ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2-4717

Yu.A.Danilov, G.I.Kuznetsov, Ya.A.Smorodinsky

STUDY OF ANALYTICAL PROPERTIES OF PENTAGON FEYNMAN GRAPH BY HOMOLOGICAL METHOD

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1969

E2-4717

Yu.A.Danilov, G.I.Kuznetsov, Ya.A.Smorodinsky

STUDY OF ANALYTICAL PROPERTIES OF PENTAGON FEYNMAN GRAPH BY HOMOLOGICAL METHOD

Направлено в журнал "Теоретическая и математическая физика".

1. Introduction

Pham, Froissart, Lascoux and Fotiadi /1-4/ have proposed to study analytical properties of Feynman integrals by methods of homology theory. All the resources of traditional approach, i.e. explicit analytical continuation, tracing of moving singularities, calculations of the intersections of singular manifolds etc., become powerless when the number of variables increases. The new approach, replacing tedious analytical calculations by investigation of topological features of the complement of singular manifolds, is almost insensitive to the number of variables and has many other advantages besides.

A detailed account of homological techniques can be found in the books by Pham /1/ and Hwa and Teplitz /5/. The original papers by French authors are inserted in the latter as appendices.

Since even elementary notions of homology theory are yet considered by physicists to be rather exotic and are not widely known, we shall formulate the basic definitions and enumerate the facts which are necessary for understanding of what follows. By doing so, we shall neglect the rigour required for methematical works and as far as possible shall restrict ourselves to visual considerations.

Let us consider a convex n-dimensional polyhedron Δ^n and provide it with an orientation ϵ. The latter means that the set of its vertices is ordered in a certain way. The orientation of any of its $(n-1)$-dimensional faces will be chosen in agreement with ϵ. For that we shall delete out from the ordered (by ϵ) set of vertices of Δ^n, all the vertices that do not belong to the face in hand. The remainder of vertices will give for this face the orientation induced by ϵ (i.e. "synchronized" with ϵ). By transition from oriented polyhedron Δ^n to the sum of all its $(n-1)$-dimensional faces with the induced orientation will be referred to as we "take a boundary" and denote it by ∂.

An element of n-dimensional chain σ will be defined as a convex n-dimensional oriented polyhedron Δ^n. A finite sum of such elements taken with integer coefficients will be called an n-dimensional chain γ:

$$\gamma = \sum_{i=1}^{k} n_i \sigma_i .$$

A boundary of a chain $\partial(\gamma)$ will be defined as

$$\partial(\gamma) = \sum_{i=1}^{k} n_i \partial(\sigma_i).$$

It may be proved that for every chain σ

$$\partial^2 \sigma = 0 .$$

A chain with zero boundary is called a cycle C:

$$\partial C = 0 .$$

A chain A may happen to be a boundary of some $(n+1)$-dimensional chain B:

$$A = \partial B .$$

Then chain A is called an exact chain. Exact chain always has zero boundary:

$$\partial A = \partial(\partial B) = \partial^2 B = 0$$

and therefore is a cycle.

The n-dimensional cycles (for brevity, n-cycles) may be multiplied by integers and added. They constitute an abelian group Z_n (a group of n - cycles). The set of exact chains forms in Z_n a subgroup B_n called the subgroup of boundaries.

Taking the quotient of the group of cycles Z_n by subgroup of boundaries B_n, it is possible to introduce into the set of n-cycles equivalence relation. The factor group

$$H_n = Z_n / B_n$$

is called the n-th homology group (of the space in question). If the space is K-dimensional, then n runs from 0 to K.

We shall be interested in the relative homology groups. The latter means that dividing the cycles into the classes, we shall consider to be equivalent (homologous) not only the cycles differing by a boundary of a chain (that is of dimension higher by unity), but also the cycles which difference belongs to some special kind of manifolds (i.e. the Landau surfaces in our case).

