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INVARIANT EQUATIONS FOR NONLINEARLY TRANSFORMING FIELDS AND CONSERVATION LAWS

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The investigation of the possible theories with interacting fields is usually made in terms of the Lagrangian formalism. This formalism, as is well known, treats arbitrary Lagrangian functions from which by variation methods, one obtains the fundamental equations and conservation laws. In this manner one has searched for the possible theories including interaction vector, spinor and scalar fields in the approximation of the dimensionless coupling constants. All internal symmetry groups which allow the interactions has been found.

There is another than Lagrangian formalism which gives the possibility to build directly invariant equations for the fields. In the case of the noninteracting fields which transform linearly under the Lorentz group, this formalism, as the Lagrangian one, is applied often independently. However, if we introduce interaction for a few exceptions it is difficult to construct the necessary equations. Besides, if we have the equations of motion to find the conservation laws we must use the Lagrangian too. The situation becomes more complicated if the interacting fields transform nonlinearly, if we want, under some internal symmetry group G . In the last years there are good many models like that discussed here[2-5].

In the present paper we consider the construction of the invariant equations for interacting fields, transforming linearly under the Lorentz group and nonlinearly under the group of internal symmetry G . Besides, it turned out to be possible to obtain all conservation laws which follow from invariance of the equations under action of the group G .

In the first section we quailify any essential and common properties of the nonlinear irreducible representations. Although the fundamental results connected with nonlinear realizations of a given continuous group are known many years ago (see for example[6,7]) in this section we introduce some new quantities which are not applied usually in the general theory and which have turned out to be very useful. In this section we accept the notation employed in[8]. The transformation properties of the investigated quantities are estimated in terms of the commutators[3]. We think that this is more transparent than the application of global transformation.

In the second section one constructs the above mentioned equations and conservation laws.

In application one illustrates the general theory on the case of the chiral group $SU(2) \times SU(2)$.

1. Properties of the Nonlinear Realizations of the Lie Group

We shall consider the case when the semisimple compact f -parameter Lie group G is given. Let its structure constants be $C^{\omega}_{\mu\nu}$. Then the metric tensor is given by

$$g_{\mu\nu} = C^{\sigma}_{\mu\rho} \; C^{\rho}_{\nu\sigma} \quad . \tag{1.1}$$

(Here and further the Greek indices run from 1 to f , the summation is assumed over repeating lower and upper indices).

We shall suppose that in the N-dimensional space X , acts any irreducible nonlinear representation $F_i(g_\mu, x_k)$ of the group G with generator functions $M^\mu_i(x_k)$. They satisfy the following commutation relations

$$[\, M^\mu \;\; M^\nu \,]_i \equiv \frac{\partial M^\mu_i}{\partial x_k} M^\mu_k - \frac{\partial M^\nu_i}{\partial x_k} M^\mu_k = C^{\mu\nu}_{\rho} \quad M^\rho_i \tag{1.2}$$

(here and further indices i, j , ..,s,t run over 1, ... N).The irreducibility of the representation means that $f \geq N$ and the rank of $\|M^\mu_i\|$ equals N . Taking into account this fact, there exist $f-N$ solutions $N^\mu_a(x_k)$ of the system

$$g_{\mu\nu} M^\mu_i (x) N^\nu_a = 0, \tag{1.3}$$

where the rank of the $\| N^\mu_a \|$ equals $f-N$. (Here and further a , b ... g run over $N+1, \dots f$). With the help of the M^μ_i and N^μ_a we may define a matrix

$$R^\mu_\nu = \begin{cases} M^\mu_i & \nu \geq N \\ N^\mu_a & \nu \geq N+1 \end{cases} \tag{1.4}$$

It is easy to see that R is nondegenerate.

Let now enumerate some properties of the nonlinear representations.

a. The symmetric matrices $\|H_{ik}\|$ and $\|h_{ab}\|$ with matrix elements

$$H_{ik} = g_{\mu\nu} M^{\mu}_{i} M^{\nu}_{k} , \qquad h_{ab} = g_{\mu\nu} N^{\mu}_{a} N^{\nu}_{b} \tag{1.5}$$

are nondegenerate. Indeed, we consider these equations

$$H_{ik} r^{k} = 0.$$

For all solutions of these equations hold also

$$H_{ik} r^{i} r^{k} = 0.$$

Hence, using the positive definiteness of the $g_{\mu\nu}$

$$M^{\mu}_{i} r^{i} = 0 . \tag{1.6}$$

Hence, $r^{i} = 0$. Similarly this may be proved for $\|h_{ab}\|$.

