СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
Дубна

E2 - 4716

M.Huszár, J.Smorodinsky

A SECOND ORDER INFINITE COMPONENT WAVE EQUATION

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1969

E2 - 4716

M.Huszár, J.Smorodinsky

A SECOND ORDER INFINITE COMPONENT WAVE EQUATION

Introduction

There is a number of no-go theorems restricting the possibility of obtaining mass spectrum by the combination of relativistic invariance with intrinsic degrees of freedom. In particular, according to O'Raifeartàigh's theorem/1/ under certain weak conditions there exists no enlargement of the Poincaré group to a finite order Lie group irreducible representations of which contain more than one mass. We can work, however, in a Hilbert space consisting of a continuous direct sum of irreducible spaces with definite masses and in some way selecting out certain mass and spin values we can obtain a spectrum. Selection of definite irreducible components characterizing the Poincaré group can be performed by requiring the fulfilment of an infinite component wave equation or that of a certain non-Lie algebraic commutation relation/2/. In most cases, spectra obtained in this way fit the experimental data not too well, though recently a linear mass-spin spectrum has been obtained by the aid of a six order equation cf./4/.

A representation of a covariant equation can be found by group-theoretical methods which proved very useful for Majorana equation as well/3/. Operators entering a covariant equation are irreducible tensors and thus they can be determined up to a reduced matrix element by the use of the Wigner-Eckart theorem. If, however, the irreducible tensor itself is a generator of the group considered, matrix elements are completely determined. Therefore if we enlarge the Poincaré group by the generators playing a part in the equation and we close the Lie algebra in some way we arrive at an explicit representation of the equation. For the Majorana equation/3/, for instance, we can close the Lie-algebra by requiring $[\Gamma_\mu, \Gamma_\nu] = -i\sigma_{\mu\nu}$ (where $\sigma_{\mu\nu}$ are the generators of the spin part of the Lorentz group). Then Γ_μ and $\sigma_{\mu\nu}$ generate the $0(3,2)$ group which leads to the Majorana representation.

In the following we shall consider a second order infinite component equation which is a natural generalization of the Majorana equation. Enlarging the Poincaré group by the generators of the equation and closing the algebra we get a group which is the semi-direct product of the Poincaré group and the $U(3,1)$ group. A representation of Γ_μ - s of the Majorana equation generates a representation of the second order equation in consideration but the converse is not true. In particular the representation for $r^\mu{}_\nu$ (cf. eq. (1)) satisfying together with $\sigma^\mu{}_\nu$ the $U(3,1)$ algebra genetares a representation of the Majorana equation only in the trivial case: $r^\mu{}_\nu = \delta^\mu{}_\nu$ that is, if eq. (1) and the Majorana equation degenerate to the Klein-Gordon and Dirac equations respectively.

The Equation

Consider the following equation

$$(p^\nu p_\mu r^\mu{}_\nu - \kappa^2)\psi(p) = 0 \tag{1}$$

($\mu, \nu = 0,1,2,3$, metric $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$), where $r^\mu{}_\nu$ mixes the components of ψ while p leaves unaltered. Under the Lorentz transformation L , $\psi(p)$ transforms as

$$T_L \psi(p) = D(L)\psi(L^{-1}p). \tag{2}$$

Infinitesimal generators of the representation T_L are

$$J_{\mu\nu} = i\left(p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}\right) + \sigma_{\mu\nu} \equiv L_{\mu\nu} + \sigma_{\mu\nu}$$

$J_{\mu\nu}$ -s obey the familiar commutation relations of the Lorentz group

$$[J_{\alpha\beta}, J_{\mu\nu}] = i(J_{\alpha\nu} g_{\beta\mu} + J_{\beta\mu} g_{\alpha\nu} - J_{\alpha\mu} g_{\beta\nu} - J_{\beta\nu} g_{\alpha\mu}).$$

The same commutators hold for $\sigma_{\mu\nu}$. In the following it will be convenient to work with the mixed generators $\sigma^\mu{}_\nu$ that satisfy

$$[\sigma^\alpha{}_\beta, \sigma^\mu{}_\nu] = i(\sigma^\alpha{}_\nu \delta^\mu{}_\beta + \sigma^\mu{}_\kappa g^{\alpha\kappa} g_{\beta\nu} - \sigma^\mu{}_\beta \delta^\alpha{}_\nu - \sigma^\kappa{}_\nu g_{\beta\kappa} g^{\alpha\mu}). \qquad (3)$$

By eq. (2) we get the condition of relativistic invariance of the eq.(1), namely:

$$[\tau^\alpha{}_\beta \sigma^\mu{}_\nu] = i(\tau^\alpha{}_\nu \delta^\mu{}_\beta - \tau^\mu{}_\beta \delta^\alpha{}_\nu + \tau^\gamma{}_\nu g^{\alpha\mu} g_{\beta\gamma} - \tau^\alpha{}_\gamma g^{\gamma\mu} g_{\nu\beta}). \qquad (4)$$