Besides the boundary operator ∂ which diminishes the dimension of the chain by 1, we shall use also the so-called coboundary operator δ that increases the dimension of a chain by 1. The action of δ can be imagined as follows.

If σ is a one-dimensional chain (an oriented line), then δ brings in correspondence with it the surface of circular cylinder which axis coincides with δ and orientation in every cross section makes with the orientation of σ the right screw. Operator δ enables us to simplify considerably the evaluation of homology groups (using the Froissard decomposition theorem).

The notions introduced above turn to be quite natural for qualitative description of the properties of integrals of the functions of many complex variables and, in particular, for classification of Feynman integrals.

In general, Feynman integrals constitute a new class of special functions of many variables which theory has not been yet developed. The topological methods enable us to clarify some general properties of such functions.

Here we shall be interested in the singularities of the integral. More exactly, we shall care for the singularities of the integrand that may become the singularities of the integral.

As it is known, the singular manifolds must clutch the integration chain ("pinch") and some of the cycles (the so-called "vanishing" cycles) must refract into the point to make such a possibility real. All the information about the singularities of the integral we shall elicit from the study of the intersection of the integration chain with vanishing cycles in terms of the so-called Kronecker indices (intersection indices).

Let M_1 and M_2 be two oriented manifolds of dimensions k and ℓ respectively in N-dimensional space (for simplicity we shall consider it to be Euclidean). Let M_1 and M_2 intersect only in the finite number of points. The dimensions k and ℓ satisfy the equation $k+\ell=N$. In every point of intersection M_1 and M_2 are supposed to be in general position, i.e. the hyperplanes tangent in the point of intersection do not intersect in any point more. If M_1 and M_2 are one-dimensional curves on a plane, then the general position means that in point of intersection the tangents (and therefore the normals) are not collinear.

The standard "synchronized" N -hedrals are said to be given in all points of the space if it is possible to pass from one of them to another by parallel shift and by homogeneous linear transformation with positive determinant.

Let us consider the subneighbourhoods U_1 and U_2 of the intersection point 0 which belong to M_1 and M_2 respectively. Let $\{v_1, v_2, \dots, v_k\}$ be a collection of k linearly independent vectors which are tangent to M_1 in 0. This collection is supposed to define in U_1 the orientation synhronized with the given one (both M_1 and M_2 are oriented). Further, let $\{v_{k+1}, \dots, v_N\}$ be a collection of ℓ linearly independent vectors which are tangent to M_2 in the point 0 and let this collection also define in U_2 the orientation agreed with the given one.

Since M_1 and M_2 are in general position in the point 0, the union of the collections

$$\{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_N\}$$

gives the basis in the whole space. If this basis is "synchronized" with the standard basis in the point 0, then the Kronecker index $K\ell$ in the point 0 is $+\ell$, otherwise $K\ell$ is $-\ell$. If M_1 and M_2 do not intersect in the given point, then $K\ell$ is defined to be 0.

The Kronecker index of the full intersection i.e. the total set of the finite number of intersection points is the sum of the indices of all its points.

Knowing the Kronecker indices of the intersection of integration chain with the vanishing cycles and using the Picard-Lefschetz theorem, we can learn what happens with the integration chain when we circle round this or that singular manifold.

Such is the general scheme of the work which is reported below. The picture drawn by us is , of course, considerably simplified and do not claim for any serious introduction to the homology theory. Our aim was only to help reader which is anfamiliar with the homology theory.

II. The Reduction of the Integral to the Standard Form

We study the analytical properties of the pentagon graph by methods of homology theory.

Let us consider the integral corresponding to Feynman graph drawn on fig.1.

