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E2 - 4397

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TIME-DEPENDENT GINZBURG-LANDAU EQUATION FOR SUPERCONDUCTING ALLOYS IN HIGH MAGNETIC FIELDS

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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Submitted to physica status solidi

x/ On leave of absence from Sektion Physik, Karl-Marx-Universität, Leipzig, DDR.

1. Introduction

Recently, Gorkov and Eliashberg [1] developed a general formalism for the calculation of the response of a superconductor to external fields in arbitrary order.[x] One part of this response (the regular part) may be expanded in powers of the order parameter Δ neglecting higher-order terms. The other (nonregular) part depends on the ratio of the order parameter to its frequency and momentum, and it is necessary to sum up some classes of diagrams. Gorkov and Eliashberg considered the derivation of the time-dependent Ginzburg-Landau equation for a superconductor containing paramagnetic impurities in a concentration near the critical one. The situation is more complicated in other cases[xx].

In the present paper we consider a superconductor containing nonmagnetic impurities in the limit of small mean free path ($\ell \ll \xi_0$) and high magnetic field ($H \approx H_{c2}$) ; the temperature can be arbitrary ($0 < T < T_c$) . Using the method of Gorkov and Eliashberg we derive the time-dependent Ginzburg-Landau equation and the differential

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[x] See also [2].

[xx] Compare also the paper by Eliashberg [3], where superconductors with a small concentration of paramagnetic impurities are treated.

equation for the vertex part from the microscopic theory. The vertex part Γ_1 plays the role of a generalized scalar potential. The reduction of the order parameter by the high magnetic field essentially simplifies the situation, because it allows to restrict the calculation of some diagram blocks for the nonregular part of the response to the lowest order in Δ. For small fields H (and $T \approx T_c$) such a restriction seems not to be possible.

The resulting Ginzburg-Landau equation is compared with the equation derived earlier by Schmid[4] (for $T \approx T_c$) and Caroli and Maki[5] without considering the nonregular part of the response. Our equation takes the simpler form of this earlier equation in the cases, in which the vertex part reduces to the scalar electromagnetic potential ($\Gamma_1 = (ie/r)\,\phi$). In section 3 we show for the case of slow motion of the lattice of the vortex lines (flux-flow) caused by a static electric field, that this simplification is compatible with the differential equation for the vertex part.

2. Equations for Δ^+ and the Vertex Part

The time-dependent Ginzburg-Landau equation is obtained from the condition of self-consistency

$$\Delta^+(1) = |g| < \tilde{\tilde{\Psi}}^+_\downarrow(1)\, \tilde{\tilde{\Psi}}^+_\uparrow(1) >, \tag{1}$$

where $\tilde{\tilde{\Psi}}$ is the electron field operator in the Heisenberg picture (t_1 real).[x]

To the superconductor are applied the external fields ϕ, $\vec{A}$ (electromagnetic potentials), Δ (superconducting order parameter treated as an external field) and Δ^+. The correlation function on the right-hand side of (1) can be expanded in powers of the exter-

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[x] We use the notation of[2]. Because we are considering here only real times, the + has the usual meaning of complex (or Hermitean) conjugate.

nal fields (compare [1] and [2]). Then equation (1) takes the form (we go over to the frequency representation)

$$\Delta^{+}_{\omega} = \frac{1}{2}|g| \int \frac{d\omega_1}{2\pi} \{ - \text{th} \frac{\beta\omega_1}{2} F^{+A}_{\omega_1,\omega_1-\omega} + \text{th} \frac{\beta(\omega_1-\omega)}{2} F^{+R}_{\omega_1,\omega_1-\omega} \} +$$

$$+ \frac{1}{2}|g| \int \frac{d\omega_1}{2\pi} \frac{d\omega_2}{2\pi} \frac{d\omega'}{2\pi} [\text{th} \frac{\beta\omega_2}{2} - \text{th} \frac{\beta(\omega_2-\omega')}{2}] \times \tag{2}$$

