ОБЪЕДИНЕННЫЙ
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Дубна

E2 - 4225

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REPRESENTATIONS OF THE LORENTZ GROUP AND GENERALIZATION OF HELICITY STATES

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1968

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Submitted to Commun. in Math. Phys.

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Introduction

The expansion of the relativistic amplitudes raised the problem of finding a simple form for the matrix elements of the irreducible representations of the Lorentz group. The most degenerate representation has been studied at first by I.S.Shapiro and by A.Z.Dolginov and I.N.Toptygin/1/ by analytic continuation of the representations of the O(4) group. Later, representations on the hyperboloid have been investigated systematically by Ya.A. Smorodinsky and N.Ya. Vilenkin/2/ introducing a number of coordinate systems on the hyperboloid and finding the spherical functions corresponding to these systems. The problem of expansion of functions on the hyperboloid has been also solved there. Further, it has been extended to the spin non-zero case (by M.A.Liberman, Ya.A.Smorodinsky and M.B.Sheftel) separating the part describing the spin by realizing the representations on the direct product of spaces of the hyperboloid and the sphere or cone/3/. The representation of the principal series has been derived in angular momentum basis by S.Ström in 1965/4/, however, due to the improper parametrization the matrix elements obtained were rather complicated. Recently, a compact form of the matrix elements has been derived in angular momentum basis by A.Sebestyén et al.[9]. In the present paper the unitary representations will be built up with parameters leading to a Lie algebra of two independent angular momenta. In terms of these parameters the representations acquire a comparatively simple form. An arbitrary element of the Lorentz group can be decomposed in the following way. We choose three Cartesian axes and define a combined transformation consisting of a rotation about an axis and of a boost along the same axis. It is shown that an arbitrary of the Lorentz group can be represented as a product of combined rotations about x_3, x_1 and once more about x_3 axes, respectively. These operations can be written in the form of subsequent rotations by certain complex Euler angles and by Euler angles complex conjugated to the previous ones. The subgroups of spatial rotations are obtained when the imaginary parts of the complex angles are zero. (Rotations with pure imaginary Euler angles, however, do not form a subgroup). It is possible to consider the real part of Euler angles as coordinates on the surface of 4-dimensional real sphere since the group of motion of this sphere is isomorphic to the 0(3) group. Then it becomes natural to consider the complex rotation group as a group of motion of the 4-dimensional complex sphere.

As a result of our calculation we get a system of functions realizing the representation of the Lorentz group, which contains 6 continuous parameters. We mention that the generators M_3 and N_3 are diagonal in our representation as well as in the so called cylindrical system used in[2]. By introducing suitable parameters on the complex sphere both our Casimir operators coincide with the Laplacean on the hyperboloid expressed in terms of cylindrical variables, however, the eigenfunctions are not identical since they satisfy different boundary conditions.

Since these functions are eigenfunctions of the square of the 4-momentum they can be considered as a wave function of the relativistic top (as far as this word has a meaning in the relativistic domain).

The state of this relativistic top has no definite angular momentum value. Instead, its wave function must be decomposed into a series of spherical harmonics in a fixed coordinate system. This operation leads to the usual series of angular momentum states, contained in a single representation of Lorentz group. Details of this problem will be treated separately.

Consider a moving particle and choose the x_3 -axis along its momentum. Then the real of the eigenvalue of the complex generator J_3 is the helicity of the particle, so it is natural to call the eigenvalue of J_3 the generalized complex helicity of the particle. This fact is exhibited by the exponential dependence of eigenfunctions on this complex eigenvalue. The imaginary part of J_3 products the boost state (which belongs really to the representation of Poincaré group). Such combination of rotation and boost turns to be a very natural basis for the representation of the Lorentz group.

1. Parametrization

Parametrization of a Lie group G can be performed by means of embedding a suitable homogeneous space X into the group. We assume the group to realize a mapping of the space X onto itself, by which we mean the fulfilment of the following requirements: a) For the unit element $e \in G, x \in X, ex=x$. b) For $g_1, g_2 \in G, x \in X, (g_1 g_2)x = g_1(g_2 x)$. c) The function gx is a continuous function of g and of x.

The homogeneity of the space X requires that for any $x_1, x_2 \in X$ there exists at least one element $g \in G$ which connects x_1 and x_2: $x_2 = gx_1$. If there are several elements $g_1, g_2, \dots$ of G for which $x_2 = g_1 x_1$ and $x_2 = g_2 x_1$ hold then elements of type $g_1^{-1} g_2 = h$ form the little group of the point x_1. Thus every point of X determines the elements of G up to the little group H, which means that points $x \in X$ represent the elements of the factor group G/H. Introducing a coordinate system in the X space we can label the elements of the G/H factor group. Repeating this procedure we obtain a parametrization whithin the H subgroup: we choose a suitable homogeneous space Y which can be embedded into H. Than a coordinate system parametrizes the factor group H/K where K is the little group of a point $y \in Y$. Finally we arrive at a Z space points of which characterize unambiguously the little group L obtained in the previous step. The homogeneous space X has an additional significance since it serves for the domain of the spherical functions with respect to the subgroup H.

One of the usual chains parameters for the homogeneous Lorentz group is as follows. For the homogeneous space we choose the upper sheet of the real double-sheeted hyperboloid $x_0^2-x_1^2-x_2^2-x_3^2=1$. Points on the hyperboloid determine the elements of the Lorentz group up to the subgroup of spatial rotations. Choosing e.g. the point $x=(1,0,0,0)$ each element of G can be decomposed as a product of spatial rotation leaving unaltered x and a displacement on the hyperboloid. Considering the rotation group H we choose the real sphere $y_1^2+y_2^2+y_3^2=1$ for the homogeneous space. The little group of the north pole is the O(2) group involving one parameter the Euler angle ψ and the remaining two parameters are θ, ϕ the polar coordinates on the sphere. Finally, realizing the O(2) group we obtain the circle $z_1^2+z_2^2=1$. No further classification required since points on the circle are unambiguously characterized by the angle ψ. Similar decomposition can be made starting from the single sheeted hyperboloid. If for the homogeneous space hyperboloid is chosen, 3 out of the 6 parameters of the Lorentz group are contained by the hyperboloid. In the following we shall use a comparatively larger homogeneous space, the three-dimensional complex sphere which involves 4 real parameters while the remainig 2 parameters are contained by the little group of a certain point on the complex sphere.

The Lie algebra of the Lorentz group can be represented by the three boost generators N_i and the generators of spatial rotations M_i. These satisfy the commutation relations

$$[M_i, M_k] = i\epsilon_{ik\ell} M_\ell, \; [N_i, N_k] = -i\epsilon_{ik\ell} M_\ell, \; [M_i, M_k] = i\epsilon_{ik\ell} N_\ell$$

where $\epsilon_{ik\ell}$ is totally antisymmetric, $\epsilon_{123}=1$ and $i,k,\ell=1,2,3$. Introduce the linear combinations in the usual way:

$$J_k = \frac{1}{2}(M_k + iN_k), \quad K_k = \frac{1}{2}(M_k - iN_k)$$

that obey the relations:

$$[J_i, J_k] = i\epsilon_{ik\ell} J_\ell, [K_i, K_k] = i\epsilon_{ik\ell} K_\ell, [J_i, K_k] = 0. \tag{1}$$

The form of these commutators suggests to write an element of the homogeneous Lorentz group in the form

$$T(g) = e^{-i\epsilon_1 J_3} e^{-i\epsilon_2 J_1} e^{-i\epsilon_3 J_3} e^{-i\epsilon_1^* K_3} e^{-i\epsilon_2^* K_1} e^{-i\epsilon_3^* K_3}, \tag{2}$$

where

$$\epsilon_k = p_k + iq_k, \quad \epsilon_k^* = p_k - iq_k \tag{3}$$

with the range[x)] of p_k and q_k

$$0 \le p_1 < 2\pi, \; 0 \le p_2 < \pi, \; 0 \le p_3 < 2\pi, \tag{4}$$
$$-\infty < q_1 < \infty, \; -\infty < q_2 < \infty, \; -\infty < q_3 < \infty.$$

Equation (2) containes two rotations by Euler angles $\epsilon_1^*, \epsilon_2^*, \epsilon_3^*$ and $\epsilon_1, \epsilon_2, \epsilon_3$, respectively. Rearranging the exponential in (2) we obtain the direct physical significance of the above parametrization in terms of the real parameters p_k and q_k. In T(g) the following subsequent operations are contained:

Boost along x_3-axis by hyperbolic angle $-q_3$
Spatial rotation about x_3-axis by angle p_3
Boost along x_1-axis by hyperbolic angle $-q_2$
Spatial rotation about x_1-axis by angle p_2
Boost along x_3-axis by hyperbolic angle $-q_1$
Spatial rotation about x_3-axis by angle p_1

---

x) This range of parameters is related to the Lorentz group. For the universal covering group SL(2,C) we should have to put $-2\pi \le p_3 < 2\pi$.