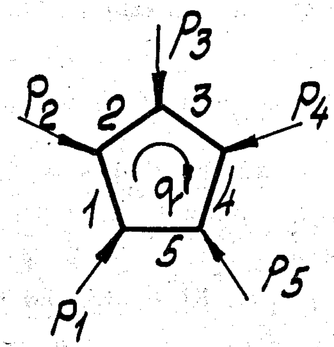

Fig.1

We shall adhere to the following notations:

p_i =external momenta.

$$p = (p_0, \vec{p}),\ \tilde{\tilde{p}} = (i p_0, \vec{p}),$$

q -internal momentum,

m_i- internal masses,

$$a_i = \sum_{s=1}^{i} p_s \quad , i = 1,2,\ldots, s \ , \quad a_5 = 0, \ a_i = \tilde{\tilde{b}}_i$$

$$Q_j = (a_j + q)^2 + m_j^2 \ , \quad Q_{j\epsilon} = Q_j + i\epsilon \ .$$

The collection of parameters $(a_i, m_j) = t$ constitutes a subset T in 25-dimensional complex space C^{25}.

As it is known, the Feynman integral can be brought to the form

$$I_1(t) = \int d^3 q \int_{-i\infty}^{i\infty} dq_0 \frac{1}{\prod_{j=1}^{5} Q_{j\epsilon}} . \tag{1}$$

Turning the contour of integration as usually we have

$$I_2(t) = \int_{R^4} d^4 q \frac{1}{\prod_{j=1}^{5} Q_{j\epsilon}} . \tag{2}$$

Here the integration is carried out over the real four-dimensional space. The form (2) is convenient for analytical investigation but is not of standard type which is necessary for application of the required theorems.

The integral

$$I(t) = \int_{\Gamma} \frac{\omega(t)}{\prod_i S_i(t)}$$

is said to be of standard type (or of standard form) if a) for $\forall t \in T$ (sign $\forall$ means "every") the equations $S_i(t, x) = 0$ define in n-dimensional complex manifold X the compact analytical subsets analytically depending upon t ; b) $\omega(t)$ is a regular external n-form on X which is holomorphic with respect to $t \in T$ (it is now an accustomed way of denoting the elementary volume); c) $\Gamma \subset X - \bigcup_i S_i(t)$ is a compact n-dimensional cycle; d) X is compact and the manifolds $S_i(t,x)=0$ are in general position when $t \in T$.

In order to bring the integral (2) in question to standard form, we carry out the following consequence of operations. First, using the mapping $x \in R^4 \to \mathfrak{X}=(x,1) \in R^5$ we imbed the R^4 into the $R^5 - \{0\}$ (the real five-dimensional space with deleted origin). Further, we put in correspondence with every $x \in R^4$ its inverse image under stereographical projection

$$\mathfrak{X} = \frac{2(x,1)}{x^2+1}$$

and pass to the sphere $x^2+(x_5-1)^2=1$ in R^5. The integration domain (a cycle) becomes by this a compact (sphere!). Finally, completing the R^5 by the point deleted before, we carry out the inclusion $R^5 \to C^5 \to CP^5$ (the complex projective s-dimensional space). The equation of the sphere in homogeneous coordinates in CP^5 is $x^2+x_5^2=x_6^2$. (The centre of the sphere is situated at the origin). After this, as it is easy to see, all the requirements for the integral of the standard form are fulfilled. The ambiant space of $\overline{X^4}$ is the closure Σ^4 of the space Σ (complex 4-dimensional sphere). The surfaces Q_j become the $S_j = X \quad \bar{P}_j^4$, where P_j^4 is an affine 4-plane in C^5 and $\bar{P}_j^4$ is a corresponding projective plane in CP^5.

The integral (2) is transformed as following

$$I_3(t) = \int_{\mathrm{Re}\,\Sigma^4} \frac{\bar{\omega}^4 \cdot (x_5+x_6)}{\prod_{j=1}^{5} \bar{P}_{j\epsilon}}, \qquad (3)$$

where

$$\bar{\Sigma}^4 = \{ q \mid q^2 = 0 \} \tag{4}$$

$$\bar{P}_{j\epsilon} = \{ q \mid \bar{p}_{j\epsilon} q = 0 \} \tag{5}$$

$$\bar{p}_j q = \bar{p}_{j_1} q_1 + \bar{p}_{j_2} q_2 + \bar{p}_{j_3} q_3 + \bar{p}_{j_4} q_4 + \bar{p}_{j_5} q_5 + \bar{p}_{j_6} q_6 \tag{6}$$

