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SOME PROPERTY OF QUANTUM FIELDS AS OPERATORVALUED DISTRIBUTIONS

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The concept of Heisenberg field is introduced according /1,2/. The space of test functions is denoted Φ . For each test function $f \subset \Phi$, defined on space time, there exists a set $A_\alpha(f)$ of linear operators acting in Hilbert space $\mathcal{H}$. The mapping $f \to A_\alpha(f)$ is linear. There exists for each α β such that $A_\alpha(f) = A_\beta(f)$.

Operators $A_\alpha(f)$ are defined on a domain D of vectors, dense in $\mathcal{H}$. Furthermore, D is a linear set containing the vacuum Ψ_0 and $A_\alpha(f)$ carry vectors in D into vectors in D , $A(f)D \subseteq D$ for each α and f .

The space of linear continuous functionals on the test functions is denoted Φ^x . If $\Psi_1, \Psi_2 \subset D$ then $(\Psi_1, A_\alpha(f)\Psi_2)$ regarded as a functional of D , belongs to Φ . Commonly it is assumed that $\Phi = S$ and $(\Psi_1, A_\alpha(f)\Psi_2)$ is a tempered distribution. S is a complete countably normed space. It is the only property of S which is essential for our consideration S_0 , we assume Φ being arbitrary complete countably normed space and $(\Psi_1, A_\alpha(f)\Psi_2)$ being an operatorvalued distribution on Φ .

For the definition and general properties of spaces Φ and Φ^x one is referred to[3]. A complete countably normed space Φ is an intersection of Banach spaces Φ_p , $\Phi_p \supseteq \Phi_{p+1}$. Then we have $\Phi^x = \bigcup_{p=1}^{\infty} \Phi_p^x$, $\Phi_{p+1}^x \supseteq \Phi_p^x$.

We shall prove that for every field $A_\alpha(f)$ there exists such p, that for arbitrary $\Psi_1, \Psi_2 \subset D$ $(\Psi_1, A_\alpha(f)\Psi_2)$ belongs to Φ_p.

This means that if we suppose $A_\alpha(f)$ to be an operatorvalued distribution on a complete countably normed space $\Phi = \bigcap_{p=1}^{\infty} \Phi$, then $A_\alpha(f)$ should be necessarily an operatorvalued distribution on some Banach space Φ_p. The separability of $\mathcal{H}$ is essential point in our proof. The separability of $\mathcal{H}$ is a consequence of separability of Φ, irreducibility of fields and countability of the set of the values of α. Irreducibility of fields means that the vacuum state is cyclic for the smeared fields, that is, polynomials in the smeared fields $P(A_{\alpha_1}(f_1)A_{\alpha_2}(f_2)\ldots)$, when applied to the vacuum state, yield a set D_0 of vectors dense in the $\mathcal{H}$, if functions $f_1, f_2 \ldots$, run through all the spaces Φ. It is evident that at every f_1 running through some sequence of functions, dense in Φ, we obtain a countably set of vectors dense in $\mathcal{H}$. This is the consequence of continuity of matrix elements of fields as distributions.

Let Ψ_k be complete orthonormal system, belonging to D. Now we prove theorem I: there exists such p that all $A_{kn}(f) = (\Psi_k, A_\alpha(f)\Psi_n)$ lie in Φ_p^x for arbitrary values k and n.

The matrix element $(\Psi, A_\beta(f_1)A_\alpha(f)A_\beta(f_1)\Psi)$, $\Psi \subset D$, is a distribution on f due to $A_\beta(f_1)\Psi$, $A^x_\beta(f_1)\Psi \subset D$.

