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POSSIBLE QUANTIZATIONS

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POSSIBLE QUANTIZATIONS

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1. The progress recently made in the para-quark model of hadrons[1] seems to indicate that quarks are para-Fermions of order 3 as Greenberg[2] has proposed some years ago. The equations of motion however are unknown and special ansatzes must be made for them. Another problem arises in hadronic physics because of the non-uniqueness of the quantization of wave equations for particles of higher spin, which may be connected with the above-mentioned difficulty. As Takahashi[3] emphasizes the well known relation between the generators of space-time translations P_μ and the field operator $\phi(x)$,

$$-i\partial_\mu \phi(x) = [\phi(x), P_\mu] \tag{1}$$

leads to different quantizations: a special choice of P_μ requires special commutation relations for $\phi(x)$ in order to make (1) consistent with the equations of motion

$$\Lambda(\partial)\, \phi(x) = 0. \tag{2}$$

For physically relevant theories P_μ must be a 4-vector with P_i and $H = -iP_4$ hermitian and furthermore there must exist a unique vacuum state $|0\rangle$ such that $H \geq E_0$ (zeropoint energy). A more appropriate form is arrived by the introduction of creation and annihilation operators for the field quanta,

$$\phi(x) = \sum_r \int d^3k \, \{ a^r(k) u^r_k(x) + b^r(k)^+ v^r_k(x) \}, \tag{3}$$

where $u^r_k(x)$ resp. $v^r_k(x)$ is a positive resp. negative frequency solution of (2) by c-number functions. With

$$P_\mu = \sum_r \int d^3k \, k_\mu \, N^r(k), \tag{4}$$

where $N^r(k)$ is the number operator of the state (k,r), it follows (only one index is written out)

$$[N_i , a^\pm_j] = \pm a^\pm_i \delta_{ij} \tag{5}$$

and an analogous equation for the b's. Further it must hold

$$[N_i , N_j] = 0 \tag{6}$$

$$N_i = N^+_i \tag{7}$$

$$N_i |0> = a^-_i |0> = 0. \tag{8}$$

The problem now consists in taking a definite form for the N_i in terms of $a^\pm_i$ and then to search for the commutation relations of the $a^\pm_i$ which are consistent with (5) - (8).

2. Such a consistency problem was treated by Wigner/4/, O-Raifeartaigh et al./5-7/ for a simple harmonic oscillator. A more detailed analysis shows that for

a) N linear in a and a^+ there is only the trivial solution $a \equiv 0$,

b) N bilinear there are two infinite families

$$N = s\,g(k)\,(a\,a^+ - k a^+ a) - E_0; \quad s = \pm 1, \; -\infty \le k \le +\infty \tag{9}$$

with

$$g(k) = \begin{cases} 1/(1-k) & k \neq 1 \\ 1/2 & k = 1 \end{cases} \quad . \tag{10}$$

The spectrum of N in this case is simple which is equivalent to [5]

$$a a^+ | n> = c_n | n>$$

with

$$| n> = N(n)(a^+)^n | 0>.$$

A representation of the a's is then given by

$$<n+1 | a^+ | n> = \sqrt{c_n}$$

and c_n follows from (9) as

$$c_n = \begin{cases} s(1-k^{n+1})(E_0 + f_n(k)) & k \neq 1, \infty \\ s(n+1)(2E_0 + n) & k = 1 \end{cases} \tag{11}$$

with

$$f_n(k) = \frac{n+1}{1-k^{n+1}} - \frac{1}{1-k} \quad . \tag{12}$$

From $c_n \geq 0$ there follow restrictions on E_0 which should not be reproduced here. It can be shown that all these different representations for a, which lead to different commutation relations, are physically equivalent to the case $k = \infty$, $s = 1$ which corresponds to the well known Bose- and the Okayama-commutation relations [7-9], but such a reduction works only for one oscillator and not for a field.

The case c) N trilinear in a and a^+ reduces to b) essentially, and for N quadrilinear there is in general no simple spectrum of N . Therefore we shall firstly try to generalize b) to a countable set of operators $a_i^{\pm}$ and N_i .

3. First of all we define a symbol (,) by

$$(A, B) = sg(k)(AB - kBA) \tag{13}$$

and remark the identities

$$[(AB)\ C] + [(BC)\ A] + [(CA)B] = 0 \tag{14}$$

$$[(AB)\ C] = (A[BC]) - ([CA]B). \tag{15}$$

Next we define the operators

$$N_{ij} = (a_j, a_i^+) \qquad N_{ij}' = (a_j^+, a_i) \tag{16a}$$

$$L_{ij} = (a_j^+, a_i^+) \qquad M_{ij} = (a_j, a_i) \tag{16b}$$

with

$$N_{ij} = N_{ji}^+ \quad , N_{ij}' = N_{ji}'^+ \, , \qquad L_{ij}^+ = M_{ji} \, . \tag{17}$$

Then we generalize eq. (5) to

$$[N_{ij}, a_m^+] = a_i^+ r_{ij}\, \delta_{jm} \tag{18a}$$

or

$$[N_{ij}, a_m^+] = a_j^+ r_{ij}\, \delta_{im} \tag{18b}$$

with

$$r_{ij} = 1 + (1 - \delta_{ij})(r(k) - 1) = r_{ji} \, , \tag{19}$$

With the help of eq. (15) we conclude, by treating the commutator $[N_{ij}, N_{mn}]$, that only $r(k) = 0$ or 1 for (18a) and $r(k) = 0$ for (18b) are possible, so we have

$$[N_{ij}, a_m^+] = a_i^+ \delta_{jm} \qquad (r = 1) \tag{20}$$

or

$$[N_{ij}, a_m^+] = a_i^+ \delta_{im} \, , \quad N_{ij} = N_i\, \delta_{ij} \qquad (r = 0) \, . \tag{20'}$$

This result follows also from the requirement of invariance of the commutation relations under infinitesimal unitary transformations/10/

$$a_i \to \tilde{\tilde{a}}_i = (\delta_{ij} + \alpha_{ji}) a_j \quad \text{with} \quad \alpha_{ij} + \alpha^*_{ji} = 0.$$

If we require invariance for every α_{ij} then eq. (20) follows and we qualify such theories with (20) as <u>admissible</u>. Theories with (20') are not admissible in this sense because they follow for $\alpha_{ij} = \alpha_i \delta_{ij}$ only.