$$\bar{\omega}^4 = \frac{\overset{4}{\omega}'}{x_5} \Big|_{\bar{\Sigma}^4} = \frac{\overset{4}{\omega}}{x_6} \Big|_{\bar{\Sigma}^4} = \Big(\sum_{j=1}^{5} \frac{x_j}{x_6 d x_j} \Big) dx_1 \Lambda\, dx_2 \Lambda \dots \Lambda\, dx_5 = \tag{7}$$

$$= \Big(\sum_{j=1}^{6} \frac{x_j}{x_5 d x_j} \Big) dx_1 \Lambda\, dx_2 \Lambda \dots \Lambda\, dx_6 ,$$

$x = (\vec{x}, x_5, x_6)$ the homogeneous coordinates in CP ,

$$q = (\vec{x}, x_5 - i x_6) \in CP^5 ,$$

$$\bar{p}_\epsilon = [2 a_j , a_j^2 + m_j^2 + i\epsilon - 1, i (a_j^2 + m_j^2 + i\epsilon + 1)]. \tag{8}$$

III. The Landau Surfaces

The function given by the integral (3) has singularities on the manifolds defined by the Landau equations

$$\sum_l \alpha_l \frac{\partial S_l}{\partial x_k} = 0 , \quad k = 1 , \dots, 6 , \tag{9}$$

$$S_l (x, t) = 0$$

implying the manifolds S_i are in nongeneral position, i.e.

$$\det(\bar{p}_j \cdot \bar{p}_k) = 0. \tag{10}$$

Here $(\bar{p}_j \cdot \bar{p}_k)$ is the scalar product of 6-vectors. Taking their explicit form, we find that

$$\frac{1}{2}(\bar{p}_j \cdot \bar{p}_k) = -[m_j^2 + m_k^2 + (a_j - a_k)^2]. \tag{11}$$

The manifolds defined by these equations will be denoted further by $L(t)$. Remark that $L(t)$ can be represented as

$$L(t) = \underset{\beta,\gamma}{} \; L_{\beta,\gamma}(t), \tag{12}$$

where L_β are some of the minors of determinant (10), $\beta \subset (1,2,3,4,5)$ and codimension γ in our case is 1.

It is easy to give the explicit form of some minors in our case.

$$L_1 \quad L_2 \quad L_3 \quad L_4 \quad L_5 \supset t \rightarrow m_1^2 m_2^2 m_3^2 m_4^2 m_5^2 = 0 \tag{13}$$

$$L_{ij} \supset t \rightarrow (a_j - a_i)^2 + (m_j \pm m_i)^2 = 0 \tag{14}$$

$$L_{ijk} \supset t \rightarrow \det{}^{ijk} M = 0, \quad \det{}^{\beta} M = \| \bar{p}_j \bar{p}_k \|_{j,k \in \beta} \tag{15}$$

$$\begin{aligned} L_{1234} \supset t \rightarrow\; & S_{23}^2 S_{34}^2 - 4m^2 S_{23} S_{34} (S_{23} + S_{34}) + \\ & + 2m^2 S_{23} S_{34} S_{15} - 3m^4 S_{15}^2 + 10 m^4 S_{23} S_{34} + 6m^6 S_{15} - 3m^8 = 0 \end{aligned} \tag{16}$$

If $S_{15}=m^2$, then the latter equation changes into the equation of hyperbola (the singularity of the square graph).