For the sake of comparison with other parametrizations we have listed in the Appendix the matrix elements of the 4 x 4 representation in terms of the above parameters.

It can be seen from (2) directly that the condition of unitarity of the representation is

$$J_i^+ = K_i \quad (5)$$

where + denotes the hermitean adjoint with respect to a certain positive definite scalar product in the Hilbert space where the representation is defined.

2. Representation of the Infinitesimal Generators

Consider the 2-parameter subgroup $H = O(2) \times O(1,1)$ which contains spatial rotation about the x_3-axis and boost along the x_3-axis. These transformations will be characterized by the spatial and hyperbolic angle p_1 and q_1 respectively with the range $0 \leq p_1 < 2\pi, -\infty < q_1 < \infty$. The differential operator representation of the generators is given by

$$M_3 = -i \frac{\partial}{\partial p_1}, \quad N_3 = i \frac{\partial}{\partial q_1}.$$

The simultaneous eigenfunctions of these generators satisfying the equations

$$M_3 f_{\mu\nu} = \mu f_{\mu\nu}, \quad N_3 f_{\mu\nu} = \nu f_{\mu\nu}$$

are

$$f_{\mu\nu}(p_1, q_1) = \frac{1}{2\pi} e^{i(\mu p_1 - \nu q_1)} \quad (6)$$

For unitary representations M_3 and N_3 are hermitean that is μ and ν are both real. Requiring the representation to be single or at most double-valued we obtain

$\mu = 0, \pm\frac{1}{2}, \pm 1, \ldots$ and the eigenvalue ν is continuous in the range $-\infty < \nu < \infty$. (7)

The eigenfunctions (6) are, of course, eigenfunctions of the generators $J_3 = \frac{1}{2}(M_3 + iN_3)$, $K_3 = \frac{1}{2}(M_3 - iN_3)$ as well:

$$J_3 f_{\mu\nu} = \frac{\mu + i\nu}{2} f_{\mu\nu}, \quad K_3 f_{\mu\nu} = \frac{\mu - i\nu}{2} f_{\mu\nu}$$

or

$$J_3 \phi_{mm^*} = m \phi_{mm^*}, \quad K_3 \phi_{mm^*} = m^* \phi_{mm^*} \quad (8)$$

with

$$m = \frac{\mu + i\nu}{2}, \quad m^* = \frac{\mu - i\nu}{2}, \quad \phi_{mm^*} \equiv f_{\mu\nu}.$$

For labelling the eigenfunction we can use m and m^* instead of μ, ν or vice versa, Introducing the combinations

$$J_\pm = J_1 \pm i J_2, \quad K_\pm = K_1 \pm i K_2$$

the familiar commutation relations are obtained:

$$[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_3$$

$$[K_3, K_\pm] = \pm K_\pm \qquad [K_+, K_-] = 2K_3 \quad (9)$$

(The remaining are zero).

We get from these relations that

$$(J_3 + K_3) J_\pm f_{\mu\nu} = (\mu \pm 1) J_\pm f_{\mu\nu}.$$

and thus the states $J_\pm f_{\mu\nu}$ are eigenfunctions of the hermitean operator $M_3 = J_3 + K_3$ with the eigenvalue $\mu \pm 1$. The similar statement is valid for the state $K_\pm f_{\mu\nu}$. The same functions $J_\pm f_{\mu\nu}$, $K_\pm f_{\mu\nu}$, however, are not eigenfunctions of the hermitean operator $N_3 = \frac{1}{i}(J_3 - K_3)$. Though, one would obtain from the commutators the relation

$\frac{1}{i}(J_3-K_3)J_{\pm}f_{\mu\nu}=(\nu\mp i)\,J_{\pm}f_{\mu\nu}$. in spite of this formal equation $J_{\pm}f_{\mu\nu}$ cannot be considered as an eigenstate since it has complex eigenvalues and thus it does not constitute an element of the unitary basis. The origin of this fact lies in the non-compactness of the $O(2)\times O(1,1)$ group. (Recently, the same problem was faced in a paper by A.Chakrabarti et al.[10] where the representation of the Poincaré group has been constructed in the non-compact Lorentz group basis. It turned out that the action of the displacement generators P_μ on the states belonging to the eigenvalue λ of M^2-N^2 gives a state with the eigenvalue $\lambda\pm i$. We shall solve this problem by expanding these undesired states in series of states of the unitary basis. As it will be seen the complex δ-function arising in the expansion will lead to no complication in the matrix elements of finite rotations). So, we can search for the representation in the form:

$$J_{\pm}f_{\mu\nu}=\int d\nu'\,A^{\pm}_{\mu}(\nu,\nu')\,f_{\mu+1,\nu'},$$
$$K_{\pm}f_{\mu\nu}=\int d\nu'\,B^{\pm}_{\mu}(\nu,\nu')\,f_{\mu+1,\nu'}, \qquad (10)$$

where the path of integration is the real ν'-axis. We obtain from (9) that $A^{\pm}_{\mu}$ and $B^{\pm}_{\mu}$ have the form[x)]

$$A^{\pm}_{\mu}(\nu,\nu')=a^{\pm}_{\mu}(i\nu)\,\delta(\nu'-\nu\pm i)$$
$$B^{\pm}_{\mu}(\nu,\nu')=b^{\pm}_{\mu}(i\nu)\,\delta(\nu'-\nu\mp i)\,. \qquad (11)$$

The complex δ-function has arisen here from requiring the unitarity (that involves the infinite dimension of the representation). If, on the other hand, one considered the compact $O(4)$ group, the δ-function would be concentrated on the real ν'-axis that would lead to the selection rule $\nu'-\nu=\pm 1$. As a consequence of the complex delta-function the operators $J_{\pm}$, $K_{\pm}$ mix the entire continuous ν' spectrum.

x) The Dirac delta of complex argument has a strict mathematical meaning. It is treated in details by textbooks on generalized functions (cf.[5]). It appears here, essentially, as the Fourier expansion of the exponential function.

For $a^{\pm}_{\mu}$, $b^{\pm}_{\mu}$ eq. (9) yields the recurrance relations

$$a^{-}_{\mu}(\lambda)\,a^{+}_{\mu-1}(\lambda-1)-a^{+}_{\mu}(\lambda)\,a^{-}_{\mu+1}(\lambda+1)=\mu+\lambda$$
$$b^{-}_{\mu}(\lambda)\,b^{+}_{\mu-1}(\lambda+1)-b^{+}_{\mu}(\lambda)\,b^{-}_{\mu+1}(\lambda-1)=\mu-\lambda$$
$$a^{+}_{\mu}(\lambda)\,b^{+}_{\mu+1}(\lambda+1)-b^{+}_{\mu}(\lambda)\,a^{+}_{\mu+1}(\lambda-1)=0 \qquad (12)$$
$$a^{+}_{\mu}(\lambda)\,b^{-}_{\mu+1}(\lambda+1)-b^{-}_{\mu}(\lambda)\,a^{+}_{\mu-1}(\lambda+1)=0$$
$$a^{-}_{\mu}(\lambda)\,b^{+}_{\mu-1}(\lambda-1)-b^{+}_{\mu}(\lambda)\,a^{-}_{\mu+1}(\lambda-1)=0$$
$$a^{-}_{\mu}(\lambda)\,b^{-}_{\mu-1}(\lambda-1)-b^{-}_{\mu}(\lambda)\,a^{-}_{\mu-1}(\lambda+1)=0\,,$$