4. Now we shall study the admissible theories. From (20) we arrive at

$$[N_{ij}, N_{mn}] = N_{in}\,\delta_{jm} - N_{mj}\,\delta_{in} \tag{21}$$

for every form of $N_{ij}(k)$, especially we have (6). Eq. (21) shows that any representation of the $a^\pm_i$ must induce a representation of the Lie algebra of the unitary group in ℓ dimensions $U(\ell)$, where ℓ is the number of different modes i. Now we can ask in what cases this Lie algebra closes in any way to the algebra of one of the subgroups of $U(\ell)$, the groups $0(2\ell+1)$, $0(2\ell)$ of $Sp(2\ell)$.

With (14) we see that

$$[L_{ij}, a_m] = -a^+_j\,\delta_{im} - [N'_{mi}, a^+_j] \tag{22}$$

and therefore conclude

$$[N'_{mi}, a^+_j] = -s'(k)\,a^+_i\,\delta_{jm} \tag{23}$$

must hold. By inspection of the commutation relations between the operators (16a) and (16b) the following requirement must be fulfilled.

$$L_{ij} + s'(k)\,L_{ji} = 0 \tag{24}$$

$$N'_{ij} + s'(k)\,N_{ji} = t(k)\,\delta_{ij} \tag{25}$$

with $s' = 1$ resp. -1 for the orthogonal resp. symplectic groups.

From (24) and (25) it follows that either $s' = k$, which means para-commutation relations /11/ or

$$[a_i, a_j]_{s'} = [a_i^+, a_j^+]_{s'} = 0 \qquad (26)$$

$$[a_i, a_j^+]_{s'} = -s \frac{1-k}{s' + k} \delta_{ij} . \qquad (27)$$

Relation (27) may be written in the usual form

$$[a_i, a_j^+]_{s'} = \delta_{ij} \qquad (27')$$

after an appropriate transformation of the a's if $s = 1$ and $|k| \nless 1$ for $s' = -1$ (Bose commutation relations) and if $s = 1$ and $|k| > 1$ or $s = -1$ and $|k| < 1$ for $s' = 1$ (Fermi commutation relations).

The requirement of an unique vacuum state $|0>$ with (8) leads with the help of eq. (21) to

$$N_{ij} | 0 > = s\, g(k)\, a_j a_i^+ | 0 > = c_{ij} | 0 > \qquad (28)$$

and from eq. (21) it follows

$$c_{ij} = E_0 \delta_{ij} = s g^{-1} c_0 \delta_{ij} . \qquad (29)$$

From eq. (25) it follows

$E_0 = -1/(1+k)$ for Fermi c.r.

$E_0 = 1/(1-k)$ for Bose c.r.

whereas in the case of para-Fermi (Bose) c. r.

$$E_0 = (\mp)\, p/2 \qquad (p = 1, 2, ..)$$

follows /11/.

In the other cases where only the unitary group is realized by the $a_i^\pm$ the requirement of an unique vacuum state and the positive definiteness of the representing Hilbert space leads to complicated polynomials in E_0 with k-dependend coefficients which in general may not be factorized. For $k = 0$ a very laborous computation secures the positive definiteness, whereas the study of a few special types of Hilbert space vectors seems to indicate that all the cases with $k \neq 0, \pm 1$ must be excluded.

5. The study of the non-admissible theories with (20') goes the same line of reasoning. One attains also

$$N_i' = s' N_i + t \qquad (30)$$

and either

$$s' = 1, \quad [L_i, a_j^+] = 0 \tag{31}$$

or

$$L_i = M_i = 0 . \tag{31'}$$

For (31) we have either Bose c. r. or $k = -1$ with

$$[N_i, a_i^{\pm}] = \pm a_i^{\pm} \quad , \quad [L_i, a_i] = -2 a_i \tag{32}$$

and for $i \neq j$ all $a_i^{\pm}$ and $a_j^{\pm}$ anticommute. By a Klein transformation[12] it is possible to reverse these anticommutation relations in normal commutation relations and one sees that (31) is equivalent to quantize every mode i separately by para-Bose c.r. For (31') in strikt analogy we have

$$[N_i, a_i^{\pm}] = \pm a_i^{\pm}, \quad N_i = [a_i^-, a_i^+] \tag{32'}$$

and for $i \neq j$ all $a_i^{\pm}$ commute with $a_j^{\pm}$. So we see that (31') is equivalent to quantize every mode i separately by para-Fermi c.r. The whole Hilbert space in both cases is the direct product of Hilbert spaces of the a_i, therefore the reduction mentioned in 2. can be applied and hence the theories with (31) and (31') are equivalent to the superstatistic theory of Roman and Aghassi[13,14] and also to the theory of Parks[15,16]. As Parks has shown this type of theories may be of interest in the many body problems, for instance in the BCS-theory of superconductivity.

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