.12}$$

where $i=1,2,\dots,n-1$,

$1 \leq k \leq n-2$ and the $\beta_i(w)$ are the functions satisfying the properties (1.6).

Thus this class of solutions possesses greater arbitrariness than the arbitrariness (1.6). In the particular case when the matrix A defining (2.7) can be decomposed into the direct product of the crossing matrices $A_1, A_2, \ldots, A_k$, such a definition of the class of solutions is justified.

Then we shall consider the (n-1) equations

$$a_i(w) = a_i(w+\beta_1(w),\ w+\beta_2(w), \ldots, w+\beta_k(w))$$

as the parametric equations of a k-dimensional hypersurface Φ_k embedded in the $(n-1)$ dimensional space a. In particular, at $k=1$ the solution of (2.5), (2.7-8) is represented by a space curve in a. In so far as the operator $G_i(a_j)$ for $a_j(w)$, the solutions of (2.5), (2.7-8), is a shift operator on the k-dimensional hypersurface Φ_k, the set of solutions of (2.5), (2.7-8) will be represented by the set of hypersurfaces Φ_k in the space a, which are:

a) invariant under the transformation (2.11), (2.8),

b) odd under inversion of all $(n-1)$ axes a_i.

Let us now consider the invariant hypersurfaces Φ_k which contain at least one of the fixed points of (2.11). Below we propose a method for the local construction of the invariant hypersurfaces Φ_k in the neighbourhood of their fixed points.

§3. The Method of the Local Construction of the Invariant Hypersurfaces

The general equation of the hypersurface Φ_k in the neighbourhood of a fixed point of (2.11) is given by the $(n-1-k)$ equations

$$
\begin{aligned}
a_{i_1} &= \phi_{i_1}(a_{j_1}, a_{j_2}, \dots, a_{j_k}) \\
a_{i_2} &= \phi_{i_2}(a_{j_1}, a_{j_2}, \dots, a_{j_k}) \\
&\dots\dots\dots\dots\dots\dots\dots \\
a_{i_{n-1-k}} &= \phi_{i_{n-1-k}}(a_{j_1}, a_{j_2}, \dots, a_{j_k}),
\end{aligned}
\tag{2.13}
$$

where none of the indices $j_1, j_2, \dots, j_k$ is equal to any of the indices $i_1, i_2, \dots, i_{n-1-k}$ x/.

By the property b) of the hypersurfaces Φ_k the functions ϕ_{i_1}, ϕ_{i_2}, ..., $\phi_{i_{n-1-k}}$ are odd under the simultaneous change of all the signs of $a_{j_1}, a_{j_2}, \dots, a_{j_k}$, i.e.

$$
\phi_{i_\ell}(-a_{j_1}, -a_{j_2}, \dots, -a_{j_k}) = -\phi_{i_\ell}(a_{j_1}, a_{j_2}, \dots, a_{j_k}). \tag{2.14}
$$

The simple form of equations (2.14) is the consequence of the substitution (2.4).

The mathematical formulation of the condition a) consists in the fulfilment of the next $(n-1-k)$ functional equations on the invariant hypersurfaces Φ_k :

x/ The necessity of writing (2.13) symbolically in this way arises from the possibility of the tangency of the hypersurface Φ_k to the maximum number k of axes (this is the requirement of the smoothness of Φ_k in the neighbourhood of the fixed point) of the non-rotated local coordinate frame of reference at the fixed point.

$$
\begin{aligned}
\alpha'_{i_\ell} &= \phi_{i_\ell}(\alpha'_{j_1}, \alpha'_{j_2}, \dots, \alpha'_{j_k}) \\
\alpha'_{i_\ell} &= G_{i_\ell}(\alpha_{j_m}, \phi_{i_q}(\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_k})) \\
\alpha'_{j_p} &= G_{j_p}(\alpha_{j_m}, \phi_{i_q}(\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_k})) \\
& 1 \le \ell,\ q \le n-1-k \\
& 1 \le p,\ m \le k\ .
\end{aligned}
\tag{2.15}
$$

Thus the finding of all subspaces in the α-space invariant under (2.11) is reduced to solving the system of functional equations (2.14-15) in $(n-1-k)$ unknown functions $\phi_{i_1}, \dots, \phi_{i_{n-1-k}}$. The method of solving such functional equations is the expansion of the functions ϕ_{i_ℓ}, G_{i_ℓ}, G_{j_p} in the Taylor series in the neighbourhood of the fixed points of the transformation (2.11).