The inverse matrices, i.e.

$$H^{ik} H_{k\ell} = \delta^{i}_{\ell} , \quad h^{ab} h_{bc} = \delta^{b}_{c}$$

are denoted as H^{ik} and h^{ab} , correspondingly.

b. The symmetric matrices $\|\Pi^{\mu\nu}\|$ and $\|\amalg^{\mu\nu}\|$ with elements

$$\Pi^{\mu\nu} = H^{ik} M^{\mu}_{i} M^{\nu}_{k} , \qquad \amalg^{\mu\nu} = h^{ab} N^{\mu}_{a} N^{\nu}_{b} \tag{1.7}$$

satisfy identically the following equalities

$$\Pi^{\mu\rho}\,\Pi^{\sigma\nu}\,g_{\rho\sigma} = \Pi^{\mu\nu} \tag{1.8}$$

$$\Pi^{\mu\rho}\,\Pi^{\sigma\nu}\,g_{\rho\sigma} = 0 \tag{1.9}$$

$$\Pi^{\mu\rho}\,\Pi^{\sigma\nu}\,g_{\rho\sigma} = \Pi^{\mu\nu} \,. \tag{1.10}$$

Besides we have

$$\begin{aligned} g_{\rho\sigma}\,\Pi^{\mu\rho}\,R^{\sigma}_{\nu} &= \delta_{\nu i}\,M^{\mu}_{i} \\ g_{\rho\sigma}\,\Pi^{\mu\rho}\,R^{\sigma}_{\nu} &= \delta_{\nu a}\,N^{\mu}_{a} \,. \end{aligned} \tag{1.11}$$

All written above equalities may be verified directly. In particular it follows from them

$$\Pi^{\mu\nu} + \Pi^{\mu\nu} = g^{\mu\nu} \,. \tag{1.12}$$

If substituting here $\Pi^{\mu\nu}$ and $\Pi^{\mu\nu}$ from (1.7) we obtain the normalization condition for the generator-functions

$$H^{ik}\,M^{\mu}_{i}\,M^{\nu}_{k} + h^{ab}\,N^{\mu}_{a}\,N^{\nu}_{b} = g^{\mu\nu} \,. \tag{1.13}$$

If we introduce a matrix $\| U^{\mu\cdot}_{\cdot\nu} \|$ with matrix elements

$$U^{\nu\cdot}_{\cdot\mu} = \begin{cases} M^{\rho}_{k}\,H^{ki}\,g_{\rho\mu} & \nu \le N \\ N^{\rho}_{b}\,h^{ba}\,g_{\rho\mu} & \nu \ge N+1 \end{cases} \tag{1.14}$$

by vertue of (1.13) we obtain

$$R^{\mu\cdot}_{\cdot\rho}\; U^{\rho\cdot}_{\cdot\nu} = \delta^{\mu}_{\nu} \tag{1.15}$$

i.e. $\| U^{\mu\cdot}_{\cdot\nu} \|$ is inverse to $\| R^{\mu\cdot}_{\cdot\nu} \|$.

c. The following identities are valid

$$C_{\rho\sigma\omega}\, N^{\rho}_{a}\, N^{\sigma}_{b}\, M^{\omega}_{i} = 0. \tag{1.16}$$

One may obtain them by multiplying the both sides of (1.2) by $g_{\mu\rho}\, g_{\nu\sigma}\, N^{\rho}_{a}\; N^{\sigma}_{b}$ using (1.3).

d. The quantities

$$d^{ab}_{c} = C_{\rho\sigma\omega}\, N^{\rho}_{a'} N^{\sigma}_{b'}\, N^{\omega}_{c}\, h^{aa'}\; h^{bb'} \tag{1.17}$$

are the structure constants of any group $S'(x)$ the algebra of which is isomorphic to some subalgebra S of G . The first part of this assertion may be proved evaluating directly Jacoby identity for the d^{ab}_{c} . To prove the second one let us consider the quantities

$$0^{\mu\nu}_{\lambda} = C^{\rho\sigma}_{\omega}\, U^{\mu}_{\rho}\, U^{\nu}_{\sigma}\, R^{\omega}_{\lambda} \,. \tag{1.18}$$

In consequence of the nondegeneracy of the R and U it is clear that $0^{\mu\nu}_{\lambda}$ are the structure constants of the algebra of a group $G'(x)$ isomorphic to the G . Immediately, one verifies that

$$0^{ab}_{c} \equiv d^{ab}_{c}$$

i.e. that we want to prove.

e. The quantities

$$\Delta_{i}^{a,k} = C_{\rho\sigma\omega} \; M_{i}^{\rho} \, N_{b}^{\sigma} \, M^{\omega} H^{bk} \; h^{ab} \tag{1.19}$$

are generators of a linear representation of $S'(x)$ in an N -dimensional space. This is proved by direct calculation of the commutation relations

$$[\Delta^{a} \Delta^{b}]_{i}^{k} \equiv \Delta_{i}^{a,\ell} \; \Delta_{\ell}^{b,k} - \Delta_{i}^{b,\ell} \; \Delta_{\ell}^{a,k} = d_{c}^{ab} \; \Delta_{i}^{c,k} \; . \tag{1.20}$$