$$\times \{ i G^{(+)R}_{\omega_1\omega_2} \Delta^{+}_{\omega'} G^{(-)A}_{\omega_2-\omega',\omega_1-\omega} - i F^{+R}_{\omega_1\omega_2} \Delta_{\omega'} F^{+A}_{\omega_2-\omega',\omega_1-\omega} -$$

$$- G^{(+)R}_{\omega_1\omega_2} \Phi^{+}_{\omega'} F^{+A}_{\omega_2-\omega',\omega_1-\omega} + F^{+R}_{\omega_1\omega_2} \Phi_{\omega'} G^{(-)A}_{\omega_2-\omega',\omega_1-\omega} \},$$

where $\Phi = e\phi + (ie/m)\vec{A}\nabla$. For the retarded and advanced Green functions in (2) we insert their power series expansions in the external fields.

The unperturbed Green functions contain the effects of the potential u of the nonmagnetic impurities. We treat this impurity scattering in the usual way (compare [6]). All diagrams have to be averaged over the positions of the impurities. The impurity scattering is considered to be isotropic characterized by the electron collision time τ:

$$\frac{1}{\tau} = \frac{1}{\pi} n m p_F |u(p_F)|^2 ; \tag{3}$$

n is the density of the impurities. In the "dirty" limit $(\ell \ll \xi_0)$ holds $\tau T_0 \ll 1$. The unperturbed Green functions are given by

$$G_0^{(\pm)R}(\omega) = \frac{1}{\omega \pm \xi + \frac{i}{2\tau}} \tag{4}$$

($\xi = v_F(p - p_F)$) and the complex conjugate expression for $G_0^{(\pm)A}$.

The regular part of the response is given by the first integral in (2) containing only advanced (or only retarded) Green functions. The integral over ω_1 is evaluated with the help of the residuum theorem. This part of the response can be expanded in powers of the order parameter without difficulties.

In our case of high magnetic field we may restrict the calculation to the first order in Δ^+, that means to the diagram of Fig.1.

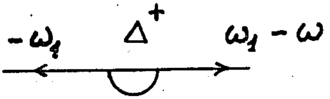

Fig.1. Diagram of first order in Δ^+ for the regular part of the response. $\leftarrow$ means $G_0^{(+)R,A}$, $\rightarrow$ means $G_0^{(-)R,A}$ and $\cup$ indicates the corrected vertex.

(If it is necessary to consider the normalization of Δ, the term of the third order in Δ has to be added of course). For the corrected vertex in Fig.1 holds the equation illustrated in Fig.2, and

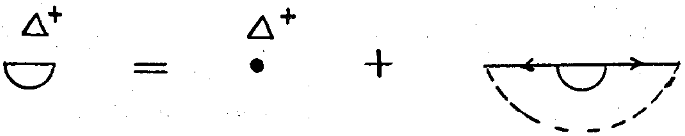

Fig.2. Equation for the correction of the Δ^+-vertex. $\bullet$ indicates the uncorrected vertex and - - - corresponds to the impurity potential u. The Green functions $\leftarrow$ and $\rightarrow$ include the corrections due to the potential Φ.

the corrected Δ^+-vertex is given by

$$(\eta^{R,A}\Delta^+) = \left(\frac{1 \mp 2i\tau\omega_1}{\mp i\tau(2\omega_1 - \omega) + \tau D\vec{Q}^{+2}}\Delta^+\right), \tag{5}$$

where $D = \tau v_F^2/3$ is the diffusion coefficient and $\vec{Q} = -i\nabla - 2e\vec{A}$. In the case of a Δ-vertex $\vec{Q}^+$ has to be substituted by $\vec{Q}$. The contributions due to the scalar potential cancel out. The calculation gives for the diagram of Fig.1 the expression (compare also [5])