where $\lambda\equiv i\nu$. The last four equations can be satisfied by the ansatz

$$a^{\pm}_{\mu}(\lambda)=a^{\pm}\left(\frac{\mu+\lambda}{2}\right),\quad b^{\pm}_{\mu}(\lambda)=b^{\pm}\left(\frac{\mu-\lambda}{2}\right)$$

while the first two lead to:

$$a^{-}\left(\frac{\mu+\lambda}{2}\right)a^{+}\left(\frac{\mu+\lambda}{2}-1\right)=j(j+1)-\frac{\mu+\lambda}{2}\left(\frac{\mu+\lambda}{2}-1\right)=j(j+1)-\frac{\mu+i\nu}{2}\left(\frac{\mu+i\nu}{2}-1\right).$$
$$\qquad (13)$$
$$b^{-}\left(\frac{\mu-\lambda}{2}\right)b^{+}\left(\frac{\mu-\lambda}{2}-1\right)=k(k+1)-\frac{\mu-\lambda}{2}\left(\frac{\mu-\lambda}{2}-1\right)=k(k+1)-\frac{\mu-i\nu}{2}\left(\frac{\mu-i\nu}{2}-1\right),$$

where $j(j+1)$ and $k(k+1)$ are certain constants within an irreducible representation. With a proper normalization of the basis functions we get:

$$a^{+}_{\mu}(i\nu)=i\sqrt{j(j+1)-\frac{\mu+i\nu}{2}\left(\frac{\mu+i\nu}{2}+1\right)},\quad a^{-}_{\mu}(i\nu)=-i\sqrt{j(j+1)-\frac{\mu+i\nu}{2}\left(\frac{\mu+i\nu}{2}-1\right)}$$
$$b^{+}_{\mu}(i\nu)=i\sqrt{k(k+1)-\frac{\mu-i\nu}{2}\left(\frac{\mu-i\nu}{2}+1\right)},\quad b^{-}_{\mu}(i\nu)=-i\sqrt{k(k+1)-\frac{\mu-i\nu}{2}\left(\frac{\mu-i\nu}{2}-1\right)}. \qquad (14)$$

Thus finally the generators have the following representation

$$J_\pm f_{\mu\nu} = J_\pm f^{jk}_{\mu\nu} = \int d\nu' \delta(\nu'-\nu\pm i)\, a^\pm_\mu(i\nu)\, f_{\mu\pm 1\nu'}$$
$$K_\pm f_{\mu\nu} = K_\pm f^{jk}_{\mu\nu} = \int d\nu' \delta(\nu'-\nu\mp i)\, b^\pm_\mu(i\nu)\, f_{\mu\pm 1\nu'}. \tag{15}$$

The condition of unitarity (5) yields:

$$a^+_\mu(i\nu-1) = b^-_{\mu+1}(i\nu)^*,\; a^-_\mu(i\nu+1) = b^+_{\mu-1}(i\nu)^*. \tag{16}$$

The two independent Casimir operators are

$$\vec{J}^2 = J_1^2 + J_2^2 + J_3^2 = \frac{1}{4}(\vec{M}^2 - \vec{N}^2 + 2i\vec{M}\vec{N})$$

$$\vec{K}^2 = K_1^2 + K_2^2 + K_3^2 = \frac{1}{4}(\vec{M}^2 - \vec{N}^2 - 2i\vec{M}\vec{N}).$$

Using (15) we get that the constants $j(j+1)$ and $k(k+1)$ in eq. (13) are just the eigenvalues of J^2 and K^2:

$$\vec{J}^2 f^{jk}_{\mu\nu} = j(j+1) f^{jk}_{\mu\nu},\quad \vec{K}^2 f^{jk}_{\mu\nu} = k(k+1) f^{jk}_{\mu\nu}. \tag{17}$$

Thus an irreducible representation is characterized by the quantities j and k. The representation (15) is invariant with respect to the substitution

$$j \to -j-1 \;,\; k \to -k-1 \tag{18}$$

so the representations characterized by (j,k) and $(-j-1,-k-1)$ are equivalent.

The necessary condition of the unitarity is

$$\vec{J}^2 = (\vec{K}^2)^+ .$$

That is

$$(j_2-k_2)(j_2+k_2) - (j_1-k_1)(j_1+k_1+1) = 0 \tag{19}$$

$$(j_2+k_2) + 2(k_1k_2 + j_1j_2) = 0, \tag{19}$$

where

$$j = j_1 + ij_2,\; k = k_1 + ik_2.$$

We have the solutions

$$\begin{aligned} &a).\quad j_1 = k_1, j_2 = k_2\\ &b).\quad j_2 = k_2 = 0,\; j_1 = k_1\\ &c).\quad j_1 = -k_1 - 1,\; j_2 = k_2\\ &d).\quad j_2 = k_2 = 0,\; j_1 = -k_1 - 1. \end{aligned} \tag{20}$$

In view of the invariance (18) solutions c) and d) can be omitted. In the case b) we have a further restriction, namely $0 \le j_1 \le 1$, $0 \le k_1 \le 1$. Cases a) and b) constitute the principal and supplementary series, respectively. Writing j and k in the form

$$j = \frac{1}{2}(\ell_0 - 1 + ip),\; k = \frac{1}{2}(\ell_0 - 1 + ip)^* \tag{21}$$

it is easily seen that we can choose p to be non-negative as a consequence of equation (18). It will be shown further that ℓ_0 has only integer or half-integer values. Thus the results of the cases a) and b) can be summarized as 1) p real non-negative, ℓ_0 integer or half integer (principal series). 2) $\ell_0 = 0$, p imaginary, $p = ip'$ with $-1 \le p' \le 1$ (supplementary series). In the following only the principal series will be treated since the supplementary series has no contribution in the harmonic analysis. The eigenvalues of the Casimir operators in terms of ℓ_0 and p are

$$j(j+1) = \frac{1}{4}(\ell_0^2 - p^2 - 1 + 2i\ell_0 p)$$
$$k(k+1) = \frac{1}{4}(\ell_0^2 - p^2 - 1 - 2i\ell_0 p). \tag{22}$$

3. Matrix Elements of the Representations

Consider a function on the Lorentz group $f(g)$ and define the action of the representation on $f(g)$ by left displacement:

$$T(g_0)f(g) = f(g_0^{-1}g). \tag{23}$$

Choosing for g_0 the six one-parameter subgroups according to the six (real) parameters we obtain from (23) the infinitesimal generators in the form of differential operators acting on the functions on the group

$$J_1 = \frac{1}{i}\left(-\sin\epsilon_1 \cot\epsilon_2 \frac{\partial}{\partial\epsilon_1} + \cos\epsilon_1 \frac{\partial}{\partial\epsilon_2} + \frac{\sin\epsilon_1}{\sin\epsilon_2}\frac{\partial}{\partial\epsilon_3}\right)$$

$$J_2 = \frac{1}{i}\left(\cos\epsilon_1 \cot\epsilon_2 \frac{\partial}{\partial\epsilon_1} + \sin\epsilon_1 \frac{\partial}{\partial\epsilon_2} - \frac{\cos\epsilon_1}{\sin\epsilon_2}\frac{\partial}{\partial\epsilon_3}\right)$$

$$J_3 = \frac{1}{i}\frac{\partial}{\partial\epsilon_1} \tag{24}$$

$$K_1 = \frac{1}{i}\left(-\sin\epsilon_1^* \cot\epsilon_2^* \frac{\partial}{\partial\epsilon_1^*} + \cos\epsilon_1^* \frac{\partial}{\partial\epsilon_2^*} + \frac{\sin\epsilon_1^*}{\sin\epsilon_2^*}\frac{\partial}{\partial\epsilon_3^*}\right)$$

$$K_2 = \frac{1}{i}\left(\cos\epsilon_1^* \cot\epsilon_2^* \frac{\partial}{\partial\epsilon_1^*} + \sin\epsilon_1^* \frac{\partial}{\partial\epsilon_2^*} - \frac{\cos\epsilon_1^*}{\sin\epsilon_2^*}\frac{\partial}{\partial\epsilon_3^*}\right)$$

$$K_3 = \frac{1}{i}\frac{\partial}{\partial\epsilon_1^*}.$$

Here

$$\epsilon_k = p_k + i q_k, \quad \epsilon_k^* = p_k - i q_k$$

and

$$\frac{\partial}{\partial\epsilon_k} = \frac{1}{2}\left(\frac{\partial}{\partial p_k} - i\frac{\partial}{\partial q_k}\right), \quad \frac{\partial}{\partial\epsilon_k^*} = \frac{1}{2}\left(\frac{\partial}{\partial p_k} + i\frac{\partial}{\partial q_k}\right).$$