From the two latter sections it follows that there exists a subgroup S of G which has a linear representation in N -dimensional space X. Hence the nonlinear representation, itself, can be chosen so that it becomes linear on subgroups. This means that a part of the generator functions corresponding to the subgroup S is chosen to be linear functions of X. In this case it is always possible to do $S'(0) \equiv S$, i.e. to choose the structure constants of G with separated subgroup S. Although this choice has certain physical meaning we shall not restrict ourselves to this assumption.

f. To formulate the transformation properties of the quantities which we have introduced above, we define a linear representation T_g of G on the space of the almost everywhere differentiable functions by

$$T_g \; f(x) T_g = f(F \; (g^{-1}, x)). \tag{1.21}$$

The form of the left-hand side allows one to find immediately commutation of the I^{μ} with the transforming quantities according to the formula

$$[I^{\mu} f] = -\left(\frac{\partial f'}{\partial g_{\mu}}\right)_{g_{\mu}=0} \quad , \tag{1.22}$$

where f' is a transformed f . In particular, using (1.21), the following relations can be found

$$[I^{\mu}\ f(x)] = -M^{\mu}_{k}\ \frac{\partial f}{\partial x_k} \tag{1.23}$$

$$[I^{\mu}\ x_{l}] = -\ M^{\mu}_{l}\ (x) \tag{1.24}$$

$$[I^{\mu}\, d\, x_{l}] = -\ \frac{\partial M^{\mu}_{l}}{\partial x_n}\, dx_{n}\ . \tag{1.25}$$

The last one can be obtained using Section e and differentiating with respect to x .

g. The quantity $f^{\nu,b}_{a}(x)$ defined by the equation

$$M^{\nu}_{n}\ \frac{\partial N^{\mu}_{a}}{\partial x_{n}} + C^{\mu\nu}_{\omega}\ \ N^{\omega}_{a} + f^{\nu,b}_{a}(x) N^{\mu}_{b} = 0 \tag{1.26}$$

satisfies also identically the equation

$$M^{\mu}_{n}\frac{\partial f^{\lambda,b}_{a}}{\partial x_{n}} - M^{\lambda}_{n}\ \frac{\partial f^{\mu,b}_{a}}{\partial x_{n}} + f^{\mu,c}_{a}\ f^{\lambda,b}_{c}\ - f^{\lambda,c}_{a}\ \ f^{\mu,b}_{c}\ + C^{\mu\lambda}_{\omega}\ f^{\omega,b}_{a} = 0. \tag{1.27}$$

To prove this assertion it is sufficient to remark that according to (1.23) and (1.26) we have

$$[I^{\mu}\ N^{\nu}_{a}] = -M^{\mu}_{n}\ \frac{\partial N^{\nu}_{a}}{\partial x_{n}} = C^{\mu\nu}_{\omega} N^{\omega}_{a} + f^{\mu,b}_{a} N^{\nu}_{b}\ . \tag{1.28}$$

Now if we write a Jacoby identity for the $[I^{\lambda}[I^{\mu} N^{\nu}_{a}]]$ using the corresponding identity for the $C^{\mu\nu}_{\omega}$ we obtain (1.27) for $f^{\mu,b}_{a}$. It is not difficult to see that equality (1.27) is the very Jacoby identity for any double commutator $[I^{\mu}[I^{\nu} v_a]]$, where v_a are defined by

$$[I^{\mu} v_a] = f^{\mu,b}_{a} v_b . \tag{1.29}$$

So, the identity (1.27) is a necessary and sufficient condition for the existence of quantities like a v_a .

h. The matrices $\|H_{ik}\|, \|H^{ik}\|, \|h^{ab}\|, \|h_{ab}\|$ satisfy the following commutation relation with generators of the representation

$$[I^{\mu} H_{ik}] = -\frac{\partial M^{\mu}_{i}}{\partial x_n} H_{nk} - \frac{\partial M^{\mu}_{k}}{\partial x_n} H_{in} \tag{1.30}$$

$$[I^{\mu} H^{ik}] = \frac{\partial M^{\mu}_{n}}{\partial x_i} H^{nk} + \frac{\partial M^{\mu}_{n}}{\partial x_k} H^{in} \tag{1.31}$$

$$[I^{\mu} h_{ab}] = f^{\mu,c}_{a} h_{cb} + f^{\mu,c}_{b} h_{ac} \tag{1.32}$$

$$[I^{\mu} h^{ab}] = -f^{\mu,a}_{c} h^{cb} - f^{\mu,b}_{c} h^{ca} , \tag{1.33}$$

where $f^{\mu,a}_{c}$ are functions introduced in (1.26). These equalities were obtained by direct evaluation using the definitions of the H_{ik}, h_{ab} and also the transformation properties of M^{μ}_{i} and N^{μ}_{a} .

The commutators of I^{μ} with inverse matrices can be written applying the well known method for finding the transformation low of the inverse matrix if the transformation low for a given one is known.