$$\frac{|g| m p_F}{2\pi^2}\left\{ \ln\frac{2\omega_D\gamma}{\pi T} + \Psi\left(\frac{1}{2}\right) - \Psi\left(\frac{1}{2} + \frac{1}{4\pi T}\frac{\partial}{\partial t} + \frac{1}{4\pi T} D\vec{Q}^{+2}\right)\right\}\Delta^+, \qquad (6)$$

where ω_D is the Debye frequency , γ is Euler's constant, and $\Psi = \Gamma'/\Gamma$ is the di-gamma function. Using the expression for the critical temperature,

$$\frac{|g| m p_F}{2\pi^2}\ln\frac{2\omega_D\gamma}{\pi T_c} = 1, \qquad (7)$$

we get for (6)

$$\Delta^+ + \frac{|g| m p_F}{2\pi^2}\left\{ \ln\frac{T_c}{T} + \Psi\left(\frac{1}{2}\right) - \right.$$
$$\left. - \Psi\left(\frac{1}{2} + \frac{1}{4\pi T}\frac{\partial}{\partial t} + \frac{1}{4\pi T} D\vec{Q}^{+2}\right)\right\}\Delta^+ . \qquad (8)$$

The nonregular part of the response, given by the second integral in (2), contains advanced and retarded Green functions together. The impurity lines connecting a retarded and an advanced Geen function lead to the factor

$$I(\omega\, k) = \frac{1}{\tau}\,\frac{1}{-i\omega + Dk^2}, \qquad (9)$$

which becomes large for small differences of the frequencies and the momenta. The equation for I is illustrated in Fig.3[x/]. The nonregu-

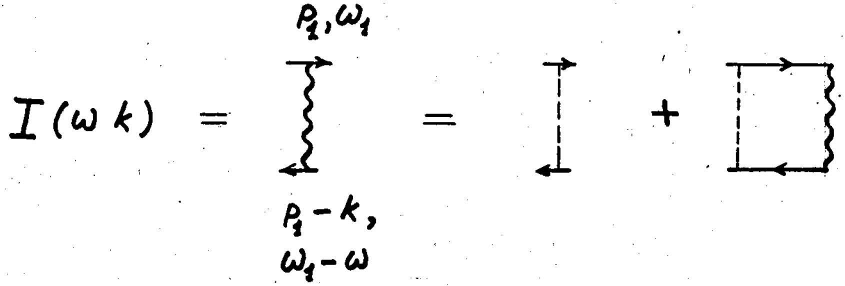

Fig.3. Equation for I. The upper lines represent retarded, the lower advanced Green functions (for the lower lines the direction of the arrow has the reversed meaning); the wavy line denotes I .

x/ Apart from slight differences we follow the diagram representation of Gorkov and Eliashberg [1,3].

lar part of the response can be expressed in terms of vertices Γ constructed in such a way, that they sum up the diagrams containing I-lines to all orders (see Figs.4 and 5). The blocks $\alpha, \beta, \gamma, \delta$

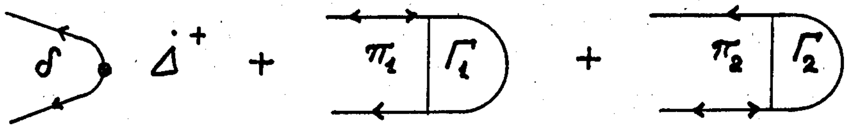

Fig.4. Nonregular part of the response.

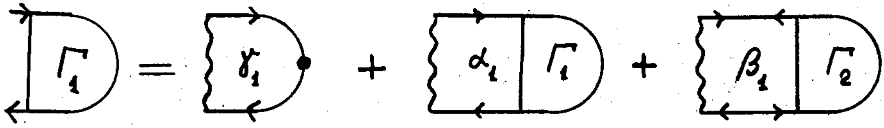

Fig.5. Equation for the vertex part Γ_1.

and π contain no I-lines. The expansion of

$$\text{th}\,\frac{\beta\omega_2}{2} - \text{th}\,\frac{\beta(\omega_2 - \omega')}{2} \approx \omega' \frac{d}{d\omega_2}\,\text{th}\,\frac{\beta\omega_2}{2} \tag{10}$$

in equation (2) leads to the appearance of a time derivative in the vertices δ and γ.