In finding the Taylor coefficients of the functions $\phi_{i_\ell}(\alpha_{j_1}, \alpha_{j_2}, \dots, \alpha_{j_k})$ from the system (2.14-15), we shall obtain the representation for the functions ϕ_{i_ℓ} in the form of a series in some k-dimensional neighbourhood of the fixed point of (2.11), whose size will be determined by the radius of convergence of the resulting series. If these series can be summed, then the functions ϕ_{i_ℓ} thus obtained will be an analytic determination of the whole hypersurface Φ_k.

The invariant subspaces thus obtained permit us to decrease the number of unknown functions $\alpha_i(w)$ in the system (2.5),(2.7-8), so that by the same token the solving of the latter is essentially simplified. In addition to this the many local properties of the solutions of (2.5),(2.7-8) (for example: the fulfilment of conditions 1) and 2) of (2.10); the presence of the poles and the zeros in $x_i(w)$, both those common to all $x_i(w)$ as well as those specific to each $x_i(w)$) can be obtained from the functions $\phi_{i\ell}$ found by solving (2.14-15), but here we shall not resort to solving the system (2.5), (2.7-8).

As already mentioned above, it is physically interesting to find the invariant hypersurfaces Φ_k containing the origin of coordinates in the α-space. The construction of the invariant hypersurfaces Φ_k containing only the other fixed points is of independent mathematical interest.

We note that the problem of finding the invariant curves in the real plane x, y was considered at the beginning of this century in the classical works[24,25] and more recently in[26], whose results have been expounded in the monograph[27]. However, the conditions under which the theorems in [26] were proved are not fulfilled for the 3x3 crossing matrix ($n-1=2$, the two-dimensional α-plane).

III. The Construction of All Physical Invariant Trajectories in the α-Plane for the Chew-Low 3x3 Matrix

The Chew-Low 3x3 matrix is of the form

$$A = \frac{1}{9}\begin{pmatrix} 1 & -8 & 16 \\ -2 & 7 & 4 \\ 4 & 4 & 1 \end{pmatrix} \qquad (3.1)$$

In connection with the definition $x_i = S_i / S_k$ in part II we introduce

$$x_1 = S_1 / S_2$$
$$x_3 = S_3 / S_2 \,. \qquad (3.2)$$

Then according to (3.2), (2.4), (2.7) and formula (2.8), in which the index n must be replaced by 2 and the indices $i, j, \{k_q\}$ assume the values 1,3, we obtain the equations for $a_1(w)$ and $a_3(w)$:

$$a_1(w+1) = G_1(a_j(w)) = \frac{-\frac{1}{3}a_1 - \frac{4}{3}a_3 - \frac{5}{3}a_1 a_3}{1 + \frac{10}{9}a_1 - \frac{11}{9}a_3 - \frac{10}{9}a_1 a_3}$$

(3.3)

$$a_3(w+1) = G_3(a_j(w)) = \frac{-\frac{2}{3}a_1 + \frac{1}{3}a_3 - \frac{1}{3}a_1 a_3}{1 + \frac{7}{9}a_1 + \frac{4}{9}a_3 + \frac{2}{9}a_1 a_3}.$$

With regard to the geometrical interpretation of these equations as the transformation of the real plane a_1, a_3, the fixed points of the transformation (3.3) for the arbitrary 3x3 matrix A are of the form:[16]

$$1)\ a_1 = a_3 = 0$$

$$2)\ a_1 = \pm\left(1 - \frac{4A_{11}A_{22}}{(A_{22} - A_{11})(A_{21} - A_{12}) + A_{11}A_{22}}\right)^{1/2}, \qquad (3.4)$$

where the sign of the root for a_3 must be taken opposite to that of a_1.

