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ON THE UNITARITY OF THE S-MATRIX IN THE METHOD OF THE QUASIPOTENTIAL

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In the basic works performed by A.A.Logunov and A.N.Tavkhelidze[1] the notion of the quasipotential was introduced to consider the bound states and the scattering of two relativistic particles. It is important that quantum field theory does not contradict such considerations and moreover by means of this theory it is possible to receive the series expansions of the quasipotential in a small parameter. Of course, such expansions are useful only for small coupling constants, for instance, in the case of electrodynamics[2].

But in the case of strong interactions it is difficult to find a direct connection between the method of quasipotential and quantum field theory. In this case we write the quasipotential equations for the system of two particles starting from various physical assumptions. For instance, in recent works[3] it was shown that the quasipotential method is valid for the description of high energy elastic scattering at large and small angles. In this case a purely imaginary smooth quasipotential of the Gaussian type was used. When the incident particle energy becomes high enough there exist also various inelastic processes and the quasipotential has an imaginary part. This work is devoted to the problem of the sign of this imaginary part.

Let us consider, at first, the generally accepted formulation of the quasipotential equations. So, we have two identical spinless particles. We use here the center-of-mass system $\vec{p}_1 + \vec{p}_2 = 0$, so we can put $\vec{p}_1 = \vec{p}$; $\vec{p}_2 = -\vec{p}$. The energy of free particle will be $\frac{E(p)}{2} = \sqrt{m^2 + p^2}$. Using these notations we write the quasipotential equation in the form

$$\{ p^2 + m^2 - (\frac{E}{2})^2 \} \phi(p) - \int V(p,q)\phi(q) \frac{dq}{\sqrt{q^2+m^2}} = 0$$

or, putting $\phi(p) = \sqrt{p^2+m^2}\,\psi(p)$ we can write also

$$\sqrt{p^2+m^2}\,\{ p^2 + m^2 - (\frac{E}{2})^2 \}\ \psi(p) = \int V(p,q)\psi(q)\,dq . \qquad (1)$$

In the case of the local potential which is often considered, the quasipotential is chosen in the form

$$V = V(p,q,E) = V(|p-q|^2, E)$$

then its Fourier-transform U depends only on the distance between the particles r and the quasipotential equation becomes similar to the Schroedinger equation and differs from it only by the kinematic factor $\sqrt{p^2+m^2}$

$$\sqrt{p^2+m^2}\,\{ p^2 + m^2 - (\frac{E}{2})^2 \}\Phi(\vec{r}) - U(\vec{r},E)\,\Phi(\vec{r}) = 0,$$

where Φ is the Fourier-transform of the wave function ψ and $p^2 = -\Delta_r$.

To find the solution which corresponds to the elastic scattering of the two particles considered, let us put in the equation (1)

$$\psi(p) = \delta(p-p') + T(p,p')\frac{1}{G\{E+i\epsilon\}}$$

$$E = E(p'), \quad \epsilon \underset{\to}{>} 0,$$

where

$$G(E) = \sqrt{p^2+m^2}\,\{ (\frac{E(p)}{2})^2 - (\frac{E(p')}{2})^2 \}.$$

Then one can obtain the scattering equation from equation (1):

$$T = V + V \frac{1}{G - i\epsilon} T. \qquad (2)$$

In this equation E is considered as a given positive parameter, $E \geq 2m$.

Starting from this equation there was established an important theorem that if V is the Hermitian operator, then the scattering matrix S , defined by the amplitude T , should be unitary $S^+ S = 1$. But when the incident particles energy E becomes high enough, in the real system, in addition to elastic scattering there exist some inelastic processes.

So, if the S -matrix describes only elastic scattering one should have the underunitarity condition for the S -matrix: $S^+ S \leq 1$ i.e. the operator $(1 - S^+ S)$ should be positive. On the other hand, the S -matrix, defined by the quasipotential equation can describe only elastic processes since it includes, in addition to incident particles term $\delta(p - p')$ the terms, proportional to $\delta(E(p) - E(p'))$ with the same energy as that of incident particles. Therefore to take into account the real processes one has to insert into the quasipotential an imaginary part.