For high magnetic fields $H \approx H_{c2}$ it is possible to restrict the calculation of the blocks α, β, γ and π to the lowest order in Δ. To show this, we compare the vertex in Fig.2 with the vertex of the third order in Fig.6. Since the nonregular part of the response contains advanced and retarded Green functions, the frequencies playing the main role in the integration over ω_1 are of the order of ϵ, determined by the relation $D\vec{Q}^2\Delta \approx \epsilon\Delta$

Fig.6. Vertex of the third order in Δ .

In the case of high magnetic fields holds $\epsilon \approx \epsilon_0 = 2DeH_{c2}$. The calculation shows that the higher-order diagrams are negligible, if the following condition[x] is satisfied:

$$\frac{\overline{|\Delta|^2}}{\epsilon^2} \ll 1. \tag{11}$$

For low temperatures we have $\epsilon_0 \approx T_c$, and for temperatures close to T_c holds $\epsilon \approx T_c(1 - T/T_c)$. Therefore, condition (11) is well satisfied for high fields H (only for temperatures $T \approx T_c$ the field has to be very close to $H_{c2}(T)$). For small fields H (and $T \approx T_c$) condition (11) is not satisfied; it seems then necessary to sum up the higher-order diagrams.

We now consider the equations for the Γ's . The diagrams for γ, α and β to lowest order in Δ are shown in Figs.7,8 and 9.

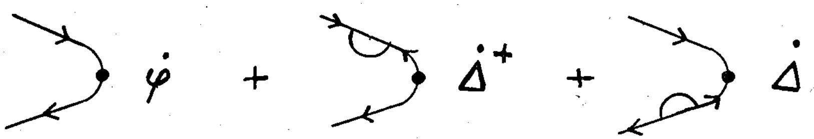

Fig.7. Diagrams for γ_1 .

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[x] This condition is analogous to the condition $\tau_s \Delta \ll 1$ used in the case of paramagnetic impurities /1,3/. (τ_s is the collision time for spin reversal).

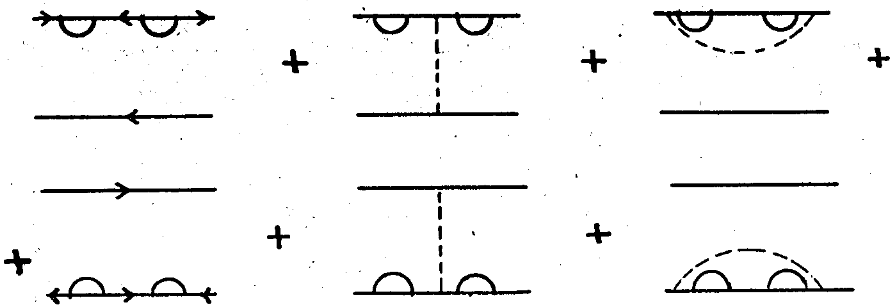

Fig. 8. Diagrams for α_1 .

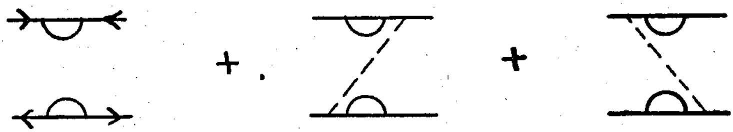

Fig. 9. Diagrams for β_1 .