From 2) of (3.4) for the matrix (3.1) we obtain the non-null fixed points

$$1)\ a_1 = \frac{3}{5}\sqrt{2}, \quad a_3 = -\frac{3}{2}\sqrt{2}, \qquad (3.5)$$

$$2)\ a_1 = -\frac{3}{5}\sqrt{2}, \quad a_3 = \frac{3}{2}\sqrt{2}.$$

According to part II, if the physical solutions of (3.3) satisfying the local conditions (2.10) are contained in the class (2.12), then they will be represented in the α -plane by the trajectories which are the curves invariant under the transformation G_i and passing through the null fixed point 1) of (3.4) and the Born point (see 2) of (2.10)):

$$\begin{pmatrix} \alpha_1 \\ \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} \\ \\ -3 \end{pmatrix} . \qquad (3.6)$$

In obtaining (3.6) from 2) of (2.10), we used the fact that the λ_i for the matrix (3.1) are of the form:

$$\lambda_i = \begin{pmatrix} -4 \\ -1 \\ 2 \end{pmatrix} \cdot c ,$$

where c is constant proportional to the square of the πN coupling constant.

With regard to (2.14-15), the functional equations on the invariant curves $\alpha_i = \phi_i(\alpha_j)$are of the form

$$G_i(\alpha_j, \phi_i(\alpha_j)) = \phi_i[G_j(\alpha_j, \phi_i(\alpha_j))] \qquad (3.7)$$

$$\phi_i(-\alpha_j) = -\phi_i(\alpha_j), \qquad (3.8)$$

the physical solutions of which are found by expanding in a Taylor series the functions $\phi_i(\alpha_j)$, G_i, G_j in the neighbourhood of the point $\alpha_j = 0$. For convenience we shall seek the invariant curves in the system of coordinates u_1 and u_3

$$u_1 = \alpha_1 + \alpha_3$$

$$u_3 = -\frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_3, \tag{3.9}$$

in which the linear approximation to (3.3)

$$\alpha_1(w+1) = -\frac{1}{3}\alpha_1(w) - \frac{4}{3}\alpha_3(w)$$

$$\alpha_3(w+1) = -\frac{2}{3}\alpha_1(w) + \frac{1}{3}\alpha_3(w)$$

is diagonal

$$u_1(w+1) = -u_1(w)$$

$$u_3(w+1) = u_3(w).$$

Equations (3.3) in the variables u_1 and u_3 are of the form:

$$u_1(w+1) = \tilde{\tilde{G}}_1(u_1(w), u_3(w))$$

$$u_3(w+1) = \tilde{\tilde{G}}_3(u_1(w), u_3(w)),$$

where $\tilde{\tilde{G}}_1$ and $\tilde{\tilde{G}}_3$ are

$$\tilde{\tilde{G}}_1(u_1, u_3) = -\frac{u_1[1+u_1+c(u_1,u_3)]}{a(u_1,u_3)b(u_1,u_3)}$$

$$\tilde{\tilde{G}}_3(u_1,u_3) = \frac{u_3 - 2u_3^2 + u_1u_3 + \frac{4}{27}u_1^2 + \frac{1}{3}(\frac{16}{9}u_1 - u_3)c(u_1,u_3) + \frac{10}{27}[c(u_1,u_3)]^2}{a(u_1,u_3)b(u_1,u_3)} \tag{3.10}$$