Another consequence from the last section is the assertion that

$$ds_1^2 = H^{ik}\, dx_i\, dx_k$$

$$ds_2^2 = h^{ab}\, v_a\, v_b \tag{1.34}$$

are invariants of the group G . In particular the first one determines the Riemmanian metric in the space x . The Christoffel symbols in this case are

$$\Gamma_{\ell}^{ik} = \frac{1}{2} g_{\rho\sigma} M_n^{\rho} (H^{ni} \frac{\partial M_{\ell}^{\sigma}}{\partial x_k} + H^{nk} \frac{\partial M_{\ell}^{\sigma}}{\partial x_i}). \tag{1.35}$$

Besides the covariant derivative of the metric tensor H_{ik} is equal to zero

$$H_{ik}^{;\ell} \equiv \frac{\partial H_{ik}}{\partial x_{\ell}} - \Gamma_{i}^{\ell n} H_{nk} - \Gamma_{k}^{\ell n} H_{in} = 0. \tag{1.36}$$

i. The matrices $\Pi^{\mu\nu}$ and $\Pi^{\mu\nu}$ satisfy the following commutation relations with I^{μ}

$$[I^{\mu}, \Pi^{\rho\sigma}] = C_{\omega}^{\mu\rho} \Pi^{\sigma\omega} + C_{\omega}^{\mu\sigma} \Pi^{\rho\omega} \tag{1.37}$$

$$[I^{\mu} \Pi^{\rho\sigma}] = C_{\omega}^{\mu\rho} \Pi^{\sigma\omega} + C_{\omega}^{\mu\sigma} \Pi^{\rho\omega}. \tag{1.38}$$

Using only the definition of $\Pi^{\rho\sigma}$ and transformation properties of M^{ρ}_{i} and N^{μ}_{a} it is easy to establish the first one. The second is obtained from (1.37) inserting $\Pi^{\rho\sigma} = g^{\rho\sigma} - \Pi^{\rho\sigma}$.

j. In the space spanned by v_a acts a representation of the subgroup S equivalent to the adjoint one. In fact the operators

$$S^{a} = g_{\mu\nu} N^{\mu}_{b} h^{ba} I^{\nu} \tag{1.39}$$

are generators of some group isomorphic to S . In fact

$$[S^{a} S^{b}] = d^{ab}_{c} S^{c} . \tag{1.40}$$

To find this commutator the following identities for $f^{\mu,a}_{b}$ obtained from definitions (1.26) and from orthogonality of the M^{μ}_{i} and N^{μ}_{a}

$$\begin{aligned} & g_{\rho\sigma} f^{\rho,c}_{a} N^{\sigma}_{b} = C_{\mu\rho\sigma} N^{\sigma}_{a} N^{\mu}_{d} N^{\rho}_{b} h^{dc} = - d^{c}_{ab} \\ & g_{\rho\sigma} f^{\rho,a}_{c} N^{\sigma}_{b} N^{\mu}_{d} h^{cd} = C^{\mu}_{\rho\sigma} N^{\rho}_{d} N^{\sigma}_{b} h^{da} \end{aligned} \tag{1.41}$$

were used. Now if one multiplies the both sides of (1.29) by $g_{\mu\nu} N^{\nu}_{d} h^{ab}$ and sums over μ one obtains

$$[S^{b} v_{a}] = g_{\mu\nu} N^{\nu}_{d} h^{db} f^{\mu,c}_{a} v_{c} , \tag{1.42}$$

and from (1.41) one has

$$[S^{b} v_{a}] = d^{cb}_{a} v_{c} . \tag{1.43}$$

This is just what we wanted to prove.

k. Now we shall consider a system of differential equations for M_i^{μ} and N_a^{μ} which are equivalent to (1.2).

Multiplying both sides of (1.2) by $M_n^{\rho} H^{nj}$ and summing over ν we obtain

$$\frac{\partial M_i^{\mu}}{\partial x_j} = H^{jm} \frac{\partial M_i^{\sigma}}{\partial x_k} M_k^{\mu} M_m^{\rho} g_{\rho\sigma} + C^{\mu}_{\rho\omega} M_i^{\omega} M^{\rho}_m H^{mj} . \tag{1.44}$$

Similarly from (1.26) it follows

$$\frac{\partial N_a^{\mu}}{\partial x_j} = - f_a^{\rho,b} N_b^{\mu} M_n^{\sigma} H^{nj} g_{\rho\sigma} + C^{\mu}_{\rho\omega} N_a^{\omega} M_n^{\rho} H^{nj} . \tag{1.45}$$