Now let us prove the theorem that for the model equation of the type (1), one should receive an underunitary elastic scattering matrix, i.e. $(1 - S^+ S) \geq 0$, if the Hermitian operator $A = \frac{1}{i}(V - V^+)$ is positive, $A \geq 0$ x/.

x/ Here and in what follows we say that a certain operator A is positive if
$$\int \phi(p) A(p, p') \phi^*(p') \, dp \, dp' \geq 0,$$
where ϕ is an arbitrary function. In the local case, mentioned above, the positivity conditions can be written simply $\mathrm{Im} V \geq 0$.

Thus, one can obtain at once, from (2)

$$\left(1 - V\frac{1}{G - i\epsilon}\right)T = V$$

$$(G - i\epsilon - V)(G - i\epsilon)^{-1}\, T = V \tag{3}$$

$$(G - i\epsilon)^{-1}\, T = (G - i\epsilon - V)^{-1}\, V\,.$$

Substituting (3) into (2)

$$T = V + V(G - i\epsilon - V)^{-1}V\,. \tag{4}$$

From (3) and (4) we get

$$T^{+}(G + i\epsilon)^{-1} = V^{+}(G + i\epsilon - V^{+})^{-1} \tag{5}$$

and

$$T^{+} = V^{+} + V^{+}(G + i\epsilon - V^{+})^{-1}V^{+}\,. \tag{6}$$

So, we have for

$$T - T^{+} = V - V^{+} + D,$$

where

$$D = V(G - i\epsilon - V)^{-1}V - V^{+}(G + i\epsilon - V^{+})^{-1}V^{+}. \tag{7}$$

To calculate D use the identity

$$(X + x)(Y + y)(Z + z) - XYZ =$$

$$= x(Y + y)(Z + z) + Xy(Z + z) + XYz \tag{8}$$

and put in it

$$X = V^+ \ , \quad Y = (G + i\epsilon - V^+)^{-1} \ , \quad Z = V^+$$

$$X + x = V \ , \quad Y + y = (G - i\epsilon - V)^{-1} \ , Z + z = V \ .$$

Consequently

$$x = V - V^+ = iA = z \ , \tag{9}$$

where A is the Hermitian operator. We remind that we put on the operator $A(E) = \frac{1}{i}(V(E) - V^+(E))$ the positivity condition

$$A(E) \geq 0 \tag{10}$$

if

$$E \geq 2m \ . \tag{11}$$

From (7) and (8) we receive

$$y = (G - i\epsilon - V)^{-1} - (G + i\epsilon - V^+)^{-1} =$$

$$= (G + i\epsilon - V^+)^{-1} (G + i\epsilon - V^+)(G - i\epsilon - V)^{-1} -$$

$$-(G + i\epsilon - V^+)^{-1} (G - i\epsilon - V)(G - i\epsilon - V)^{-1} =$$

$$= 2i\epsilon (G + i\epsilon - V^+)^{-1} (G - i\epsilon - V)^{-1} + i(G + i\epsilon - V^+)^{-1} A (G - i\epsilon - V)^{-1}$$

Therefore

$$T - T^+ = V - V^+ + iA(G - i\epsilon - V)^{-1} V \ +$$

$$+ V^+ (G + i\epsilon - V^+)^{-1} (2i\epsilon + iA)(G - i\epsilon - V)^{-1} + V^+ (G + i\epsilon - V^+)^{-1} iA$$

or, taking into account (3) and (5)

$$T - T^+ = iA + iA(G - i\epsilon)^{-1} T + T^+ (G + i\epsilon)^{-1} iA +$$

$$+T^{+}(G+i\epsilon)^{-1}2i\epsilon\,(G-i\epsilon)^{-1}T+T^{+}\,(G+i\epsilon)^{-1}\,iA(G-i\epsilon)^{-1}\,T=$$

$$=[\,1+T^{+}\,(G+i\epsilon)^{-1}\,]\,iA\,[\,1+(G-i\epsilon)^{-1}\,T]+2i\pi T^{+}\delta\,(G)T\,.$$