Here:

$$c(u_1,u_3) = -u_3^2 + \frac{1}{3}u_1u_3 + \frac{2}{9}u_1^2,$$

$$a(u_1,u_3) = 1 + \frac{1}{3}u_1 - \frac{7}{3}u_3 - \frac{10}{9}c(u_1,u_3),$$

$$b(u_1,u_3) = 1 + \frac{2}{3}u_1 - \frac{1}{3}u_3 + \frac{2}{9}c(u_1,u_3).$$

Equations (3.7-8) on the invariant curves $u_1=\phi_1(u_3)$ and $u_3=\phi_3(u_1)$ are of the form:

$$1)\ \tilde{\tilde{G}}_1(\phi_1(u_3),u_3) = \phi_1[\tilde{\tilde{G}}_3(\phi_1(u_3),u_3)]$$

$$\phi_1(-u_3) = -\phi_1(u_3);$$

$$(3.11)$$

$$2)\ \tilde{\tilde{G}}_3(u_1,\phi_3(u_1)) = \phi_3[\tilde{\tilde{G}}_1(u_1,\phi_3(u_1))]$$

$$\phi_3(-u_1) = -\phi_3(u_1).$$

The first pair of equations of (3.11) permits us to find the curves non-tangent to the axis u_1 and the second pair to find those non-tangent to the axis u_3.

Because of the oddness of the functions $\phi_1(u_3)$ and $\phi_3(u_1)$ all the even derivatives of these functions at zero argument must be equal to zero

$$\phi_1^{(2k)}(0) = 0. \tag{3.12}$$

Let us now find the solution of 1) of (3.11). We have $\phi_1(0) = 0$. Differentiating 1) of (3.11) with respect to u_3, we obtain:

$$\frac{d\tilde{\tilde{G}}_1(\phi_1(u_3), u_3)}{d u_3} = \phi_1' [\tilde{\tilde{G}}_3(\phi_1(u_3), u_3)] \frac{d\tilde{\tilde{G}}_3(\phi_1(u_3), u_3)}{d u_3}. \qquad (3.13)$$

From (3.13) and (3.10) we find

$$-\phi_1'(0) = \phi_1'(0), \text{ i.e. } \underline{\phi_1'}(0) = 0.$$

By the method of mathematical induction, it may be proved that all the higher derivatives at zero argument are equal to zero.

From this it follows that

$$u_1 = \phi_1(u_3) \equiv 0 \qquad \text{and } a_1(w) = -a_3(w). \quad (3.14)$$

It is not difficult to demonstrate, by differentiating the third equation of (3.11), that it does not possess any solutions. Indeed, by once differentiating 2) of (3.11) we obtain

$$\phi_3'(0) = -\phi_3'(0).$$

From this, $\phi_3'(0) = 0$. But differentiating twice, we obtain a contradiction

$$\frac{8}{27} = 0.$$

We remark that in finding solutions of (3.11) we assumed implicitly that the point $u_j = 0$ is a regular point of the functions $\phi_i(u_j)$. However, with regard to the threshold behaviour 1) of (2.10), the definition (3.9) and the inverse function theorem (see, Markushevich[22], ch. IV, §5), the point $u_j=0$ may be, in general, an algebraic branch point of finite order $2m$, where $m \geq \ell$. Then the function $\phi_i(u_j)$ will have the representation

$$u_i = \phi_i(u_j) = \sum_{k=0}^{\infty} c_k \, u_j^{\frac{2k+1}{2m+1}}. \tag{3.15}$$

One can demonstrate from (3.11) that all the c_k for $k<m$ will be equal to zero, so that the expansion in (3.15) begins with u_j. Introducing $v_j = u_j^{\frac{1}{2m+1}}$ and expanding (3.11) in the Taylor series in the neighbourhood of $v_j = 0$, it is easily shown that the unique algebraic curve is the one defined by (3.14).

Thus, only the single curve (3.14) exists satisfying the threshold behaviour 1) of (2.10).

This curve corresponds the familiar functional relation between the functions $S_1(w)$, $S_2(w)$ and $S_3(w)$ in Rotherlutner's solution for the 3x3 Chew-Low matrix

$$S_2^2 = S_1 S_3. \tag{3.16}$$

It is easy to see that the curve (3.14) does not pass through the Born point (3.6) and this means that it does not define the physical solution of equations (1.1). This fact has been noted from formula (3.16) in ref.[21].

Thus we have proved that the Chew-Low equations (1.1) with the 3x3 crossing matrix (3.1) do not possess solutions in the class (2.12) under consideration.