$$
\begin{aligned}
L_{12345} \supset t \to\ & 4m^{10} - 5m^8 \sum_{i=1}^{5} s_{i\,i+1} + m^6 \sum_{i=1}^{5} s^2_{i\,i+1} + 7m^6 \sum_{i=1}^{5} s_{i\,i+1}\, s_{i+1\,i+2} + \\
& + 2m^6 \sum_{i=1}^{5} s_{i\,i+1} s_{i+2\,i+4} - 2m^4 \sum_{i=1}^{5} \left(s^2_{i\,i+1}\, s_{i+1\,i+2} + s^2_{i+1\,i+2}\, s_{i\,i+1} \right) - \\
& - 2m^4 \sum_{i=1}^{5} s_{i\,i+1}\, s_{i+1\,i+2}\, s_{i+3\,i+4} - m^4 \sum_{i=1}^{5} s_{i\,i+1}\, s_{i+1\,i+2}\, s_{i+2\,i+3} + \\
& + m^2 \sum_{i=1}^{5} s^2_{i\,i+1}\, s^2_{i+1\,i+2} - m^2 \sum_{i=1}^{5} s_{i\,i+1}\, s^2_{i+1\,i+2}\, s_{i+2\,i+3} + \\
& + 2m^2 \sum_{i=1}^{5} s_{i\,i+1}\, s_{i+1\,i+2}\, s_{i+2\,i+3}\, s_{i+3\,i+4} - s_{12}\, s_{23}\, s_{34}\, s_{45}\, s_{51}
\end{aligned}
\tag{17}
$$

(If the value of any index is more than 5, then it must be taken modulo 5).

As usual, $s_{ik} = -\ (+p_i\ ,+p_k\ )$.

(Due to the complexity of the general expression the equations of the hypersurfaces L_{1234} and L_{12345} are given here for the case of equal external and internal masses).

IV. Homology Groups

In order to study the analytical properties of our graph it is necessary to evaluate the homology groups of the complement $X - \bigcup_{j=1}^{5} \bar{P}_j$, i.e. to enumerate the independent cycles of the corresponding dimensions and to establish the relations between them.

Let us use for this purpose the Froissart theorem (decomposition theorem):

$$H_q^c(\bar{\Sigma} - \quad_{j=1}^{5} \bar{P}_j) = H_q^c(\Sigma^4 - \quad_{j=1}^{4} P_j) = \quad_{\beta \subset \{1,2,3,4\}} + \delta^{|\beta|} H^c_{q-|\beta|}({}^{\beta}P \quad \Sigma^4), \tag{18}$$

where H_q^c is a compact q-dimensional homology group ${}^{\beta}P = P_{i_1} \ldots P_{i_\beta}$. In our case P_5 is the 4-plane at the infinity and $q = 4$. It is known that if $q = p, 0$, then $H_q^c(\Sigma^p) = Z$. $H_0^c(\Sigma^0) = Z \times Z$. The homology group in all other cases is zero. (Here Σ^p is p-dimensional sphere; Z as usual, means free Abelian group of the integers with respect to addition).

Knowing this, we get by Froissart theorem:

$$\begin{aligned}
H_4^c(\Sigma^4 - \quad_{j=1}^{4} P_j) = {} & H_4^c(\Sigma^4) + \delta^{(1)} H_3^c(P_1 \quad \Sigma^4) + \\
& + \delta^{(2)} H_3^c(P_2 \quad \Sigma^4) + \delta^{(3)} H_3^c(P_3 \quad \Sigma^4) + \delta^{(4)} H_3^c(P_4 \quad \Sigma^4) + \\
& + \delta^{(1)}\delta^{(2)} H_2^c({}^{12}P \quad \Sigma^4) + \delta^{(1)}\delta^{(3)} H_2^c({}^{13}P \quad \Sigma^4) + \delta^{(1)}\delta^{(4)} H_2^c({}^{14}P \quad \Sigma^4) + \\
& + \delta^{(2)}\delta^{(3)} H_2^c({}^{23}P \quad \Sigma^4) + \delta^{(2)}\delta^{(4)} H_2^c({}^{24}P \quad \Sigma^4) + \delta^{(3)}\delta^{(4)} H_2^c({}^{34}P \quad \Sigma^4) + \\
& + \delta^{(1)}\delta^{(2)}\delta^{(3)} H_1^c({}^{123}P \quad \Sigma^4) +
\end{aligned} \tag{19}$$