Generators (24) fulfill the commutation relations (1). The Casimir operators can be obtained from (24):

$$-\vec{J}^2 = \frac{1}{\sin^2\epsilon_2}\left(\frac{\partial^2}{\partial\epsilon_1^2} + \frac{\partial^2}{\partial\epsilon_3^2} - 2\cos\epsilon_2 \frac{\partial}{\partial\epsilon_1}\frac{\partial}{\partial\epsilon_3}\right) + \frac{\partial^2}{\partial\epsilon_2^2} + \cot\epsilon_2 \frac{\partial}{\partial\epsilon_2}$$

$$-\vec{K}^2 = \frac{1}{\sin^2\epsilon_2^*}\left(\frac{\partial^2}{\partial\epsilon_1^{*2}} + \frac{\partial^2}{\partial\epsilon_3^{*2}} - 2\cos\epsilon_2^* \frac{\partial}{\partial\epsilon_1^*}\frac{\partial}{\partial\epsilon_3^*}\right) + \frac{\partial^2}{\partial\epsilon_2^{*2}} + \cot\epsilon_2^* \frac{\partial}{\partial\epsilon_2^*}. \tag{25}$$

Matrix elements of the unitary irreducible representations are the simultaneous eigenfunctions of (25), that is

$$[\vec{J}^2 - j(j+1)]\, T(g)^{jj^*}_{mm^*;nn^*} = 0$$

$$[\vec{K}^2 - j^*(j^*+1)]\, T(g)^{jj^*}_{mm^*;nn^*} = 0. \tag{26}$$

The representation can be factored out in the form

$$T^{jj^*}_{mm^*;nn^*} = e^{-i(\epsilon_1 m + \epsilon_1^* m^* + \epsilon_3 n + \epsilon_3^* n^*)} R^{jj^*}_{mm^*;nn^*}(\cos\epsilon_2, \cos\epsilon_2^*) =$$

$$= e^{-i(p_3\kappa - q_3\lambda + p_1\mu - q_1\nu)} R^{jj^*}_{mm^*;nn^*}(\cos\epsilon_2, \cos\epsilon_2^*), \tag{27}$$

where

$$m = \frac{1}{2}(\mu + i\nu), \quad n = \frac{1}{2}(\kappa + i\lambda). \tag{28}$$

Substituting (27) into (26) and introducing the variable $z=\cos\epsilon_2$, $z^*=\cos\epsilon_2^*$ (26) reduces to

$$\left[(1-z^2)\frac{d^2}{dz^2}-2z\frac{d}{dz}-\frac{m^2+n^2-2mnz}{1-z^2}+j(j+1)\right]R^{jj^*}_{mm^*;nn^*}(z,z^*)=0 \qquad (29)$$

$$\left[(1-z^{*2})\frac{d^2}{dz^{*2}}-2z^*\frac{d}{dz^*}-\frac{m^{*2}+n^{*2}-2m^*n^*z^*}{1-z^{*2}}+j^*(j^*+1)\right]R^{jj^*}_{mm^*;nn^*}(z,z^*)=0. \qquad (30)$$

Let us restrict ourselves temporally to the case

$$\mathrm{Re}\,(m+n)\geq 0\ ,\quad \mathrm{Re}\,(m-n)\geq 0\ . \qquad (31)$$

It will be convenient to take for the two independent solutions of equation (29) the following ones:x)

$$P^j_{mn}(z)=n^j_{mn}\left(\frac{1-z}{2}\right)^{\frac{m-n}{2}}\left(\frac{1+z}{2}\right)^{\frac{m+n}{2}}F\left(-j+m,j+m+1,m-n+1;\frac{1-z}{2}\right) \qquad (32)$$

$$Q^j_{mn}=n^j_{mn}\left(\frac{1-z}{2}\right)^{-\frac{m-n}{2}}\left(\frac{1+z}{2}\right)^{\frac{m+n}{2}}F\left(-j+n,j+n+1,-m+n+1;\frac{1-z}{2}\right)=$$

$$(33)$$

$$=n^j_{mn}\left(\frac{1-z}{2}\right)^{-\frac{m-n}{2}}\left(\frac{1+z}{2}\right)^{-\frac{m+n}{2}}F\left(-j-m,j-m+1,-m+n+1;\frac{1-z}{2}\right),$$

x) The solutions of the first and second kind of the equation (29) have been studied by Andrews and Gunson/6/. Solutions (32) and (33) are related to the functions d^{mn}_j and e^{mn}_j of ref./6/ as

$d^{mn}_j=P^j_{mn}$, $e^{mn}_j=\frac{\pi}{2\sin\pi(j-m)}\left[e^{\mp i\pi(j-m)}P^j_{mn}(z)-P^j_{m,-n}(-z)\right]$ with $+$ for $\mathrm{Im}\,z\lessgtr 0$

(As to $P^j_{m,-n}(-z)$ see section 5 of the present paper).

where

$$n^j_{mn}=\frac{1}{\Gamma(1+m-n)}\sqrt{\frac{\Gamma(j+m+1)\Gamma(j-n+1)}{\Gamma(j+n+1)\Gamma(j-m+1)}}.$$

The general solution of equations (29) and (30) can be written in the form

$$R^{jj^*}_{mm^*;nn^*}=c_1P^{j^*}_{m^*n^*}(z^*)Q^j_{mn}(z)+c_2P^j_{mn}(z)Q^{j^*}_{m^*n^*}(z^*)+c_3P^j_{mn}(z)P^{j^*}_{m^*n^*}(z^*)+$$
$$+c_4Q^j_{mn}(z)Q^{j^*}_{m^*n^*}(z^*)\ , \qquad (34)$$

where c_1, c_2, c_3, c_4 are some constants (independent of z and z^*). Equations (29) and (30) have three singular (branch) points: $z=1,-1,\infty$, the cuts will be directed from ± 1 outwards, that is from -1 to $-\infty$ and from 1 to ∞.

It is seen immediately that $c_4=0$ since this term is singular at $z=1$. Investigating the behaviour of eq. (34) as $z\to -1$ and $z\to\infty$ we obtain two equations for the three constants c_1, c_2, c_3. Taking into account, however, that to the unit element of the group has to correspond the unit matrix, that is

$$\lim_{z\to 1} R^{jj^*}_{mm^*;nn^*}=\delta_{\mu\kappa}\,\delta(\nu-\lambda)\ ,$$

it is easily proved that this can be satisfied only by putting $c_3=0$ and so there remain two constants c_1 and c_2 to be determined.

Requiring the finiteness at the points $z=-1$, $z=\infty$ we get the equations

$$c_1A+c_2A^*=0$$
$$c_1B+c_2B^*=0\ , \qquad (35)$$

where

$$A = \frac{\Gamma(m^*-n^*+1)\Gamma(-m+n+1)}{\Gamma(-j^*+m^*)\Gamma(j^*+m^*+1)\Gamma(-j+n)\Gamma(j+n+1)}$$

(36)

$$B = e^{i\pi(n-m)} \frac{\Gamma(m^*-n^*+1)\Gamma(-m+n+1)}{\Gamma(j^*-n^*+1)\Gamma(j^*+m^*+1)\Gamma(j-m+1)\Gamma(j+n+1)} .$$

Equation (35) has non-trivial solution for c_1 and c_2 if the determinant vanishes, that is

$$e^{i\pi(\mu-\kappa)} \frac{\sin\pi(j-n)\sin\pi(j^*-m^*)}{\sin\pi(j^*-n^*)\sin\pi(j-m)} = 1 , \tag{37}$$

where μ and κ are given by the equation (28).