Let us designate

$$\Omega=1+(G-i\epsilon)^{-1}\,T\,. \tag{12}$$

Then

$$(iT)+(iT)^{+}+2\pi(iT)^{+}\delta(G)(iT)+\Omega^{+}A\,\Omega=0. \tag{13}$$

Introducing the operator $\mathcal{E}$ so that for arbitrary function ϕ

$$\mathcal{E}\phi\,(p)=E(p)\,\phi(p)$$

one can see immediately

$$\delta\{G(E)\}=\frac{4}{E^{2}}\,\delta(\mathcal{E}-E)$$

and write (13) in the form

$$(iT)+(iT)^{+}+\frac{8\pi}{E^{2}}(iT)^{+}\delta(\mathcal{E}-E)(iT)+\Omega^{+}A\Omega=0.$$

Note, that the operator $\Omega^{+}A\,\Omega\geq 0$ for $E\geq 2m$ since $A(E)\geq 0$. Rewrite it in the matrix p -representation

$$(iT)_{p,p',E}+(iT)^{+}_{p,p',E}+$$

$$+\frac{8\pi}{E^{2}}\int(iT)_{p,p'',E}\,\delta(E-E(p''))(iT)_{p'',p',E}\,dp''+ \tag{14}$$

$$+(\Omega^{+}A\Omega)_{p,p',E}=0.$$

Put here $E = E(p')$ and multiply (14) by $\delta(E(p) - E(p'))$. Then, having designated

$$\frac{8\pi}{E^2}(iT)_{p,p',E(p)} \; \delta(E(p) - E(p')) = r(p,p')$$

we receive

$$r(p,p') + r^+(p,p') + \int r^+(p,p'')\, r(p'',p')\, dp'' +$$

$$+ (\Omega^+ A\Omega)_{p,p',E}\; \delta(E(p) - E(p')) = 0. \qquad (15)$$

Here we have to show that the operator, represented by the matrix

$$(\Omega^+ A\Omega)_{p,p',E} \;\; \delta(E(p) - E(p')) \qquad (16)$$

is positive, i.e. for arbitrary function $h(p)$

$$\iint (\Omega^+ A\Omega)_{p,p',E(p)}\; \delta(E(p) - E(p'))\, h^*(p) h(p')\, dp\, dp' \geq 0.$$

Really, since the operator $(\Omega^+ A\Omega)_{p,p',E(p)} \geq 0$, then

$$\iint (\Omega^+ A\Omega)_{p,p',E(p)}\; g^*(p) g(p')\, dp\, dp' \geq 0 \quad \text{for} \quad E \geq 2m.$$

Put here $g(p) = \delta(E(p) - E) h(p)$ and note that

$$\delta(E(p) - E)\; \delta(E(p') - E) =$$

$$= \delta(E(p) - E)\; \delta(E(p') - E(p)).$$

Therefore

$$\iint [(\Omega^+ A\Omega)_{p,p',E(p)}\; \delta(E(p') - E(p))]\, h^*(p')\, h(p)\, \delta(E(p) - E)\, dp\, dp' \geq 0. \qquad (17)$$

Strictly speaking, this inequality is fulfilled only for $E \geq 2m$. But for $E < 2m$ $\delta(E(p)-E) = 0$ and (17) is fulfilled identically. Therefore inequality (17) is true for all real E. Then having integrated it over E, we receive

$$\iint (\Omega^{+} A \Omega)_{p,p',E(p')}\, \delta(E(p)-E(p'))\, h^{*}(p')\, h(p)\, dp\, dp' \geq 0$$

that means the positivity of the operator represented by the matrix (16). Then from (15) we receive

$$r + r^{+} + r^{+} r \leq 0 \qquad \text{i.e.} \qquad S^{+} S \leq 1, \tag{18}$$

where

$$S = 1 + r$$

or, in the matrix representation

$$S(p,p') = \delta(p-p') + \frac{8\pi}{E^2}(iT)_{p,p',E(p)}\, \delta(E(p)-E(p')).$$

Thus, if we have the condition (10) then the elastic S-matrix should be underunitary.

Of course, if $A(E) \equiv 0$, i.e. V is the Hermitian operator, $r + r^{+} + r^{+} r = 0$ and $S^{+} S = 1$.

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