The last equation may be obtained also from (1.2) multiplying by N and using (1.3). By simple calculations the equalities (1.44) and (1.45) are reduced to the form

$$\frac{\partial M_i^{\mu}}{\partial x_j} = \Lambda_i^{jk} M_k^{\mu} + \Delta_i^{b,j} N_b^{\mu} , \tag{1.46}$$

$$\frac{\partial N_a^{\mu}}{\partial x_j} = F_a^{j,b} N_b^{\mu} + \Delta_n^{b,k} H^{nj} h_{ab} M_k^{\mu} , \tag{1.47}$$

where

$$\Lambda_i^{jk} = \Gamma_i^{jk} + \frac{1}{2} C_{\rho\sigma\omega} M_n^{\rho} M_m^{\sigma} M_i^{\omega} H^{nk} H^{mj}$$

$$F_a^{j,b} = - f_a^{\rho,b} M_n^{\sigma} g_{\rho\sigma} H^{nj} .$$

Consider the system

$$\frac{\partial y_i}{\partial x_j} = \Gamma_i^{jk} \; y_j + \Delta_i^{b,j} y_b$$

$$\frac{\partial y_b}{\partial x_j} = F_a^{j,b} \; y_b + \Delta_n^{b,k} H^{nj} \; h_{ba} y_k \; . \tag{1.48}$$

From the results of the last section it follows that the system has f -linearly independent solutions (it is over determined) for the desired functions. As that kind of solution one may choose

$$y_j^{\mu} = M_j^{\mu}, \quad y_a^{\mu} = N_a^{\mu}. \tag{1.49}$$

Then the complete solution of the investigated system has the form

$$y_i = c_{\mu} \, y_i^{\mu} \; , \quad y_a = c_{\mu} \, y_a^{\mu} \; . \tag{1.50}$$

In general as f -linearly independent solutions we may take

$$y_i^{\mu} = C_{\nu}^{\mu} \; M_i^{\mu} \, , \qquad y_a^{\mu} = C_{\nu}^{\mu} \, N_a^{\nu} \, , \tag{1.51}$$

where C_{ν}^{μ} is a nondegenerate matrix. But the covariance of these solutions holds only if C_{ν}^{μ} belongs to the adjoint representation of the group G . So the covariant set of linearly independent solutions of (1.49) is unique up to the trivial equivalence.

2. Invariant Equations

The properties of the nonlinear representations of the group G considered above allow one to build up the invariant equations for the fields transforming according to a given representation. Strictly speaking we shall assume that a set of fields $x_i(s_A)$ (s_A are coordinates in the Minkowsky space-time, A,B,.. =0,1, 2,3) transforming nonlinearly according to equality (1.24) is given. To assure the Lorentz covariance of the above mentioned relations it is sufficient to suppose that the Lozentz group and group G are taken in a direct product.

We also assume that in addition to x_i a set of arbitrary fields ψ_α is given. They transform under arbitrary but fixed representation of the Lozentz group and group G . In general the commutation relations with the generators of G -group can be written as

$$[I^\mu \psi_\alpha] = \phi_\alpha^{\mu,\beta}(x)\, \psi_\beta \tag{2.1}$$

$$[I^\mu \psi^\alpha] = - \phi^{\mu,\alpha}_{\ \ \beta}(x)\, \psi^\beta, \tag{2.2}$$

where ψ^α is a contravariant to ψ_α . From the above written equalities it follows that $\psi_\alpha \psi^\alpha$ is an invariant of G group. The functions $\phi_\alpha^{\mu,\beta}$ identically satisfy equation (1.27).

Now we show that with the help of x_i and ψ_α it is always possible to construct quantities of the type u_i , v_a i.e. which transform according to formulae

$$[I^\mu u_i] = - \frac{\partial M^\mu_k}{\partial x_n} u_n \quad , \tag{2.3}$$

$$[I^{\mu} v_a] = f_a^{\mu,b} v_b \quad . \tag{2.4}$$

This construction allows us to investigate only u_i and v_a instead of x_i, ψ_a and their derivatives. As we shall see further, the quantities of the kind of u_i and v_a are not unambiguously defined if x_i and ψ_a are known. Arbitrarity in the choice of u_i and v_a corresponds to that of a Lagrangian function in theory with "Lagrangians". Besides (2.3) and (2.4) we write down the transformation properties of the covariant quantities

$$u^i = H^{ik} u_k, \quad v^a = h^{ab} v_b \tag{2.5}$$

$$[I^{\mu} u^i] = \frac{\partial M_n^{\mu}}{\partial x_i} u^n \tag{2.6}$$

$$[I^{\mu} v^a] = -f_b^{\mu,a} v^b . \tag{2.7}$$

The possibility of constructing u_i and v_a follows from some complementary assertions.

1. If there are known u_i and v_a then

$$\begin{aligned} &\Delta_i^{a,k} v_a u_k \\ &h_{ab} \Delta_i^{b,k} u^i u_k \end{aligned} \tag{2.8}$$

are again quantities like u_i and v_a, correspondingly. (The proof follows from direct calculations).

2. Comparing (1.22) and (2.7) we remark that

$$\frac{\partial x_i}{\partial s_A} \equiv \partial^A x_i \tag{2.9}$$

is of the kind of u_l (for the moment we are not interested in the Lorentz structure).