We have the following possibilities:

a) $\mu-\kappa$ is half integer. Then (37) generally has no solution which means that (35) has the trivial solution $c_1 = c_2 = 0$ only, and the matrix elements vanish. This corresponds to the fact that by eq. (10) J_+ and J_- increases and decreases the values of μ by 1.

b) $\mu-\kappa$ is even. Then (37) reduces to

$$\sin\pi\left(\ell_0 - \frac{\mu+\kappa}{2}\right) \operatorname{sh}\pi \frac{\nu-\lambda}{2} = 0$$

or

$$\ell_0 - \frac{\mu+\kappa}{2} = \text{integer}$$

and thus ℓ_0 is integer if κ integer and ℓ_0 is half-integer if κ is half-integer .

c) $\mu-\kappa$ is odd. Then (37) yields:

$$\cos\pi\left(\ell_0 - \frac{\mu+\kappa}{2}\right) \operatorname{ch}\pi \frac{\nu-\lambda}{2} = 0 .$$

So

$$\ell_0 - \frac{\mu+\kappa}{2} = \text{half-integer.}$$

Or

ℓ_0 is integer if κ integer and ℓ_0 is half-integer if κ half-integer.

We have concluded that ℓ_0 is integer if the eigenvalue of J_3+K_3 is integer and ℓ_0 is half-integer if the eigenvalue of J_3+K_3 is half-integer. In the latter case the representations are double valued.

Solving the eq. (35) and choosing a suitable normalization factor the R-functions are obtained in the form

$$R^{jj^*}_{mm^*,nn^*} = \frac{N^j_{mn}}{4i\sqrt{\sin\pi(m-n)\sin\pi(m^*-n^*)}}\left[C^j_{mn} P^{j^*}_{m^*n^*}(z^*) Q^j_{mn}(z) - C^{j^*}_{m^*n^*} P^j_{mn}(z) Q^{j^*}_{m^*n^*}(z^*)\right], \tag{38}$$

where

$$C^j_{mn} = \frac{\sin\pi(m-n)\sin\pi(j-m)}{\pi(m-n)\sin\pi(j-n)} \Gamma(m-n+1)^2 \frac{\Gamma(j-m+1)\Gamma(j+n+1)}{\Gamma(j-n+1)\Gamma(j+m+1)}$$

and

$$N^j_{mn} = \sqrt{\frac{\sin\pi(j-n)\sin\pi(j^*-n^*)}{\sin\pi(j-m)\sin\pi(j^*-m^*)}} \quad (\mathrm{Re}(m+n) \geq 0 , \ \mathrm{Re}(m-n) \geq 0) .$$

It is seen that the constants N^j_{mn}, C^j_{mn} and n^j_{mn} have the following property

$$C^j_{mm} = 1, \quad N^j_{mm} = 1, \quad n^j_{mm} = 1 . \tag{39}$$

We note that if incidentally $m-n = 1,2,3\ldots$ the Q-function becomes infinite, however the factor $\sin\pi(m-n)$ in C^j_{mn} is zero of the same order and therefore the R-function in eq. (38) remains finite. (In view of the identity $\sin\pi u = \pi/\Gamma(u)\Gamma(1-u)$ the above statement can be repeated for the factor $1/\Gamma(1-m+n)$ instead of $\sin\pi(m-n)$. It would have been possible to define the Q-function with the co-factor $1/\Gamma(1-m+n)$ from the very beginning as it is done in the theory of spherical functions. It is remarkable that this factor is produced automatically by the regularity requirements on the z-plane).

The generalization of eq.(38) for arbitrary (allowed by eq. 7) values of m and n is straightforward. Instead of listing the symmetry properties of the P and Q functions for different values of sign $\mathrm{Re}(m+n)$ and sign $\mathrm{Re}(m-n)$ (as it is done for real m and n in ref./6/) we give P and Q for arbitrary m and n in a unified form. To this end the symbol $\|u\|$ will be introduced (u is an arbitrary complex number) which is defined by

$$\|u\| = \begin{cases} u & \text{if } \mathrm{Re}\, u \geq 0 \\ -u & \text{if } \mathrm{Re}\, u < 0 \end{cases} .$$

Using this symbol the general form of P and Q is

$$P^j_{mn}(z) = n^j_{mn}\left(\frac{1-z}{2}\right)^{\frac{M-N}{2}}\left(\frac{1+z}{2}\right)^{\frac{M+N}{2}} F\left(-j+M, j+M+1, 1+M-N; \frac{1-z}{2}\right) \quad (40)$$

$$Q^j_{mn}(z) = n^j_{mn}\left(\frac{1-z}{2}\right)^{-\frac{M-N}{2}}\left(\frac{1+z}{2}\right)^{\frac{M+N}{2}} F\left(-j+N, j+N+1, 1-M+N; \frac{1-z}{2}\right) =$$

$$= n^j_{mn}\left(\frac{1-z}{2}\right)^{-\frac{M-N}{2}}\left(\frac{1+z}{2}\right)^{-\frac{M+N}{2}} F\left(-j-M, j-M+1, 1-M+N; \frac{1-z}{2}\right), \quad (41)$$

where

$$n^j_{mn} = \frac{1}{\Gamma(1+M-N)}\sqrt{\frac{\Gamma(j+M+1)\Gamma(j-N+1)}{\Gamma(j+N+1)\Gamma(j-M+1)}}$$

and

$$M = \frac{1}{2}(\|m+n\| + \|m-n\|), \quad N = \frac{1}{2}(\|m+n\| - \|m-n\|) .$$

Equation (27) remains valid and instead of (38) we get now

$$R^{jj^*}_{mm^*;nn^*} = \frac{N^j_{mn}}{4i\sqrt{\sin\pi(M-N)\sin\pi(M^*-N^*)}}\left[C^j_{mn}P^{j^*}_{m^*n^*}(z^*)Q^j_{mn}(z) - C^{j^*}_{m^*n^*}P^j_{mn}(z)Q^{j^*}_{m^*n^*}(z^*)\right] \quad (42)$$

where

$$C^j_{mn} = \frac{\sin\pi(M-N)\sin\pi(j-M)}{\pi(M-N)\sin\pi(j-N)}\,\Gamma(j+M-N)^2\,\frac{\Gamma(j-M+1)\Gamma(j+N+1)}{\Gamma(j-N+1)\Gamma(j+M+1)}$$

$$N^j_{mn} = \sqrt{\frac{\sin\pi(j-N)\sin\pi(j^*-N^*)}{\sin\pi(j-M)\sin\pi(j^*-M^*)}} :$$

Let g be an element of the Lorentz group characterized by the parameters $g = (\epsilon_1, \epsilon_2; \epsilon_3, \epsilon_1^*, \epsilon_2^*, \epsilon_3^*)$. Then the element inverse to it has the parameters $g^{-1} = (-\epsilon_3, -\epsilon_2, -\epsilon_1, -\epsilon_3^*, -\epsilon_2^*, \epsilon_1^*)$. The unitarity of the matrix $T^{jj^*}_{mm^*;nn^*}(\epsilon, \epsilon^*)$ (see eqs.(27) and (42)) can be checked in the form

$$T(g^{-1})^{jj^*}_{mm^*;nn^*} = [T(g)^{jj^*}_{nn^*;mm^*}]^* .$$

4. Behaviour at the Singular Points

The form of the P and Q functions given by equations (40) and (41) yields directly the asymptotic expansion of R at $z=1$. The leading term at this point is

$$R^{jj^*}_{mm^*;nn^*} \approx \frac{N^j_{mn}|n^j_{mn}|^2}{4i\sqrt{\sin\pi(m-n)\sin\pi(m^*-n^*)}}\left[C^j_{mn} r^{-\frac{m-n}{2}} r^{*\frac{m^*-n^*}{2}} - C^{j^*}_{m^*n^*} r^{\frac{m-n}{2}} r^{*-\frac{m^*-n^*}{2}}\right] \quad (43)$$

where

$$r = \frac{1-z}{2} .$$

In order to investigate the limit $r \to 0$ the following lemma will be proved. If k and ℓ are integers (or half-integers) x and x_0 real, then

$$\lim_{r\to 0} A_{k\ell}(x-x_0, r) \equiv \lim_{r\to 0} \frac{|r|^{-i\frac{x-x_0}{2}} - |r|^{i\frac{x-x_0}{2}}}{4i\sqrt{\sin^2\pi\frac{k-\ell}{2}\, ch^2\pi\frac{x-x_0}{2} + \cos^2\pi\frac{k-\ell}{2}\, sh^2\pi\frac{x-x_0}{2}}} = \delta_{k\ell}\delta(x-x_0).$$