We get

$$\Gamma_1(\omega_1, \omega k) = \frac{1}{r}\,\frac{1}{-i\omega + Dk^2}\{\omega e\phi + r(\eta^R\Delta)\dot{\Delta}^+ - r(\eta^A\Delta^+)\dot{\Delta} + \\ + \alpha_1\Gamma_1 + \beta_1\Gamma_2\}_{\omega k} \tag{12}$$

(and an analogous equation for Γ_2), where we used a gauge where $\nabla\vec{A} = 0$. $(1/r)\alpha$ and $(1/r)\beta$ are of the order $\overline{|\Delta|^2}/\epsilon_0$; therefore the α and β are only essential in equation (12) for frequency and momentum differences of the order ω , $Dk^2 \lesssim \overline{|\Delta|^2}/\epsilon_0 \ll \epsilon_0$. The calculation gives

$$\beta_1(\omega_1) = \beta_2(-\omega_1) = -\alpha_1(\omega_1) = -\alpha_2(-\omega_1) = \\ = r^2(\eta^R\Delta)\Delta^+ + r^2(\eta^A\Delta^+)\Delta \, . \tag{13}$$

The ω in the η may be neglected here. Thus

$$\Gamma_1(\omega_1, \omega k) = -\Gamma_2(-\omega_1, \omega k) \tag{14}$$

and we get

$$\Gamma_1(\omega_1, \omega k) = \frac{1}{r}\,\frac{1}{-i\omega + Dk^2}\{\omega e\phi + \\ + r(\eta^R\Delta)[\dot{\Delta}^+ - r\Delta^+(\Gamma_1(\omega_1) + \Gamma_1(-\omega_1))] - \\ - r(\eta^A\Delta^+)[\dot{\Delta} + r\Delta(\Gamma_1(\omega_1) + \Gamma_1(-\omega_1))]\}_{\omega k} \, . \tag{15}$$

Equation (15) becomes in r,t -representation a differential equation for the vertex part Γ_1 .

We now consider the nonregular part of the response illustrated in Fig.4. The block δ gives no contribution, because the poles (for ξ) of the Green functions lie in the same half-plane. The diagram for the block π_1 is shown in Fig.10.

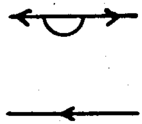

Fig.10. Diagram for π_1 .

Thus we get for the nonregular part of the response the expression

$$\frac{|g| m p_F r^3}{2\pi^2} \int d\omega_1 \left(\frac{d}{d\omega_1} \mathrm{th} \frac{\beta\omega_1}{2} \right) (\eta^R \Delta^+) \Gamma_1(\omega_1, \vec{r}\, t). \tag{16}$$

On substituting the expressions (8) and (16) into (2), we obtain the time dependent Ginzburg-Landau equation

$$\left\{ \ln \frac{T_c}{T} + \Psi\left(\frac{1}{2}\right) - \Psi\left(\frac{1}{2} + \frac{1}{4\pi T} \frac{\partial}{\partial t} + \frac{1}{4\pi T} D\vec{Q}^{+2}\right) \right\} \Delta^+ + \tag{17}$$

$$+ r^2 \int d\omega_1 \left(\frac{d}{d\omega_1} \mathrm{th} \frac{\beta\omega_1}{2} \right) (\eta^R \Delta^+) \Gamma_1 = 0.$$

With the help of the same method we can calculate the response of the charge density ρ of the electrons. The result is

$$\rho = -\frac{e^2 m p_F}{\pi^2} \phi + \frac{ie\, m p_F r}{\pi^2} \int d\omega_1 \left(\frac{d}{d\omega_1} f^<(\omega_1) \right) \Gamma_1 , \tag{18}$$

where

$$f^<(\omega) = \frac{1}{e^{\beta\omega} + 1} . \tag{19}$$

The first term in (18) is the regular , the second the nonregular part of the response.