$$+\delta^{(1)}\delta^{(2)}\delta^{(4)}H_1^{o}({}^{123}P \;.\; \Sigma^4) + \delta^{(1)}\delta^{(3)}\delta^{(4)}H_1^{o}({}^{134}P \quad \Sigma^4) +$$

$$+\delta^{(2)}\delta^{(3)}\delta^{(4)}H_1^{o}({}^{124}P \quad \Sigma^4) + \delta^{(1)}\delta^{(2)}\delta^{(3)}\delta^{(4)}H \;({}^{1234}P \quad \Sigma^4) =$$

$$= \overset{15}{+} Z + Z \times Z \;,$$

$$H_3(\Sigma^4 - \mathop{}\limits_{j=1}^{4} P_j) = H_3^{o}(\Sigma^4) + \delta^{i} H_2^{o}({}^{i}P \quad \Sigma^4) +$$

$$+\delta^{ij} H_1^{o}({}^{ij}P \quad \Sigma^4) + \delta^{ijk} H_0^{o}({}^{ijk}P \quad \Sigma^4) = \overset{4}{+} Z \;, \tag{20}$$

$$H_2^{o}(\Sigma^4 - \mathop{}\limits_{j=1}^{4} P_j) = H_2^{o}(\Sigma^4) + \delta^{i} H_1^{o}({}^{i}P \quad \Sigma^4) +$$

$$+\delta^{ij} H_0^{o}({}^{ij}P \quad \Sigma^4) = \overset{6}{+} Z \;, \tag{21}$$

$$H_1^{o}(\Sigma^4 - \mathop{}\limits_{j=1}^{4} P_j) = H_1^{o}(\Sigma^4) + \delta^{i} H_0^{o}({}^{i}P \quad \Sigma^4) = \overset{4}{+} Z \;, \tag{22}$$

$$H_0^{o}(\Sigma^4 - \mathop{}\limits_{j=1}^{4} P_j) = H_0^{o}(\Sigma^4) = Z \;, \tag{23}$$

i.e.

$$H_q^{o}(\bar{\Sigma}^4 - \mathop{}\limits_{j=1}^{5} \bar{P}_j) = (0^1,\ 1^4,\ 2^6,\ 3^4,\ 4^{17}) \tag{24}$$

respectively for $q = 0, 1, 2, 3, 4$.

In the formulas (19)-(23) given above we took into consideration the dimension of ${}^{\beta}P \quad \Sigma^4$ which is $4-\beta$.

Let $\vec{e}_{\alpha}$ be the generators of the group

$$H^{c}_{4}(\Sigma^{4} - \bigcup_{j=1}^{4} P_j), \quad \alpha \subset \{1,2,3,4\};$$

The vanishing cycles e, α will be of interest for us, too. (The detailed account of the method of construction of e, α is given in/7/). With the generators of the homology group they are connected by the relation

$$\vec{e}_{\alpha} = \delta^{\alpha} \partial^{\alpha} e_{\alpha} \quad . \tag{25}$$

Below we shall need the four-dimensional cycles

$$\Gamma = \mathrm{Re}\,\Sigma^{4}, \vec{e}_{i}, \vec{e}_{ij}, \vec{e}_{ijk}, \vec{e}_{1234}$$

only.

V. The Picard-Lefschetz Theorem

Using the traditional approach to the investigation of the analytical properties of the Feynman graphs, it is rather difficult to establish which of the singularities belongs to this or that sheet of the Rimann surface. The homological method enables us to solve such a problem easily.

The answer is given by the following Picard-Lefschetz theorem

$$\Gamma' = \Gamma + n\, e_{\alpha} \tag{26}$$

which shows the variation of the integration cycle under the circling in the positive sense round the singular manifold L_{α} .