Really, be $\phi(x)$ an element of the class of functions K x) which has a support not containing the point x_0, then as a consequence of the Riemann-Lebesgue lemma we have

$$\lim_{r\to 0} \int A_{k\ell}(x-x_0, r)\phi(x)\, dx = 0 .$$

On the other hand using the formula

$$\int_{-\infty}^{\infty} \frac{\sin ax}{sh\,\beta x}\, dx = \frac{\pi}{\beta}\, th\, \frac{a\pi}{2\beta} \qquad (\mathrm{Re}\,\beta > 0)$$

we get

$$\lim_{r\to 0} \int_{-\infty}^{\infty} A_{k\ell}(x-x_0, r)\, dx = \begin{cases} 0 & \text{if } k \neq \ell \\ 1 & \text{if } k = \ell \end{cases}$$

which proves the lemma.

Rewrite eq. (43) in the form

$$R^{jj^*}_{mm^*;\,nn^*} \approx \frac{N^j_{mn}|n^j_{mn}|^2}{4\pi i\sqrt{(\frac{\mu-\kappa}{2})^2 + (\frac{\nu-\lambda}{2})^2}}\left(|r|^{-i\frac{\nu-\lambda}{2}} - |r|^{i\frac{\nu-\lambda}{2}}\right) +$$

$$+ \frac{N^j_{mn}|n^j_{mn}|^2}{4\pi i\sqrt{(\frac{\mu-\kappa}{2})^2 + (\frac{\nu-\lambda}{2})^2}}\left[(C^j_{mn}-1)|r|^{-i\frac{\nu-\lambda}{2}} - (C^j_{m^*n^*}-1)|r|^{i\frac{\nu-\lambda}{2}}\right].$$

---

x) $\phi \in K$ if it has continuous derivatives of any order and zero outside a bounded region. See /5/.

According to the lemma and eq. (39) the first term tends to $\delta_{\mu\kappa}\delta(\nu-\lambda)$ while the second tends to zero as $r \to 0$. Comparing this result with eq. (27) we find that

$$T^{jj^*}_{mm^*;\,nn^*}(\epsilon,\epsilon^*)\Big|_{\epsilon=0} = \delta_{\mu\kappa}\delta(\nu-\lambda) .$$

Using the analytic continuation of the hypergeometric function we have

$$P^j_{mn}(z) = \alpha_1 P^j_{m,-n}(-z) + \alpha_2 Q^j_{m,-n}(-z)$$

$$Q^j_{mn}(z) = \beta_1 P^j_{m,-n}(-z) + \beta_2 Q^j_{m,-n}(-z) ,$$

where

$$\alpha_1 = \frac{\Gamma(1+M+N)\,\Gamma(-M-N)}{\Gamma(-j-N)\,\Gamma(j+N+1)} \qquad \alpha_2 = \frac{\Gamma(1+M+N)\Gamma(j-N+1)\Gamma(M+N)}{\Gamma(j+N+1)\,\Gamma(-j+M)\,\Gamma(j+M+1)}$$

$$\beta_1 = \frac{\Gamma(1+M+N)\Gamma(j-N+1)\Gamma(1-M+N)\Gamma(-M-N)}{\Gamma(1+M-N)\Gamma(j+N+1)\Gamma(j-M+1)\Gamma(-j-M)} \quad \beta_2 = \frac{\Gamma(1-M+N)\Gamma(1+M+N)\Gamma(j-N+1)\Gamma(M+N)}{\Gamma(1+M-N)\Gamma(j+N+1)^2\,\Gamma(-j+N)} .$$

Substituting these expressions of P and Q into eq. (42) we get the behaviour of the R-function at $z = -1$:

$$R^{jj^*}_{mm^*;\,nn^*}(z,z^*) = \frac{N^j_{mn}}{4i\sqrt{\sin\pi(M-N)\sin\pi(M^*-N^*)}}\{C^j_{mn}[\alpha_1^*\beta_1 P^{j^*}_{m^*,-n^*}(-z^*)P^j_{m,-n}(-z) +$$

$$+\alpha_2^*\beta_2 Q^{j^*}_{m^*,-n^*}(-z^*)P^j_{m,-n}(-z) + \alpha_1^*\beta_2 P^{j^*}_{m^*,-n^*}(-z^*)Q^j_{m,-n}(-z)] - \text{compl.conj.}\} . \qquad (44)$$

For our purposes it will be sufficient to give only the leading terms of the asymptotic expansion at $z = \infty$. (As to the derivation of the exact formula use eqs. 2.9 (34) and 2.9 (42) of ref. /8/)

$$R^{jj^*}_{mm^*;nn^*} \approx \frac{N^j_{mn} |a^j_{mn}|^2 e^{-i\pi\sigma\frac{1}{2}(M-N-M^*+N^*)}}{4i\sqrt{\sin\pi(M-N)\sin\pi(M^*-N^*)}} \Big[\big(\gamma_1 e^{i\pi\sigma(M-N)} - \gamma_2 e^{-i\pi\sigma(M^*-N^*)}\big)\left(\frac{z-1}{2}\right)^{-j-1}\left(\frac{z^*-1}{2}\right)^{j^*} - \tag{45}$$

$$-\big(\gamma_1^* e^{-i\pi\sigma(M^*-N^*)} - \gamma_2^* e^{i\pi\sigma(M-N)}\big)\left(\frac{z-1}{2}\right)^{j}\left(\frac{z^*-1}{2}\right)^{-j^*-1}\Big], \quad (|z| \gg 1, |\arg(z-1)| < \pi)$$

$$\gamma_1 = C^j_{mn} \frac{\Gamma(1+M^*-N^*)\Gamma(2j^*+1)\Gamma(1-M+N)\Gamma(-2j-1)}{\Gamma(j^*-N^*+1)\Gamma(j^*+M^*+1)\Gamma(-j-M)\Gamma(-j+N)}, \quad \gamma_2 = C^{j^*}_{m^*n^*} \frac{\Gamma(1-M^*+N^*)\Gamma(2j^*+1)\Gamma(1+M-N)\Gamma(-2j-1)}{\Gamma(j^*-M^*+1)\Gamma(j^*+N^*+1)\Gamma(-j-N)\Gamma(-j+M)}$$

$$(\sigma \equiv \text{sign Im}(z-1)).$$

5. Harmonic Analysis of Function on the Group

By making use of equation (24) we obtain the following (left or right) Haar measure in terms of the parameters (3)

$$\int dg f(g) = \frac{1}{2} \int dp_1 dq_1 dp_2 dq_2 dp_3 dq_3 (\text{ch } 2q_2 - \cos 2p_2) f(p,q) = \tag{46}$$

$$= \int dp_1 dq_1 dp_2 dq_2 dp_3 dq_3 \sin\epsilon_2 \sin\epsilon_2^* \; f(p,q)$$

where the limits of integration are given by eq. (4). The scalar product for the principal series is defined by

$$(\phi, \psi) = \int dg \, \phi^*(g) \psi(g). \tag{47}$$

By the aid of the asymptotic formula (45) the following orthogonality and completeness relations can be derived:

$$\int dg \left(T^{j'j'^*}_{m'm'^*;n'n'^*}\right)^* T^{jj^*}_{mm^*;nn^*} = \frac{32\pi^4}{(2j+1)(2j^*+1)} \delta_{\mu'\mu}\delta_{\kappa'\kappa}\delta(\nu'-\nu)\delta(\lambda'-\lambda)\delta_{\ell_0'\ell_0}\delta(p'-p) \tag{48}$$

$$(p' \geq 0, \; p \geq 0)$$

where

$$j = \frac{1}{2}(\ell_0 - 1 + ip), \quad m = \frac{\mu + i\nu}{2}, \quad n = \frac{\kappa + i\lambda}{2}$$