For a given representation of the Lorentz group solutions of (4) yield all the possible second order equations of type (1). We shall not consider the problem in this general form instead we enlarge the Poincaré group with the generators $\tau^\alpha{}_\beta$. In order to obtain a closed Lie algebra we have to prescribe the commutator of $\tau^\alpha{}_\beta$-s. It is natural to require the fulfilment of the following relations

$$[\tau^\alpha{}_\beta, \tau^\mu{}_\nu] = i(\sigma^\mu{}_\beta \delta^\alpha{}_\nu + \sigma^\gamma{}_\beta g_{\nu\gamma} g^{\alpha\mu} - \sigma^\alpha{}_\nu \delta^\mu{}_\beta - \sigma^\alpha{}_\gamma g^{\mu\gamma} g_{\beta\nu}). \qquad (5)$$

Assuming the generators $\sigma^\mu{}_\nu$. $\tau^\mu{}_\nu$ to be canonically conjugated to real parameters we conclude that eqs. (3), (4), (5) constitute the Lie algebra of the $U(3,1)$ group. For unitary representations $\sigma^\mu{}_\nu$ and $\tau^\mu{}_\nu$ are both Hermitean. Consider the transformations U leaving invariant the bilinear form $z^{0*}z^0 - z^{1*}z^1 - z^{2*}z^2 - z^{3*}z^3$. For U we obtain

$$U^+ g U = g \ . \qquad (6)$$

In terms of the infinitesimal generators X (6) reads:

$$g X^+ g = X \qquad (7)$$

and thus generators $X^0{}_0$, $X^i{}_k$ are Hermitean while $X^0{}_i$, $X^i{}_0$ anti-Hermitean. ($i, k = 1,2,3$). For a basis canonically conjugated to real parameters we have two types of generators satisfying eq.(7), those with pure imaginary elements

$$(\sigma^\mu{}_\nu)^a{}_\beta = i(g^{\mu a} g_{\nu\beta} - \delta^\mu{}_\beta \delta^a{}_\nu) \tag{8}$$

and those with real elements

$$(\tau^\mu{}_\nu)^a{}_\beta = g^{\mu a} g_{\nu\beta} + \delta^\mu{}_\beta \delta^a{}_\nu . \tag{9}$$

Generators (8) and (9) belong to real parameters and obey the commutators (3), (4), (5) that proves that (3), (4), (5) are the commutation relations of the U(3.1) group.

It will be convenient to use Weyl basis introducing

$$\lambda^\mu{}_\nu = \frac{1}{2}(\tau^\mu{}_\nu + i\sigma^\mu{}_\nu). \tag{10}$$

Inverse formulas read

$$\sigma^\mu{}_\nu = i(g_{\nu a} g^{\mu\beta} \lambda^a{}_\beta - \lambda^\mu{}_\nu)$$

$$\tau^\mu{}_\nu = g_{\nu a} g^{\mu\beta} \lambda^a{}_\beta + \lambda^\mu{}_\nu . \tag{11}$$

From (8), (9) we obtain the 4x4 representation of $\lambda^\mu{}_\nu$ - s that will be denoted by $\Lambda^\mu{}_\nu$

$$(\Lambda^\mu{}_\nu)^a{}_\beta = \delta^\mu{}_\beta \delta^a{}_\nu . \tag{12}$$

Since we have introduced linear combinations with complex coefficients parameters belonging to $\lambda^\mu{}_\nu$ will not be real but instead we have that parameters of $\lambda^i{}_k$ and $\lambda^k{}_i$ and similarly those of $\lambda^0{}_i$ and $\lambda^i{}_0$ ($i \neq k$, $i, k = 1,2,3$) are complex conjugated to each other while parameters of $\lambda^\mu{}_\mu$ (no sum) are real. Condition of the unitarity for $\lambda^\mu{}_\nu$ - s reads

$$\lambda^{0+}_{0} = \lambda^{0}_{0}, \quad \lambda^{1+}_{k} = \lambda^{k}_{1}, \quad \lambda^{1+}_{0} = -\lambda^{0}_{1}. \tag{13}$$

Using (3), (4), (5) we get the commutators for the Weyl basis

$$[\lambda^{a}_{\beta}, \lambda^{\mu}_{\nu}] = \delta^{a}_{\nu} \lambda^{\mu}_{\beta} - \delta^{\mu}_{\beta}\lambda^{a}_{\nu}. \tag{14}$$

Tower Representation and Mass Spectrum

The Poincaré group P represented by generators P_{μ}, $J_{\mu\nu}$ has been completed by generators τ^{μ}_{ν} and thus the total algebra obtained is the semi-direct sum of the Poincaré and U(3,1) algebras

$$\Sigma = P \oplus L(3,1).$$

This can be equivalently expressed as

$$\Sigma = P^{L} \oplus L(3,1)^{S},$$

where P^{L} is the "orbital" Poincaré group generated by p_{μ}, $L_{\mu\nu}$ and $L(3,1)^{S}$ is the U(3,1) algebra of the spin part generated by σ^{μ}_{ν} and τ^{μ}_{ν} or by λ^{μ}_{ν}.