Here $n = (-)^{(\ell - a+1)(\ell - a+2)} \quad KI(\Gamma, e_a)$.

For one-loop graphs $\ell = 4$.

The main role in formula (26) is played by the Kronecker indices. Their values for the pentagon graph are given in Table 1. Since the explicit expressions for Σ^4 and P_i are known, the evaluation of the KI's is not very difficult.

KI	e_i	e_{ij}	e_{ijk}	e_{1234}	e_{12345}
Γ	1	1	1	1	1
$\vec{e}_i$	0	1	1	1	1
$\vec{e}_{ij}$	0	-2	-1	-1	-1
$\vec{e}_{ijk}$	0	0	0	-1	-1
$\vec{e}_{1234}$	0	0	0	+2	+1

(In principle we could apply the ready formulas by E.Cartan/6/. See also/7/). Let us add some remarks to clarify the physical contents of our results. We shall take in turn the Landau surfaces and look after the variation of contour under the circling round them. To begin with, let us take L : Circling round it 1,2,.., k times, we get

$$\Gamma \to \Gamma + \vec{e}_i \to \Gamma + \vec{e}_i + \vec{e}_i \to \ldots \to \Gamma + k\vec{e}_i . \qquad (27)$$

Consequently, L_i has the logarithmic singularity (the same for the L_{ijk}).

Circling round L_{ij} , we shall see another picture. After the double circling we shall return to the original contour, i.e.

$$\Gamma \to \Gamma + \vec{e}_{ij} \to \Gamma + \vec{e}_{ij} + \vec{e}_{ij} - 2\vec{e}_{ij} = \Gamma. \tag{28}$$

This means that L_{ij} (as L_{1234} too) contains the square-root singularity.

Since the dimension of the integration cycle (equal to 4) is by 1 less than the number of propagators (which is five) and the variation of the contour under the circling round L_{12345} is equal to zero though the Kronecker index is nonzero, we can drow a conclusion that the surface L_{12345} possesses the pole singularity.

VI. The Relations Between the Absorptive Parts

The homological method enables us to derive the relations between the absorptive parts of amplitude with similar simplicity. Really, the surface $\overline{P}_5$ was selected (considered to be at the infinity). That is why we did not put any attention to the relative cycles which ends belong to $\overline{P}_5$. They can be taken into consideration by replacing $\alpha \subset \{1,2,3,4\}$ by $\beta \subset \{0,1,2,3,4\}$ (further we shall write index 0 instead of index 5), i.e. by replacing $\vec{e}_\alpha$ by $\vec{e}_\beta$. For this let us introduce the dual basis $\{{}^\alpha e\}$:

$$\begin{aligned} {}^\Gamma e &= e_{,0} \\ {}^i e &= e_{,0i} - e_{,0} \\ {}^{ik} e &= e_{,0ik} - e_{,ik} \\ {}^{ijk} e &= e_{,ijk} - e_{,0ijk} \\ {}^{1234} e &= e_{,1234} - e_{,01234}. \end{aligned} \tag{29}$$

Then

$$\vec{e}_{\beta} = \sum_{\alpha} \langle \vec{e}_{\beta} / {}^{\alpha}e \rangle \; \vec{e}_{\alpha} . \tag{30}$$

The transition matrix $\langle \vec{e}_{\beta} / {}^{\alpha}e \rangle$ can be easily evaluated according to (29) and to Table 1. It turns to be

$$\begin{array}{cccccc}
 & {}^{\Gamma}e & {}^{i}e & {}^{ik}e & {}^{ijk}e & {}^{1234}e \\
\vec{e}_{0} & 0 & 1 & 1 & -1 & -1 \\
\vec{e}_{0i} & 0 & -2 & -1 & 1 & 1 \\
\vec{e}_{0ik} & 0 & 0 & 0 & 1 & 1 \\
\vec{e}_{0ijk} & 0 & 0 & 0 & -2 & -1 \\
\vec{e}_{01234} & 0 & 0 & 0 & 0 & 0
\end{array} \tag{31}$$