(the factor $(2j+1)(2j^*+1)$ on the right side plays the role of "dimension" of the representation)

$$\frac{1}{32\pi^4} \sum_{\ell_0=-\infty}^{\infty} \sum_{\mu,\kappa=-\infty}^{\infty} \int_0^{\infty} dp \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\lambda (2j+1)(2j^*+1)\left(T(g')^{jj^*}_{mm^*;nn^*}\right)^* T(g)^{jj^*}_{mm^*;nn^*} = \delta(g'-g), \tag{49}$$

where $\delta(g'-g)$ is defined by

$$\int dg' \delta(g'-g) f(g') = f(g).$$

Explicitly:

$$\delta(g'-g) = \delta(p_1'-p_1)\delta(q_1'-q_1)\delta(\cos p_2' \text{ ch } q_2' - \cos p_2 \text{ ch } q_2) \times$$

$$\times \delta(\sin p_2' \text{ sh } q_2' - \sin p_2 \text{ sh } q_2)\delta(p_3'-p_3)\delta(q_3'-q_3).$$

All the summations in eq. (49) must be carried out over integer and half-integer values.

Equations (48) and (49) give the following formula of Fourier expansion of a square integrable function on the group

$$f(g) = \frac{1}{32\pi^4} \sum_{\ell_0=-\infty}^{\infty} \sum_{\mu,\kappa=-\infty}^{\infty} \int_0^{\infty} dp \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\lambda \, (2j+1)(2j^*+1) F(j,j^*;m,m^*;n,n^*) T(g)^{jj^*}_{mm^*;nn^*}$$

The inversion formula reads

$$F(j,j^*;m,m^*;n,n^*) = \int dg\, f(g)\left(T(g)^{jj^*}_{mm^*;nn^*}\right)^* . \tag{51}$$

Related convergence and other subtle questions see ref. /7/.

6. Spherical Functions

Spherical functions with respect to a subgroup H of the group G are defined on a certain homogeneous space X which has a certain fixed point x having the little group H. As it has been discussed in the section 2 each point of the homogeneous space X characterizes the factor group G/H. Elements of the group G can be characterized by an element of H and by a point of X. We have to find a homogeneous space with the above properties for $G = L^{\uparrow}_{+}$, $H = 0(2) \times 0(1,1)$.

Consider the antisymmetric tensor formed by the three-vectors $\vec{x}, \vec{y}$:

$$S_{\mu\nu} = \begin{bmatrix} 0 & -y_1 & -y_2 & -y_3 \\ y_1 & 0 & x_3 & -x_2 \\ y_2 & -x_3 & 0 & x_1 \\ y_3 & x_2 & -x_1 & 0 \end{bmatrix} \tag{52}$$

Under the Lorentz transformation $g_\alpha^{\ \beta}$, $S_{\mu\nu}$ transforms as

$$S'_{\mu\nu} = g_\mu^{\ \alpha} g_\nu^{\ \beta} S_{\alpha\beta} \tag{53}$$

Let us form a complex three-dimensional sphere from the quantities $z_k = x_k + i y_k$, $z_k^* = x_k - i y_k$:

$$\vec{z}^{\,2} = z_1^2 + z_2^2 + z_3^2 = \vec{x}^2 - \vec{y}^2 + 2i\vec{x}\vec{y} = r^2$$
$$\vec{z}^{*2} = z_1^{*2} + z_2^{*2} + z_3^{*2} = \vec{x}^2 - \vec{y}^2 - 2i\vec{x}\vec{y} = r^{*2}. \tag{54}$$

Points on this sphere are characterized by the quantities $\vec{x}$ and $\vec{y}$ or by $\vec{z}$ and $\vec{z}^*$. As it is well known both the quantities $\vec{x}^2 - \vec{y}^2$, $\vec{x}\vec{y}$ are invariant under the Lorentz transformation (53) and thus the surface of the sphere (54) is invariant. We mention that it can be shown that eq. (53) describes the transformation of z_i coinciding with the three-dimensional representation (in Cartesian basis) of the rotation group but instead of real Euler angles we have to put complex ones. Thus, the most general Lorentz transformation of an antisymmetric tensor (complex vector) can be performed by the familiar technique of spatial rotations.

The homogeneity of the space X can be proved simply by showing that each point on the sphere can be transformed into the point $(0,0,r)$. (We exclude the case when $\vec{x}$ and $\vec{y}$ has the same length and are perpendicular to each other. In this case $\vec{x}^2 - \vec{y}^2$ and $\vec{x}\vec{y}$ possess the above property in any frame of reference. At the same time both the invariants $\vec{x}^2 - \vec{y}^2$, $\vec{x}\vec{y}$ become zero and the complex sphere is deformed to a complex sphere of zero radius, which is actually the intersection of two real cones. The "north pole" of this surface is the origin that must be excluded).

It is seen from the invariance of $\vec{x}\vec{y}$ that if $\mathrm{Re}\, r$ and $\mathrm{Im}\, r$ have the same (opposite) singns than each $\vec{x}$ and $\vec{y}$ on the sphere $(\vec{x} + i\vec{y})^2 = r^2$ form acute (obtuse) angle. In other words, in the course of Lorentz transformations $\vec{x}$ and $\vec{y}$ cannot pass the perpendicular position. This fact is well known also from electrodynamics.

Consider the subgroup H consisting of spatial rotations about the third axis and boosts along the third axis:

$$h = \begin{bmatrix} \mathrm{ch}\, q_1 & 0 & 0 & \mathrm{sh}\, q_1 \\ 0 & \cos p_1 & \sin p_1 & 0 \\ 0 & -\sin p_1 & \cos p_1 & 0 \\ \mathrm{sh}\, q_1 & 0 & 0 & \mathrm{ch}\, q_1 \end{bmatrix} \quad (0 \le p_1 < 2\pi,\ -\infty < q_1 < \infty).$$

Substituting this into equation (53) it is readily shown that the subgroup H constitutes the little group of the point $z_0 = (0,0,r): h z_0 = z_0$. (And conversely: each element of the Lorentz group leaving the point z_0 unaltered, has the above form). So the complex sphere possesses all the required properties and can be considered as a domain of the spherical functions.

Introducing polar coordinates in the usual way

$$z_1 = r \sin\theta \cos\phi \qquad z_1^* = r^* \sin\theta^* \cos\phi^*$$

$$z_2 = r \sin\theta \sin\phi \qquad z_2^* = r^* \sin\theta^* \sin\phi^*$$

$$z_3 = r \cos\theta \qquad z_3^* = r^* \cos\theta^* \qquad (r \neq 0),$$

by the complex angles $\theta = \theta_1 + i\theta_2$, $\phi = \phi_1 + i\phi_2$ we have labelled the G/H factor group by 4(real) parameters. The remaining 2 angles are contained by the subgroup H . Representing the infinitesimal generators by differential operators on the complex sphere we arrive at the Casimir operators x):

$$-J^2 = \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta}$$

$$-K^2 = \frac{1}{\sin^2\theta^*}\frac{\partial^2}{\partial\phi^{*2}} + \frac{\partial^2}{\partial\theta^{*2}} + \cot\theta^*\frac{\partial}{\partial\theta^*}.$$

---

x) The Casimir operator of the group can be treated in an analogous geometic way. Writing an element of the SL(2,C)group in the form

$$\begin{bmatrix} z_0 + i z_3 & z_1 + i z_2 \\ -z_1 + i z_2 & z_0 - i z_3 \end{bmatrix}$$

we obtain $z_0^2 + z_1^2 + z_2^2 + z_3^2 = 1$, $z_0^{*2} + z_1^{*2} + z_2^{*2} + z_3^{*2} = 1$ and thus the parameter space of the group is topologically homeömorphic to the four dimensional complex sphere. Introducing the parameters

$z_0 = \cos\frac{\epsilon_2}{2}\cos\frac{\epsilon_3+\epsilon_1}{2}$, $z_1 = \sin\frac{\epsilon_2}{2}\sin\frac{\epsilon_3-\epsilon_1}{2}$, $z_2 = \sin\frac{\epsilon_2}{2}\cos\frac{\epsilon_3-\epsilon_1}{2}$, $z_3 = \cos\frac{\epsilon_2}{2}\sin\frac{\epsilon_3+\epsilon_1}{2}$,

the Laplaceans $\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}\, g^{\mu\nu}\partial_\nu$, $\frac{1}{\sqrt{g^*}}\partial_\mu^*\sqrt{g^*}\, g^{\mu\nu*}\partial_\nu^*$ on this sphere yield the Casimir operators (25).