3. If the fields b_μ transforming linearly under adjoint representation of G are given, then

$$u_l \approx M_l^\mu b_\mu$$

$$v_a \approx N_a^\mu b_\mu .$$

This is proved by direct calculations using the equality

$$[I^\mu b_\nu] = C_\nu^{\mu\omega} b_\omega .$$

4. Thus, as in the above sections it may be shown that if ψ_a^1 and ψ_a^2 transform with respect to (2.1) then

$$g_{\mu\nu} N_a^\mu \psi^{1a} \phi_a^{\nu,\beta} \Psi^2_\beta$$

is of the kind of v_a again. A quantity of the type of u_l which does not content derivatives of the fields is not always nonvanishing and we do not write it in general.

The four items formulated above allow (if the fields x_l and ψa are known) to construct all possible quantities such as u_l and v_a. If moreover we want to include derivatives of any order of these fields, we must indicate a method to build up the differential expressions transforming themselves like u_l and v_a. It is not difficult to verify that

$$\partial^A u_l + \Gamma_l^{kn} u_k \partial^A x_n \qquad \partial^A u_l + \Lambda_l^{kn} u_k \partial^A x_n \tag{2.10}$$

are quantities of the sort of u_l. Similarly

$$\partial^A v_a + F_a^{n,b} v_b \partial^A x_n \tag{2.11}$$

is a quantity of the type of v_a. In these equalities Γ_l^{kn} Λ_l^{kn} and $F_a^{n,b}$ have been introduced in the above paragraph (see (1.35) (1.46) and (1.47)). This is already enough to construct arbitrary u_l and v_a. However, this construction can be simplified remarking that the expression

$$\partial^A \psi_a + L_a^{n,\beta} \psi_\beta \partial^A x_n \tag{2.12}$$

transforms as ψ_a, if:

$$L_a^{n,\beta} = g_{\mu\nu} M_m^{\mu} \phi_a^{\nu,\beta} H^{mn}. \tag{2.13}$$

Therefore (2.12) can be considered together with ψ_a when we apply the procedure described in the item (4). We shall call the second expression in (2.10) and (2.11), (2.12) invariant derivatives of the quantities u_l, v_a, ψ_a, correspondigly. One may find the simplest equations if we let to vanish the enumerated above co-derivatives. However, we shall drop out the equation which arises according to this procedure from (2.12) because if v_a is build up by the prescription of the fourth item then the vanishing of (2.11) leads to the simultaneous vanishing of (2.12) too.

In that case however it is easy to check that in general a conservation law corresponding to this equations does not exist because the curvature tensor obtained from the affine connections $\Lambda_l^{k\ell}$ is not vanishing identically.

To clarify the above said let us suppose that there exist $J_A(u_l, v_a)$ so that

$$\partial^A J_A = 0. \tag{2.14}$$

If we want a covariance of the equality it is necessary to express $\partial^A J_A$ in terms of the covariant quantities. This is possible to do if the equations for u_i and v_a admit integrating multipliers. If a_{ik} is one of them which corresponds to the equation for u_i then it turns out that a_{ik} must satisfy the following system

$$\frac{\partial a_{ik}}{\partial x_n} = \Lambda_i^{n\ell} a_{\ell k} \quad . \tag{2.15}$$

It has a nontrivial solution only if the corresponding curvature tensor equals zero. So this treatment is not suitable to apply in construction of the equations always admitting conservation laws. Now the way in which the invariant equations should be constructed is clear. Indeed, we shall complete above written co-derivatives for u_i and v_a so that they always admit integrating multipliers. The most general expressions of that kind are

$$P^A u^i_A \equiv \partial^A u^i_A + \Lambda^{ki}_n u^n_A \partial^A x_k + \Delta^{ik}_a v^a_A \partial^A x_k \tag{2.16}$$

$$P^A v^a_A \equiv \partial^A v^a_A + F^{k,a}_b v^b_A \partial^A x_k + \Delta^{a,k}_n u^n_A \partial^A x_k \, . \tag{2.17}$$

As we noted before u_i and v_a have any Lorentz structure and to write the above expressions we need that u_i and v_a have at least one vector index A (which we write down). The quantities $P^A u^i_A$, $P^A v^a_A$ depend explicitly only on the first derivatives, but as it is easy to see, in fact, the order of the derivatives is not restricted because u_i and v_a themselves can be build up with the help of arbitrary order derivatives of the fields x_i and ψ_a . The both expressions (2.16) and (2.17) can be united if we introduce

$$W_A^{\nu} = \begin{cases} u_A^i & \nu = i \leq N \\ v_A^a & \nu = a \geq N+1 \end{cases} \tag{2.18}$$

then instead of (2.16) and (2.12) we have

$$P^A W_A^{\nu} = \partial^A W_A^{\nu} + Z^{k,\nu}_{\mu} \; W_A^{\mu} \; \partial^A x_k \; , \tag{2.19}$$

where $Z^{k,\nu}_{\mu}$ is expressed by $\Lambda_i^{k\ell}, F_a^{k,b}$, $\Delta_i^{a,k}$.
The main property of (2.19) is the following