Mass oparator p^{2} is clearly an invariant operator while the Pauli-Lubanski spin operator fails to commute with τ^{μ}_{ν}. So an irreducible representation contains one mass and several spin values in complete accordance with O'Raifeartaigh's theorem.

For labelling the states we can choose diagonal p_{μ} as well. Furthermore we can give four Casimir operators $C^{(3,1)}_{\mu}$ of the L(3,1) algebra, three Casimir operators of L(3) algebra $C^{(3)}_{i}$, the spin value s^{2} and its 3. component s_{3}. Thus finally states are labelled: $|p^{2}, p_{\mu}, C^{(3,1)}_{\mu}, C^{(3)}_{i}, s^{2}, s_{3}>$.

In the following we shall built up the tower representation of the L(3,1) algebra/3/. For this reason consider a column built up from creation and annihilation operators $(a_{0}, b^{+}_{1}, b^{+}_{2}, b^{+}_{3}) = \phi^{\mu}$

and its adjoint $\phi^{+}_{\mu} = (a^{+}_{0}\, b_1\, b_2\, b_3)$. Defining $\bar{\phi} = \phi^{+} g$ we shall require fulfilment of the relations

$$[\phi^{\mu}, \bar{\phi}_{\nu}] = \delta^{\mu}_{\nu}. \tag{15}$$

Eq. (15) is equivalent to the usual Bose-type commutators $[a_0, a^{+}_{0}] = 1$, $[b_i, b^{+}_{i}] = \delta_{ik}$, $[a_0, b_i] = 0$. The advantage of the use of this operators lies in the fact that sandwiching the 4x4 dimensional generators by $\bar{\phi}$ and ϕ we obtain generators satisfying Hermiticity requirements and thus a finite dimensional non-unitary representation can be transformed to a certain class of unitary representations. We shall work in λ^{μ}_{ν} basis 4x4 representation of which has been found in eq.(12)

$$\lambda^{\mu}_{\nu} = \bar{\phi}\, \Lambda^{\mu}_{\nu} \phi = \bar{\phi}_{\nu} \phi^{\mu} . \tag{16}$$

Casimir operators of the U(3,1) group are $C^{(3,1)}_{1} = \lambda^{\mu}_{\mu} \equiv \lambda$, $C^{(3,1)}_{2} = \lambda^{\mu}_{\nu} \lambda^{\nu}_{\mu} = \lambda\,(\lambda + 3)$,

$$C^{(3,1)}_{3} = \lambda^{\mu}_{\nu} \lambda^{\nu}_{\kappa} \lambda^{\kappa}_{\mu} = \lambda(\lambda+3)^2, \quad C^{(3,1)}_{4} = \lambda^{\mu}_{\nu} \lambda^{\nu}_{\kappa} \lambda^{\kappa}_{\alpha} \lambda^{\alpha}_{\mu} = \lambda(\lambda + 3)^3. \tag{17}$$

States transforming according to irreducible representations can be built up in the following way

$$\psi^{n_a; n_1 n_2 n_3} = \frac{1}{\sqrt{n_a! n_1! n_2! n_3!}} (a^{+}_{0})^{n_a} (b^{+}_{1})^{n_1} (b^{+}_{2})^{n_2} (b^{+}_{3})^{n_3} \psi^{0;000} . \tag{18}$$

Since $\lambda = a^{+}_{0} a_0 - b^{+}_{i} b_i - 3 \equiv N - 3$ $\quad (N \equiv n_a - n_b \,, n_b \equiv n_1 + n_2 + n_3)$ therefore according to (17) irreducible infinite dimensional representations are characterized by λ or by N and thus irreducible spaces are characterized by a fixed difference of numbers of a and b quanta

$$N = n_a - n_b . \tag{19}$$

Reduction $U(3,1) \supset U(3) \times U(1)$ is obtained fixing n_b since all the Casimir operators of $U(3)$, $C_1^{(3)}$, $C_2^{(3)}$, $C_3^{(3)}$ are unambiguously determined by n_b . In the following we shall consider representations of $U(3,1)$ with positive N since $N=0,-1,-2,\ldots$ values are irrelevant from physical point of view. Denoting by $S^{(n_a,n_b)}$ irreducible spaces with definite n_a and n_b values the $U(3)\times U(1)$ content for different values of N is

$$
\begin{aligned}
S^{N=1} &= S^{(1,0)} \oplus S^{(2,1)} \oplus S^{(3,2)} \oplus \cdots \\
S^{N=2} &= S^{(2,0)} \oplus S^{(3,1)} \oplus S^{(4,2)} \oplus \cdots \\
S^{N=3} &= S^{(3,0)} \oplus S^{(4,1)} \oplus S^{(5,2)} \oplus \cdots
\end{aligned}
\tag{20}
$$

etc.