However, in ref.[19,20] the theorem establishing the existence of a solution of equations (1.1) has been proved by assuming that the πN coupling constant is sufficiently small. In so far as the smallness of the coupling constant is not essential in the equivalent consideration of (2.5), (2.7-8) with conditions (2.10), then we establish the following fact:

if the Chew-Low equations (1.1) have a solution, then the class of solutions (2.12) is not exhaustive.

The detailed treatment of all the physical invariant curves and surfaces in the 3-dimensional α-space $\left(x_i = \frac{S_i}{S_2} = \frac{1-\alpha_i}{1+\alpha_i},\ i=1,3,4\right)$ for equations (2.5) (2.7-8) with the 4x4 Chew-Low matrix shows that the unique surfaces are:

$$1)\quad \alpha_1 = \frac{\alpha_3 - \alpha_4}{1-\alpha_3\alpha_4} \qquad (3.17)$$

$$2)\quad \alpha_3 = 0,$$

and the unique curves are:

$$1)\quad \alpha_1 = -\alpha_4,\quad \alpha_3 = 0$$

$$2)\quad \alpha_1 = 0,\quad \alpha_3 = \alpha_4 \qquad (3.18)$$

$$3)\quad \alpha_4 = 0,\quad \alpha_3 = \alpha_1 .$$

The surface 1) of (3.17) corresponds to the familiar Rothelutner's solution with the two arbitrary functions $\beta'_1(w)$ and $\beta_2(w)$, while all the solutions of the equations with the 3x3 Chew-Low matrix are contained in the plane $\alpha_3 = 0$.

The curve 1) of (3.18) is the intersection of the surface 1) of (3.17) with the plane 2) of (3.17) and, as has been shown above, is the single invariant curve for the 3x3 Chew-Low matrix. The curves 2) and 3) of (3.18) are contained on the surface 1) of (3.17), and the solutions corresponding to 1) of (3.17) and (2-3) of (3.18) are the direct product of the solutions $S^{(1)}(w)$ and $S^{(2)}(w)$ for the 2x2 crossing matrix.

The surface 1) of (3.17) does not contain the point 2) of (2.10) guaranteeing the correct Born behaviour. From this it follows that if the Chew-Low equations with the 4x4 crossing matrix have a solution, then it possesses a greater arbitrariness and is not contained in the class of solutions (2.12) under consideration.

The authors are grateful to D.V.Shirkov and Ia.A.Smorodinsky for a valuable discussion.

Appendix

Let us transform (2.6) to a more convenient form. The denominator in (2.6) is obtained from the numerator by replacing the minus sign by a plus sign before the matrix elements $A_{k\ell}$ with $k=n$. Therefore we shall make some manipulations of the numerator.

Firstly, we make use of the identity:

$$\prod_{k=1}^{n-1}(1+a_k)=\frac{1}{n-1}\sum_{j=1}^{n-1}(1+a_j)\prod_{k\neq j}^{n-1}(1+a_k). \qquad (A.1)$$

Then the numerator in (2.6) will reduce to the form

$$\sum_{j=1}^{n-1}\left[A_{1j}-A_{nj}+\frac{A_{1n}-A_{nn}}{n-1}\right]\prod_{k\neq j}^{n-1}(1+a_k)-\sum_{j=1}^{n-1}\left[A_{1j}-A_{nj}-\frac{A_{1n}-A_{nn}}{n-1}\right]a_j\prod_{k\neq j}^{n-1}(1+a_k) \quad (A.2)$$

Further, let us represent the product of the $(n-2)$ factors $\prod_{k\neq j}^{n-1}(1+a_k)$ in (A.2) in the form of the sums:

$$\prod_{k\neq j}^{n-1}(1+a_k)=1+\sum_{q=1}^{n-2}\sum_{\{k_q,j\}}a_{k_1}a_{k_2}\dots a_{k_q}. \qquad (A.3)$$

The symbol $\sum_{\{k_q,j\}}$ is introduced for convenience and denotes the summation over the ordered system of q elements a_k with the indices $k_1<k_2<\dots<k_{q-1}<k_q$ of which none is equal to j and which take the values from 1 to $(n-1)$.