The spherical functions are simultaneous eigenfunctions of $\vec{J}^2$ and $\vec{K}^2$

$$[\vec{J}^2 - j(j+1)]\, f^{jj^*}_{mm^*}(\theta,\phi) = 0$$

$$[\vec{K}^2 - j^*(j^*+1)]\, f^{jj^*}_{mm^*}(\theta,\phi) = 0.$$

The well behaved solutions of these equations are the functions: $(T^{jj^*}_{m,m^*;0,0}(\phi,\theta,0))^*$. Choosing a suitable normalization factor we obtain

$$f^{jj^*}_{mm^*}(\theta,\phi) = \sqrt{\frac{8\pi^2}{(2j+1)(2j^*+1)}}\,(T^{jj^*}_{m,m^*;0,0}(\phi,\theta,0))^*. \qquad (55)$$

Introducing the notations

$$p^j_m(z) = P_{m\rho}(z) \qquad q^j_m(z) = Q^j_{m,0}(z)$$

(see eqs. (32) (33)) the following recurrance relations can be derived

$$\sqrt{1-z^2}\frac{dp^j_m}{dz} = -\frac{mz}{\sqrt{1-z^2}}p^j_m + \sqrt{(j-m)(j+m+1)}\,p^j_{m+1} = \frac{mz}{\sqrt{1-z^2}}p^j_m - \sqrt{(j+m)(j-m+1)}\,p^j_{m-1}$$

$$\sqrt{1-z^2}\frac{dq^j_m}{dz} = -\frac{mz}{\sqrt{1-z^2}}q^j_m + \frac{m(m+1)}{\sqrt{(j-m)(j+m+1)}}q^j_{m+1} = \frac{mz}{\sqrt{1-z^2}}q^j_m - \frac{[(j+m)(j-m+1)]^{3/2}}{m(m-1)}q^j_{m-1}.$$

By the aid of these formulas and eq. (24) it is straightforward to show that the spherical functions (55) fulfill the equation (15), so they transform according to the irreducible unitary representation of the Lorentz group.

The scalar product of functions ϕ and ψ on the complex sphere we define as

$$\int d\Omega\,\phi^*(\Omega)\psi(\Omega) = \frac{1}{2}\int d\phi_1\, d\phi_2\, d\theta_1\, d\theta_2\, (\mathrm{ch}\, 2\theta_2 - \cos 2\theta_1)\phi^*(\Omega)\psi(\Omega)$$

with the limits of integration

$$0 \leq \phi_1 < 2\pi \qquad -\infty < \phi_2 < \infty$$

$$0 \leq \theta_1 < \pi \qquad -\infty < \theta_2 < \infty.$$

The orthogonality and completeness relations for the spherical functions read:

$$\int d\Omega \, (f^{j'j'*}_{m'm'*} (\theta,\phi))^* f^{jj*}_{mm*}(\theta,\phi) = \delta_{\mu'\mu}\, \delta(\nu'-\nu)\, \delta_{\ell_0' \ell_0}\, \delta(p'-p) ,$$

$$(p' \geq 0 , \; p \geq 0)$$

$$\sum_{\ell=-\infty}^{\infty} \sum_{\mu=-\infty}^{\infty} \int_0^{\infty} dp \int_{-\infty}^{\infty} d\nu \, (f^{jj*}_{mm*}(\theta',\phi'))^* f^{jj*}_{mm*}(\theta,\phi) = \delta(\Omega'-\Omega) ,$$

where $\delta(\Omega'-\Omega)$ is the δ -function on the complex sphere, namely

$$\delta(\Omega'-\Omega) = \delta(\phi_1'-\phi_1)\delta(\phi_2'-\phi_2)\delta(\cos\theta_1' \operatorname{ch}\theta_2' - \cos\theta_1 \operatorname{ch}\theta_2)\delta(\sin\theta_1' \operatorname{sh}\theta_2' - \sin\theta_1 \operatorname{sh}\theta_2).$$

Acknowledgements. Authors wish to express their gratitude to doctors M. Lieberman and M. Sheftel for a number of stimulating discussions.

Appendix

The space-time coordinates transform in the following way

$$x^{\mu'} = g^{\mu}_{\nu} \, x^{\nu} .$$

The matrix elements g^{μ}_{ν} in terms of the six parameters used throughout this paper have the form:

$$g^0_{\ 0} = \operatorname{ch} q_1 \operatorname{ch} q_2 \operatorname{ch} q_3 + \operatorname{sh} q_1 \cos p_2 \operatorname{sh} q_3$$

$$g^1_{\ 1} = \cos p_1 \operatorname{ch} q_2 \cos p_3 - \sin p_1 \cos p_2 \sin p_3$$

$$g^2_{\ 2} = -\sin p_1 \operatorname{ch} q_2 \sin p_3 + \cos p_1 \cos p_2 \cos p_3$$

$$g^3_{\ 3} = \operatorname{sh} q_1 \operatorname{ch} q_2 \operatorname{sh} q_3 + \operatorname{ch} q_1 \cos p_2 \operatorname{ch} q_3$$

$$g^1_{\ 0} = -\operatorname{sh} q_1 \sin p_2 \sin p_3 - \operatorname{ch} q_1 \operatorname{sh} q_2 \cos p_3$$

$$g^0_{\ 1} = -\cos p_1 \operatorname{sh} q_2 \operatorname{ch} q_3 - \sin p_1 \sin p_2 \operatorname{sh} q_3$$

$$g^2_{\ 0} = \operatorname{ch} q_1 \operatorname{sh} q_2 \sin p_3 - \operatorname{sh} q_1 \sin p_2 \cos p_3$$

$$g^0_{\ 2} = -\sin p_1 \operatorname{sh} q_2 \operatorname{ch} q_3 + \cos p_1 \sin p_2 \operatorname{sh} q_3$$

$$g^3_{\ 0} = -\operatorname{ch} q_1 \operatorname{ch} q_2 \operatorname{sh} q_3 - \operatorname{sh} q_1 \cos p_2 \operatorname{ch} q_3$$

$$g^0_{\ 3} = -\operatorname{sh} q_1 \operatorname{ch} q_2 \operatorname{ch} q_3 - \operatorname{ch} q_1 \cos p_2 \operatorname{sh} q_3$$

$$g^2_{\ 1} = -\cos p_1 \operatorname{ch} q_2 \sin p_3 - \sin p_1 \cos p_2 \cos p_3$$

$$g^1_{\ 2} = \sin p_1 \operatorname{ch} q_2 \cos p_3 + \cos p_1 \cos p_2 \sin p_3$$

$$g^3_{\ 2} = -\cos p_1 \sin p_2 \operatorname{ch} q_3 + \sin p_1 \operatorname{sh} q_2 \operatorname{sh} q_3$$

$$g^2_{\ 3} = \operatorname{ch} q_1 \sin p_2 \cos p_3 - \operatorname{sh} q_1 \operatorname{sh} q_2 \sin p_3$$

$$g^1_{\ 3} = \operatorname{ch} q_1 \sin p_2 \sin p_3 + \operatorname{sh} q_1 \operatorname{sh} q_2 \cos p_3$$

$$g^3_{\ 1} = \sin p_1 \sin p_2 \operatorname{ch} q_3 + \cos p_1 \operatorname{sh} q_2 \operatorname{sh} q_3 .$$

It can be checked that $\det g = +1$ and $g^0_0 > 0$ as it must be for the $L^{\uparrow}_{+}$ group. It seems to the authors that this parametrization is the simplest one whenever combined boost and rotation transformation are required.

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Received by Publishing Department
on December 25, 1968.