$$R^{\mu}_{\nu} P^A W_A^{\nu} = \partial^A (R^{\mu}_{\nu} \; W_A^{\nu}) \tag{2.20}$$

i.e. the matrix (1.4) is an integrating multiplier for $P^A W_A^{\nu}$. If we take as an equation of motion

$$P^A W_A^{\nu} = 0 \tag{2.21}$$

then

$$\partial^A J_A^{\mu} = 0 \tag{2.22}$$

and hence $J_A^{\mu} \equiv R^{\mu\cdot}_{\cdot\nu} W_A^{\nu} = M^{\mu}_i u_A^i + N_a^{\mu} v_A^a$ is a conserving quantity corresponding to (2.21). It is clear that a current J_A^{μ} transforms under adjoint representation of the group G. As we noted before, the matrix $R^{\mu\cdot}_{\cdot\nu}$ is nondegenerate and therefore if we have a given current J_A^{μ} then from it u_i and v_a as well as the equations of notion could be found uniquely. This situation is very interesting but it is not a constraint in the theory. The fields x_i are in some sense privileged because in fact only their equations are determined exactly if the current is given, while the equation for ψ_a themselves is not unique. This is a consequence of the bilinear structure of u_l and v_a built from ψ_α and ψ^α. To equation (2.21) there corresponds a set of equations for ψ_α. Let us clarify this by an example. If we choose u^l_A and v^a_A in the form

$$u^l_A = \partial_A x_k H^{kl}$$

$$v^a_A = g_{\mu\nu} N^\mu_b \phi^{\nu,\beta}_\alpha \psi_\beta 0_A \psi^\alpha h^{ba} \tag{2.23}$$

then it is not difficult to see that the last term in (2.17) identically vanishes, hence

$$P^A v^a_A \equiv \partial^A (g_{\mu\nu} N^\mu_b \phi^{\nu,\beta}_\alpha \psi_\beta 0_A \psi^\alpha h^{ba}) + \\ + F^{k,a}_b (g_{\mu\nu} N^\mu_c \phi^{\nu,\beta}_\alpha \psi_\beta 0_A \psi^\alpha h^{cb}) \partial^A x_k = 0 . \tag{2.24}$$

Now if we write in detail (2.24) we remark that all ψ_α satisfying equations

$$0_A \partial^A \psi_\alpha + 0_A L^{k,\beta}_\alpha \psi_\beta \partial^A x_k = A_\alpha \tag{2.25}$$

satisfy (2.24) too (Here A_α is an arbitrary quantity of the type of ψ_α). In fact substituting (2.25) in (2.24) we conclude that A_α is not definite unambiguously).

We shall illustrate this situation once more on the example considered in the Application.

Thus to a given current there corresponds only one equation for $x_l(s_A)$ and one set of equations for ψ_α fixed by (2.19) which can be considered as a "generating equation".

At the end of this paper we will draw our attention to the following fact. In general, the field theory constructed by the fields nonlinearly transforming under the group G is the theory of interaction between x_i and ψ_a. However if we omitted ψ_a we would obtain a theory only for the x_i (taking $u_i^A = \partial^A x_i$; $v_a^A = 0$), but a resulting theory would never be noninteracting by the reason of the remaining self-interaction. For the ψ_a fields the situation is very different from the above mentioned one because if the fields x_i were absent then it would not be possible to construct any equation for the fields ψ_a themselves. This particularity singles out the fields x_i as fundamental ones, responsible for the whole theory.

The disadvantage of this theory consisits in that at present we are able to choose as fundamental fields only scalar Lorentz fields. Therefore it would be interesting to investigate the problem of introduction of other Lorentz fields as fundamental ones.

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Application

Let us illustrate the general theory on the example of the group $SU(2) \times SU(2)$ with generators T_k and X_k commuting as usual

$$\begin{aligned}
[T_k \, T_\ell] &= i\,\epsilon_{k\ell n}\, T_n \\
[T_k \, X_\ell] &= i\,\epsilon_{k\ell n}\, X_n \\
[X_k \, X_\ell] &= i\,\epsilon_{k\ell n}\, T_n \, .
\end{aligned} \tag{A.1}$$

Consider a three dimensional non-linear representation of this group in form[9]

$$
M^{\mu}_{k} \equiv \begin{cases} T^{i}_{k} = -i\epsilon_{ik\ell}\, z_{\ell} & \mu \equiv i \leq 3 \\ X^{i}_{k} = i\left(\frac{1}{2}(1-z^{2})\delta_{ik} + z_{i} z_{k}\right); & \mu \equiv i > 3 \end{cases} \quad . \tag{A.2}
$$

(Here indices $i, k, \ell ..$ take values 1,2,3, $z_i = g\,\pi_i$, where π_i are pion functions and $g = \frac{f_0}{m_\pi}$ have dimensionality of length).