Taking into account (A.3) we transform the first sum in (A.2)

$$\sum_{j=1}^{n-1}\left[A_{1j}-A_{nj}+\frac{A_{1n}-A_{nn}}{n-1}\right]\left(1+\sum_{q=1}^{n-2}\sum_{\{k_q,j\}}a_{k_1}a_{k_2}\dots a_{k_q}\right)= \qquad (A.4)$$

$$=\sum_{j=1}^{n}(A_{1j}-A_{nj})+\sum_{q=1}^{n-2}\sum_{\{k_q\}}a_{k_1}a_{k_2}\dots a_{k_q}\left[\sum_{\substack{j=1\\ j\neq\{k_q\}}}^{n-1}(A_{1j}-A_{nj})\right]+(A_{1n}-A_{nn})\sum_{q=1}^{n-2}\frac{n-1-q}{n-1}\sum_{\{k_q\}}a_{k_1}\dots a_{k_q}.$$

Here the symbol $\sum_{\{k_q\}}$ denotes the same as $\sum_{\{k_q,j\}}$, but with the difference that the indices $\{k_q\}$ take all values from 1 to $(n-1)$.

The symbol $\sum_{j=1, j\neq\{k_q\}}^{n-1}$ denotes the summation over j from 1 to $(n-1)$, excluding values of the indices of the system $\{k_q\}$. The second sum in (A.2) is transformed as follows taking into account (A.3):

$$\sum_{j=1}^{n-1} [A_{1j}-A_{nj}-\frac{A_{1n}-A_{nn}}{n-1}]\, a_j\, (1+\sum_{q=1}^{n-2} \sum_{\{k_q,j\}} a_{k_1} a_{k_2} \ldots a_{k_q}) =$$

$$= \sum_{q=1}^{n-1} \sum_{\{k_q\}} a_{k_1} a_{k_2} \ldots a_{k_q} [\sum_{j=\{k_q\}} (A_{1j}-A_{nj})] - \qquad (A.5)$$

$$-\frac{A_{1n}-A_{nn}}{n-1} \sum_{q=1}^{n-1} q \sum_{\{k_q\}} a_{k_1} a_{k_2} \ldots a_{k_q} .$$

Subtracting (A.5) from (A.4) we obtain (A.2)

$$(A.2) = \sum_{j=1}^{n} (A_{1j}-A_{nj}) + \sum_{q=1}^{n-1} \sum_{\{k_q\}} a_{k_1} \ldots a_{k_q} [\sum_{j\neq\{k_q\}} (A_{1j}-A_{nj}) - \sum_{j=\{k_q\}} (A_{1j}-A_{nj}) + (A_{1n}-A_{nn})] .$$

The denominator is obtained by making the substitutions $A_{nj}\to -A_{nj}$, $A_{nn}\to -A_{nn}$ in this expression. Finally, using the property (1.3) of the matrix A and the identity

$$\sum_{j\neq\{k_q\}}^{n-1} A_{1j} + \sum_{j=\{k_q\}}^{n-1} A_{1j} + A_{1n} = \sum_{j=1}^{n} A_{1j} = 1 , \qquad (A.6)$$

we arrive at the formula (2.8):

$$G_i(a_j(w)) = \frac{-\sum_{q=1}^{n-1}\sum_{\{k_q\}}^{n-1} a_{k_1} a_{k_2} \dots a_{k_q} \left[\sum_{j=\{k_q\}} (A_{ij} - A_{nj}) \right]}{1 + \sum_{q=1}^{n-1}\sum_{\{k_q\}}^{n-1} a_{k_1} a_{k_2} \dots a_{k_q} \left[1 - \sum_{j=\{k_q\}} (A_{ij} + A_{nj}) \right]} \qquad (A.7)$$

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Received by Publishing Department
on October 8, 1969.