and contains all the information about the relations between absorptive parts. The elements $\vec{e}_i$, $\vec{e}_{ik}$, ... of the basis correspond to the graphs in which the i-th propagator, the i-th and the k-th propagators etc. are replaced by δ^4-functions. Therefore, to every of the developments

$$\vec{e}_0 = \sum_{i=1}^{4} \vec{e}_i + \vec{e}_{12} + \vec{e}_{13} + \vec{e}_{23} + \vec{e}_{14} + \vec{e}_{24} + \vec{e}_{34} - \vec{e}_{123} - \vec{e}_{124} - \vec{e}_{234} - \vec{e}_{134} - \vec{e}_{1234}$$

$$\vec{e}_{0i} = - 2\vec{e}_i - \sum \vec{e}_{ik} + \sum \vec{e}_{ikm} + \vec{e}_{1234}$$

$$\vec{e}_{0ik} = \sum_{j} \vec{e}_{ikj} \qquad (\text{for example, } \vec{e}_{012} = \vec{e}_{123} + \vec{e}_{124}) \tag{32}$$

$$\vec{e}_{0ijk} = -2\vec{e}_{ijk}$$

there correspond a certain relation between the graphs, i.e. a relation between absorptive parts. For example, the latter development means that

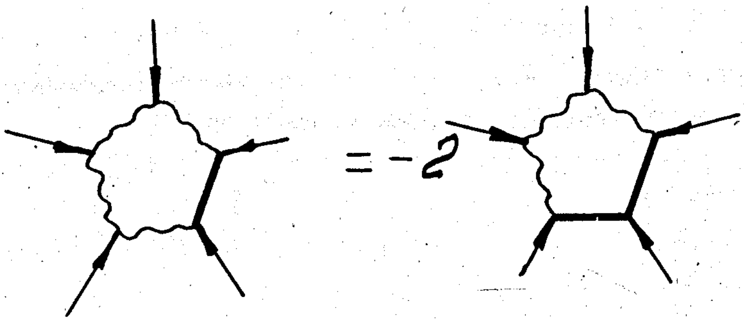

Fig. 2

The wavy line corresponds to the particle on the mass shell, i.e. to the δ^4 - function.

V. Summary

Let us formulate briefly the results of our work. First, we have derived the equations of the Landau surfaces and, using the simple algebraic operations, established the type of the singularities (the formulas (27) and (28)) of the integral. Second, we have evaluated the homology group $H_4^c\,(\,\overline{\Sigma}^4 - \overset{5}{\underset{j=1}{\ }}\ \overline{P}_j\,)$ and found the number of independent contours, i.e. the number of functions defined by the given Feynman integral (the number in question turns to be equal to 16). Third, having found the four types of relations between the integration cycles (contours) (formula (32)), we have got the relations between the absorptive parts of the amplitude.

References

1. F.Pham. Introduction a l'Etude topologique des singularities des Landau. Gauthier-Villar, Paris, 1967.
2. F.Pham. Bull. Soc. math., France, 93, 333 (1965).
3. D.Fotiady, F.Pham. Analytic Study of Some Feynman Graphs by Homological Method. Part I, partly reproduced as Appendix in /5/.
4. D.Fotiady, M.Froissart, J.Lascoux, F.Pham. Analytic Properties of Some Integrals over a Complex Manifold, Centre de Physique Theoretique de l'Ecole Polytechnique, Paris, 1964.
5. R.Hwa, V.Teplitz. Homology and Feynman Integrals, W.A.Benjamin. Inc. New York, Amsterdam, 1966.
6. E.Cartan. Oeuvres completes. Pt. 1, T.2, p.1227, Gauthier-Villar, Paris, 1952.
7. J.B.Boyling. Journ. of Math. Phys., 7, N.10, 1749 (1966).

Received by Publishing Department
on September 25, 1969