The metric tensor $g^{\mu\nu}$ equals

$$
g^{\mu\nu} = C^{\mu\rho}_{\sigma}\, C^{\nu\sigma}_{\rho} = 4\,\delta_{\mu\nu}
$$

Then it is not difficult to check that the quantities N^{μ}_{ρ} can be chosen in the form

$$
N^{\mu}_{\ell} \equiv \begin{cases} X^{i}_{\ell} = i\left(\frac{1}{2}(1-z^{2})\,\delta_{i\ell} + z_{i} z_{\ell}\right) & \mu = i \leq 3 \\ T^{i} = -i\epsilon_{i\ell n}\, z_{n} & \mu = i+3 \end{cases} \quad . \tag{A.3}
$$

The transformation law of ψ_a has the form

$$
\begin{aligned}
[T_i \psi] &= -\frac{1}{2}\sigma_i \psi \\
[X_i \psi] &= -\frac{1}{2}(z \times \sigma)_i\, \psi \quad .
\end{aligned} \tag{A.4}
$$

By means of direct calculations one obtains also all the remaining quantities necessary for building up both the invariant equations and conservation laws

$$H_{k\ell}=h_{k\ell} = -(1+z^2)^2\delta_{k\ell} \; ; \quad H^{k\ell}=h^{k\ell} = -\frac{\delta_{k\ell}}{(1+z^2)^2}$$

$$F_j^{\ell,m} = \Lambda_j^{\ell m} = \frac{2}{1+z^2}(z_m\delta_{j\ell} + z_\ell\delta_{mj} - z_j\delta_{m\ell})$$

$$\Delta_\ell^{m,n} = -\frac{2}{1+z^2}\epsilon_{m\ell n} \tag{A.5}$$

$$f_n^{n,m} = \begin{cases} i\,\epsilon_{anm} & \mu = a \leq 3 \\ i(z_m\delta_{na} - z_n\delta_{ma} - z_a\delta_{mn}); & \mu = a > 3 \end{cases} ,$$

From these expressions it follows that quantities u_A^i and v_A^a transform uniformly. However P-invariance of the theory demands that v_A^m be a vector and u_A^m axial vector.

The simplest choice of u_A^i and v_A^i is the following

$$u_A^i = \frac{\bar{\psi}\gamma_A\gamma_5\sigma_i\psi}{1+z^2} - \frac{\partial_A z_i}{g^2(1+z^2)^2}$$

$$v_A^i = -\frac{\bar{\psi}\gamma_A\sigma_i\psi}{1+z^2} \; . \tag{A.6}$$

If now we insert these expressions in the current

$$J^{\mu,A} = -i\,(M_i^\mu u^{i,A} + N_a^\mu v^{a,A})$$

(here i is taken to assure hermicity of the current) we shall obtain

$$V_A = -\frac{(\partial_A \vec{z} \times \vec{z})}{g^2(1+z^2)^2} + \frac{1}{1+z^2}\bar{\psi}\gamma_A \{ (\vec{\sigma}\times\vec{z})\gamma_5 - \frac{1}{2}(1+z^2)\vec{\sigma} -$$

$$-\vec{z}(\vec{z}\vec{\sigma}) \} \psi$$

$$A_A = \frac{1-z^2}{2g^2(1+z^2)^2}\partial_A\vec{z} + \frac{(\vec{z}\partial_A\vec{z})}{g^2(1+z^2)^2}\vec{z} +$$

$$+\frac{1}{1+z^2}\bar{\psi}\gamma_A \{ (\vec{\sigma}\times\vec{z}) - \frac{1}{2}(1-z^2)\gamma_5\vec{\sigma} - \vec{z}(\vec{\sigma}\vec{z})\gamma_5 \} \psi . \tag{A.7}$$

After some reconstruction equations of motion are

$$\frac{\Box\vec{z}}{g^2(1+z^2)} - \frac{4(\vec{z}\partial^A\vec{z})}{g^2(1+z^2)^2}\partial_A\vec{z} + \frac{2(\partial_A\vec{z}\partial^A\vec{z})}{g^2(1+z^2)^2}\vec{z} - 2im\bar{\psi}\vec{\sigma}\gamma_5\psi = 0$$

$$\partial^A(\bar{\psi}\gamma_5\vec{\sigma}\psi) = \frac{2}{1+z^2}\bar{\psi}\gamma_A\{\gamma_5(\partial^A\vec{z}\times\vec{\sigma}) - (\vec{z}\vec{\sigma})\partial^A\vec{z} + (\vec{\sigma}\partial^A\vec{z})\vec{z}\}\psi . \tag{A.8}$$

As we have remarked the second equation defines a set of equations for the fields ψ . As an example of that sort of equations one can take the following

$$i\gamma_A\partial^A\phi\psi - m\psi + \frac{\gamma_A}{1+z^2}[(\vec{\sigma}\times\vec{z})\partial^A\vec{z} - i\gamma_5(\vec{\sigma}\partial^A\vec{z})]\psi =$$

$$= (\vec{b}^A\vec{\sigma})\gamma_A\psi + ib\psi ,$$

here b_1^A and b are arbitrary vector and scalar fields, correspondigly.

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