3. Discussion

In the general case the vertex part Γ_1 depends not only on the frequency and momentum differences ω and k, but also on the integration variable ω_1. We restrict our discussion here to the special cases, in which the dependence of Γ_1 on ω_1 may be neglected. Using the condition of electric neutrality of the system,

$$\rho = 0, \tag{20}$$

and neglecting in (18) the ω_1-dependence of Γ_1, we find

$$\Gamma_1 = \frac{ie}{\tau}\phi, \tag{21}$$

so that in these cases expression (21) has to be compatible with the differential equation (15). By virtue of (21) the time-dependent Ginzburg-Landau equation (17) reduces to

$$\begin{aligned}\{ \ln \frac{T_c}{T} + \Psi(\tfrac{1}{2}) - \Psi\left(\tfrac{1}{2} + \frac{1}{4\pi T}\frac{\partial}{\partial t} + \frac{1}{4\pi T} D\vec{Q}^{+2}\right) + \\ + \frac{2ie}{4\pi T}\phi\, \Psi'\left(\tfrac{1}{2} + \frac{1}{4\pi T} D\vec{Q}^{+2}\right) \} \Delta^+ = 0.\end{aligned} \tag{22}$$

Equations equivalent to (22) were already derived by Schmid[4] (for $T \approx T_c$) and Caroli and Maki[5] without considering the non-regular part of the response[x].

An interesting problem in view of recent experiments is the application of the nonequilibrium theory to the slow motion of the lattice of the vortex lines (flux-flow) caused by a static electric field. The flux-flow resistivity calculated by Schmid[4] and Caroli and Maki[5] on the basis of equation (22) is in agreement with the experiments[xx].

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[x] From the point of view of the phenomenological nonequilibrium thermodynamics equation (22) is considered in[7].

[xx] Compare the experiments on $In + 1.5$ at. % Bi alloys for $H = 0.73 H_{c2}$ by Cape and Silvera[8] and earlier work cited in[4,5].

In connection with this application we consider the compatibility of (21) with the differential equation (15) for the flux-flow regime. It is easily seen that contributions to Γ_1 coming from (15) for large $k(Dk^2 \gg \overline{|\Delta|^2}/\epsilon_0)$ are negligible. Thus it is sufficient for the validity of (21) that the second and the third term on the right-hand side of (15) vanish for small $k(Dk^2 \lessgtr \overline{|\Delta|^2}/\epsilon_0 \ll \epsilon_0)$:

$$\Delta[\dot{\Delta}^+ - 2ie\phi\Delta^+]\Big|_{k\ \text{small}} = 0. \tag{23}$$

The (approximately constant) electric field is described by $\phi = -Ex$ and the magnetic field by $\vec{A} = (0, Hx, 0)$; since we are only considering the linear equation for Δ^+ , we put $H = H_{c2}$. Equation (22) can be written in the form

$$\left\{\frac{\partial}{\partial t} - 2ie\phi + D\vec{Q}^{+2}\right\}\Delta^+ = \epsilon_0\Delta^+ , \tag{24a}$$

$$\ln\frac{T_c}{T} + \Psi\left(\frac{1}{2}\right) - \Psi\left(\frac{1}{2} + \frac{\epsilon_0}{4\pi T}\right) = 0. \tag{24b}$$

The solution of (24a) is the moving Abrikosov solution (compare [4,5])

$$\Delta^+ = \sum_n C_n^+ \exp\left\{iqn\left(y + \frac{E}{H}t\right) - eH\left(x + \frac{nq}{2eH} + \frac{iE}{4eH^2D}\right)^2\right\}. \tag{25}$$

We insert this solution into (23) and get

$$\begin{aligned}\Delta[\dot{\Delta}^+ - 2ie\phi\Delta^+]\Big|_{k\ \text{small}} &\approx \\ \approx \sum_n |C_n|^2 \int_{-\infty}^{\infty} dx\, 2ieEx\, e^{-2eHx^2} &= 0,\end{aligned} \tag{26}$$

so that for the flux-flow regime the validity of (21) is proven.

Acknowledgements

I would like to thank Professor G.M.Eliashberg , Professor L.P.Gorkov and Professor D.N.Zubarev for valuable discussions.

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Received by Publishing Department
on April 4, 1969.