The function $f_1 \subset \Phi$ is arbitrary one but fixed. Now we have: $(\Psi, A_\beta(f_1)A_\alpha(f)A^x_\beta(f_1)\Psi) =$

$$= \sum_{k,n} (\Psi, A_\beta(f_1)\Psi_k)(\Psi_n, A_\beta(f_1)\Psi)\, A_{kn}(f) =$$

$$= \sum_{k,n} c_k\, c^x_n\, A_{kn}(f), \qquad c^x_k = (\Psi_k, A_\alpha(f)\Psi)$$

The double of series of distributions converges weakly in Φ^{x}. We shall use a theorem[3]: a sequence of distributions converges in a weak sense in Φ^{x} only if all members of sequence are belonging to the same Banach space Φ_p^{x}. In order to apply this theorem we must prove that we can suppose all $C_k \neq 0$. Let Ψ'_k be some orthonormal system, $\Psi'_k \subset D$, such that every vector Ψ_n is a finite linear combination of Ψ'_k. It is evident that if all distributions $(\Psi'_k, A_\alpha(f)\Psi'_n)$ are belonging to some Φ_p^{x}, then all distributions $(\Psi_k, A_\alpha(f)\Psi_n)$ are belonging to the same Φ_p^{x}.

Let $f_1 \subset \Phi$ possess the property that $C_k = (\Psi, A_\beta(f_1)\Psi_k) \neq 0$ for infinite set of values k.

Now it is possible to establish one to one correspondence between all $\Psi^{(1)}_{\ell_k}$, such that $C_{\ell_k} = 0$, and all $\Psi^{(2)}_{m_k}$, such that $C_{m_k} \neq 0$. Let us introduce the following vectors:

$$\Psi'_{2n-1} = \frac{\Psi^{(1)}_{\ell_n} + \Psi^{(2)}_{m_n}}{\sqrt{2}}$$

$$\Psi'_{2n} = \frac{\Psi^{(1)}_{\ell_n} - \Psi^{(2)}_{m_n}}{\sqrt{2}}$$

Vectors Ψ'_n form a complete orthonormal system. All Ψ'_n are belonging to D. It is evident, that $C'_k = (\Psi, A_\beta(\Psi_1)\Psi'_k) \neq 0$ for every k. Now the number n_{f_1} is introduced such that $(\Psi, A_\beta(f_1)\Psi_k) = 0 \quad k > n_{f_1}$. We have $n_f = n_{af}$. Let a sequence f_k converge to f in Φ. Then only two cases are possible: 1) numbers n_{f_ℓ} are bounded from above by some H, 2) there exists $f' \subset \Phi$ such that $n_{f'} = \infty$. We have:

$$\phi_k = f_{k+1} - f_k \qquad \sum_k \phi_k = f$$

$$(\Psi, A_\beta(\phi_\ell)\Psi_n) = (\phi, A_\beta(f_{\ell+1})\Psi_n) - (\Psi, A_\beta(f_\ell)\Psi_n).$$

We can choose such a sequence f_k, that $n_{\phi_\ell} \geq n_{f_{\ell+1}}$.
If $n_{\phi_\ell} \leq n_{f_{\ell+1}}$ then

$$(\Psi, A_\alpha(\phi_\ell)\Psi_{n_{f_{\ell+1}}}) = (\Psi, A_\alpha(f_{\ell+1})\Psi_{n_{f_{\ell+1}}}) - (\Psi, A_\alpha(f_\ell)\Psi_{n_{f_{\ell+1}}}) = 0.$$

In consequence of continuity of the multiplication on c - numbers in Φ, a sequence $f'_k = \alpha_k f_k$, $\alpha_k \subset (1-\frac{1}{2^k}, 1+\frac{1}{2^k})$ converges to f. We denote $\phi'_k = f'_{k+1} - f'_k$, $\sum_k \phi'_k = f$.
Numbers α_k Can be always chosen in such a way that $n_{\phi'_k} \geq n_{f'_{k+1}} = n_{f_{k+1}}$. In consequence of continuity of multiplication on c - numbers in Φ, if the series $\sum_k \phi'_k$ converges, then every series $\sum_k \beta_k \phi'_k$, $\beta_k \subset (1-\frac{1}{2^k}, 1+\frac{1}{2^k})$. m $\sum \beta_k \phi'_k = f'$ converges too. We accept all β_m, $m < \ell$, to be chosen so that $C_{n_{\phi_m}} = (\Psi, A_\alpha(f')\Psi_{n_{\phi_m}}) \neq 0$
and