Further reduction of the subspaces labelled by n_b can be made with respect to the $O(3)$ subgroup of $U(3)$ generated by $\sigma^i{}_k$ $(i,k=1,2,3)$. In the rest frame the invariant operator labelling the $O(3)$ subspaces constitues the spin value. As it is seen from (18) we have $U(3)$ representation spanned by symmetric tensors with n_1 indices 1 , n_2 indices 2 , n_3 indices 3 . $O(3)$ invariant spaces are those with zero traces. Therefore we get the reduction:

$$
S^{n_b} = S^{s=n_b} + S^{s=n_b-2} + S^{s=n_b-4} + \ldots + \begin{cases} S^{s=0} \\ S^{s=1} \end{cases} . \tag{21}
$$

Consider now equation (1) in the rest frame $p=(m,0,0,0)$

$$
(2m^2 \lambda^0_0 - \kappa^2)\psi(p) = 0 . \tag{22}
$$

The algebra Σ considered has two invariant oparators P^2 and N . Reducing it with respect to the Poincaré group we get one mass $m^2 = \kappa^2/2n_a$ and several spin values (eq. (20) (21)). In order to obtain mass-spin spectrum we consider a reducible representation which is a continuous direct sum of irreducible spaces $S^{m,N}$.

Reducing it with respect to P eq. (22) selects out certain m and s values

$$S^{N} = \int \oplus \, dm \, S^{(m,N)} = \int \oplus \, dm \sum_{n_b} \oplus \, S^{m,N,n_b} = \int \oplus \, dm \sum_{n_b} \oplus \sum_{s} \oplus \, S^{m,N,n_b,s} = \tag{23}$$

$$= \int \oplus \, dm \sum_{n_b} \oplus \sum_{s} \oplus \, \delta\left(m - \frac{\kappa}{\sqrt{2(N+n_b)}}\right) S^{mNn_bs} = \sum_{n_b} \oplus \sum_{s} \oplus \, S^{m_{N,n_b},N,n_b,s} .$$

Summarizing the results of eqs. (20), (21), (23) the following table is obtained

n_b	s	$m/\varkappa$ N=1	$m/\varkappa$ N=2	$m/\varkappa$ N=3	...
0	0	$1/\sqrt{2}$	$1/\sqrt{4}$	$1/\sqrt{6}$	...
1	1	$1/\sqrt{4}$	$1/\sqrt{6}$	$1/\sqrt{8}$	...
2	0,2	$1/\sqrt{6}$	$1/\sqrt{8}$	$1/\sqrt{10}$	...
3	1,3	$1/\sqrt{8}$	$1/\sqrt{10}$	$1/\sqrt{12}$	...
4	0,2,4	$1/\sqrt{10}$	$1/\sqrt{12}$	$1/\sqrt{14}$	...
...					

The spectrum is degenerate in the sense that for a given mass value there are several values of spin.

Finally we discuss a formal relation to the three dimensional harmonic oscillator. The existence of such a connection is not surprising since U(3,1) is the spectrum generating group of the oscillator. The first order Casimir operator of the U(3,1) group is (eq. (17))

$$C_1^{(3,1)} = \lambda^{\mu}{}_{\mu} = a_0^+ a_0 - b_1^+ b_1 - b_2^+ b_2 - b_3^+ b_3 - 3 = \lambda$$

It is related to the Hamiltonian of the ocillator by

$$H = a_0^+ a_0 - (\lambda + \frac{3}{2}) = \lambda_0^0 - (\lambda + \frac{3}{2}) .$$

It is worth to mention that while the eigenvalues of H are determined by λ_0^0 the mass spectrum of the infinite component equation has had a straightforward connection with $(\lambda_0^0)^{-1}$ which led to an undesired tendency of the mass-spin spectrum.

Acknowledgements: Authors wish to express gratitude to R.P.Zaikov for a number of valuable discussions.

References

1. L.O'Raifeartaigh. Phys. Rev.Lett., 14, 575 (1969).
2. A.Böhm. Phys. Rev., 175, 1767 (1968) .
3. D.Tz.Stoyanov, I.T.Todorov. Journ. Math.Phys., 9, 2146 (1968).
4. R.Casalbuoni, R.Gatto, G.Longhi. Preprint , Instituto di Fisica dell' Universita Padova, (1969).

Received by Publishing Department
on September 25, 1969.