$$C_{n_{\phi_\ell}} = (\Psi \; A_\alpha(f')\Psi_{n_{\phi_\ell}}) = 0.$$

Due to this we can always choose $\beta_\ell \subset (1-\frac{1}{2^\ell}, 1+\frac{1}{2^\ell})$ so, that all $C_{n_{\phi_m}} \neq 0$, $m \leq \ell$. Therefore sequence β_k can be chosen so that all $C_{n_\phi} = (\Psi, A_\alpha(f')\Psi_{n_{\phi_\ell}}) \neq 0$, for every ℓ.
If numbers n_{ϕ_ℓ} are not bounded, then $n_{f'} = \infty$.

Let a sequence $\chi_k \subset D$ have the property that $n_{\chi_k} \to \infty$. Because of continuity of multiplication on c - numbers in Φ, we can choose sequence $\gamma_k \neq 0$ so, that $\chi'_k = \gamma_k \chi_k \to 0$. We have $n_{\chi'_k} = n_{\chi_k}$. Thus only two cases are possible: 1) there exists such $f' \subset \Phi$, that $n_{f'} = \infty$, 2) all numbers n_f are bounded from above by some H, when f runs through the all space Φ. In the second case the set of the vectors $A(f)\Psi$, f running through the all space Φ, is finite dimensional for each $\Psi \subset D$, for example $\Psi = \Psi_0$. This is meaningless.

So it was proved that there exists such $f_1 \subset \Phi$ that all $C_n = (\Psi, A_\alpha(f_1)\Psi_n) \neq 0$.

We introduce the notations:

$$\sum_{n=1}^{H} C_n^x A_{kn}(f) = B_{kH}(f), \quad \sum_{n=1}^{\infty} C_n^x A_{kn}(f) = B_k(f).$$

We have:

$$(\Psi, A_\beta(f_1) A_\alpha(f) A_\beta^x(f_1)\Psi) = \sum_{k,n} C_k C_n^x A_{kn}(f) =$$

$$= \sum_k C_k B_k(f).$$

The series $\sum_k C_k B_k(f)$ converges weakly in Φ^x. Because of continuity of multiplication on c-numbers and addition in Φ^x, we can fing for every $B_k(f)$ such a weak neigbourhood U_k, that $B_k(f) \subset U_k$ and every series $\sum_k C_k B'_k(f)$, converges weakly in Φ^x for arbitrary choice of $B'_k(f) \subset U_k$. There exists such p that all U_k are belonging to Φ_p^x. If such p doesn't exist, we can find such a sequence $B'_k(f)$, that $B'_k(f) \subset U_k$ and $B'_k(f) \not\subset \Phi_k$. The series $\sum_k C_k B'_k(f)$ must converge weakly in Φ^x. The series can converge weakly in Φ^x only if all $B'_k(f)$ are belonging to the same Φ_p^x, because all $C_k \neq 0$. Thus we can find for every k such H_k, that all $B_{kH}(f), A_{kr}(f)$ are belonging to Φ_p^x when $H > H_k$, $n > H_k$, because all $C_n^x \neq 0$.

This property is valid for every complete orthonormal system Ψ_k belonging to D and such that all $C_k = (\Psi, A_\alpha(f)\ \Psi_k) \neq 0$. If all $A(f)$ aren't belonging to the same Φ_p^x., then we can choose such a sequence of pairs k_i, n'_i, that $A_{k'_i n'_i} \not\subset \Phi_i^x$ $n_i < H_{k_i}$, if $i > p$.

The complete orthonormal system Ψ'_k is defined as follows. We choose some number ℓ_i such that $n_i + \ell_i > H_{k_i}$

$$\Psi'_k = \begin{cases} \Psi_k & k \neq n_i \quad k \neq n_i + \ell_i \\ \cos\alpha_i \Psi_{n_i} + \sin\alpha_i \Psi_{n_i+\ell_i} & k = n_i \\ \sin\alpha_i \Psi_{n_i} - \cos\alpha_i \Psi_{n_i+\ell_i} & k = n_i + \ell_i \end{cases}.$$

It is evident that all $C'_k = (\Psi, A_\beta(f_1)\Psi'_k) \neq 0$, if $tg\,a_1 \neq -\frac{C_{n_1}}{C_{n_1+\ell_1}}$, $tg\,a'_1 \neq \frac{C_{n_1+\ell_1}}{C_{n_1}}$.

All
$$B'_{kH}(f) = \sum_{n=1}^{H} C'^x_n A'_{kn}(f) = \sum_{n=1}^{H} C'^x_n (\Psi'_k, A_a(f)\Psi'_n)$$

are belonging to the same Φ^x_p, necessarily, because all $C'_k \neq 0$. Let us take $i > p$, $i > p_1$

$$B'_{k_1 H}(f) = \sum_{n=1}^{H} C^{x'}_n A'_{k_1 n}(f) = \sum_{n=1}^{H'} b_n(a) A_{k_1 n}(f) \quad k_1 \neq n_1 \quad k_1 \neq n_1 + C_1$$
$$B'_{k_1 H}(f) = \sum_{n=1}^{H'} b_n(a) A_{k_1 n}(f) + \sum_{n=1}^{H'} C_n(a) A_{k_1+\ell_1 n}(f) .$$
$$k_1 = n_1 \qquad k_1 = n_1 + \ell_1$$

All $A_{kn}(f)$ are belonging to the same Φ^x_p, if $B'_k n > H_{k_1}$. Therefore, if $B'_{k_1 H_{k_1}}(f) \not\subset \Phi^x_1$, then all $B'_{k_1 H}(f) \not\subset \Phi^x_1$, $H > H_{k_1}$, are not belonging to Φ^x_1. We accept that all $a_m, m < i$ are chosen. $b_n(a)$ is belinear combination of $\cos a_k$, and $\sin a_k$. If for a certain value of a_1: $B_{k_1} H_{k_1} \subset \Phi^x_1$ we can always choose another value of a_1 so, that $B_{k_1} H_{k_1} \not\subset \Phi^x_1$. Therefore the sequence a_1 can be chosen in such a way that all $B'_{kH}(f) \not\subset \Phi_1$ if $H > H_{k_1}$. Thus we obtained the contradiction, as the result of the supposition that there exists such a sequence k_1, n_1, that $A_{k_1 n_{11}}(f) \not\subset \Phi^x_1$.

There Theorem I is proved. Now we prove the following generalization of this theorem.

Theorem II. There exists such p, that for arbitrary,

$$\Psi_1, \Psi_2 \subset D, \ (\Psi_1, A_a(f)\Psi_2) \subset \Phi^x_p .$$

If the assertion in the theorem is not correct then there exists such a sequence $\chi_k \subset D$, that $(\chi_{2n-1}, A_a(f)\chi_{2n}) \not\subset \Phi_n^x$.

It is evident that χ_k can be chosen by different ways. Vectors $\chi'_k, \chi''_k \subset D$ are arbitrary. If for λ_1, some values of λ_2, and

$$(\chi_{2n-1} + \lambda_1 \chi', A_a(f)(\chi_{2n} + \lambda_2 \chi'')) \subset \Phi_n^x$$

then for each another values of λ_1, and λ_2

$$(\chi_{2n-1} + \lambda_1 \chi', A_a(f)(\chi_{2n} + \lambda_2 \chi'')) \not\subset \Phi_n^x .$$

It is evident that χ_k can be chosen so that every subsequence of χ_k is lineary independent. The sequence $\Psi_k = \sum_{n=1}^{k} a_{k_n} \chi_k$ is orthonormal. Using the arbitraryness in choice of χ_k we can choose them so that $(\Psi_{2n-1}, A_a(f)\Psi_{2n}) \not\subset \Phi_n^x$.

If sequence Ψ_n isn't complete, we can join to it some another sequence of vectors from D. So we obtained such complete orthonormal sequence Ψ'_k, that $(\Psi'_k, A_a(f)\Psi'_{k+1}) \not\subset \Phi_{pk}^x$ $p_k \to \infty$.

This is contradictory to theorem I. Theorem II is proved.

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