ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
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ИССЛЕДОВАНИЙ

Дубна

E2 - 4148

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ANALYTIC REPRESENTATIONS OF TWO-POINT FUNCTIONS WITH NONCANONICAL LIGHT CONE SINGULARITIES

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1968

E2 - 4148

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1. Introduction

The interaction modifies the canonical light cone singularity of free two-point functions. This modification is described formally by the wave renormalization constant Z_3^{-1}. Such description, however, becomes mathematically meaningless if Z_3^{-1} is infinite. Treating the results of perturbation theory as a guide, one should expect that Z_3^{-1} in all four-dimensional nontrivial examples of interacting local fields is infinite, and it is necessary to look for another methods of describing the short-distance behaviour of two-point functions.

The renormalization factor Z_3^{-1} describes the modification of the free field functions. New approach should therefore introduce the methods of description of short distance singularities without nay reference to the free field solutions. In this paper we introduce a class of analytic functions, constructed in accordance with the analytic properties of VEV in Wightman scheme[1]. These functions, similarly like in Kallen-Lehman spectral formula, can be used for construction of spectral representation with integrable spectral functions, describing two-point functions with noncanonical singularities.

Using the boundary prescriptions [1-3] one can relate our analytic functions with the two-point functions characterized by some standard nonintegrable Kallen-Lehmann spectral functions. Some particular examples of such two-point functions have been studied in the framework of distribution theory by Stenmann [4]. Güttinger [5], Pffaffelhuber [6], Güttinger and Rieckers [7], and Vladimirov [8]. In this paper we introduce larger classes of such functions, sufficient for introduction the spectral decompositions of any commutator function having the singularities $\delta^{(k)}(x^2)$ and $(x^2)_+^{-k}$ $(k=1,2,\ldots)$ and we use the tools of the theory of analytic functions, what makes all operations unique and well defined[x)].

In last years one of the most fashionable subjects in axiomatic field theory is to study the inequivalent representations of canonical commutation relations [10,11]. It has to be stressed, however, that such method is justified if the wave renormalization constant Z_3^{-1} is finite. In relativistic quantum field theory, therefore, one can not escape from the conclusion that the interaction modifies the algebraic structure of the equal time limits. Because the correct way of calculating the equal time limit leads to one-to-one correspondence between the equal time singularities and the singularities of the four-dimensional com-

x) It has to be mentioned, however, that not all problems of Loretnz invariant distributions can be solved by such approach. In a full treatment of Lorentz-invariant distributions the analytic methods have to be supplemented with the discussion of so-called Gording mapping [9] of invariant four-dimensional distributions, and the discussion of inverse Garding mapping. For an extensive treatment of these mapping see [7].

mutator function on the light cone x), we conclude that studying of light cone singularities for at least the lowest Green's functions represents a programme of classification of interactions in Wightman scheme. In this paper we discuss the two point functions: the case of three point function will be treated in other publication.

In this paper we discuss only the example of scalar neutral Wightman field, but the results can be easily extended to the nonscalar field. Particularly interesting is the case of vector field, because of recent discussions about the validity of current algebra assumptions in the general framework of relativistic quantum field theory. In our considerations we assume that only the ultra-violet divergencies are present, i.e. Kallen-Lehmann spectral function is locally integrable. It is interesitng to mention that the infrared divergencies can be excluded formally by means of the Wightman postulate of positiveness of metric in the space of states [12].

The analytic representations, describing commutator functions with the singularities $\delta^{(k)}(x^2)$ $(k=0,1,2,..)$ and $(x^2)_+^{-\ell}$ $(\ell=1,2...)$ and depending on the continuous parameter m analogous to the mass variable in the free field case, are introduced in Sect.2. In Sect.3 we present the connection between some standard nonintegrable spectral functions and our analytic representations. In Sect.4 we introduce the numerical parameters- wave renormalizati-

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x) The best example is provided by the free field case, where the delta singularity on the light cone implies the canonical commutation relations. The argument can be extended also to large class of other light cone singularities. The ambiguity, consisting in the presence of distributions with point support $x=0$, is nonphysical.

on constants, unrenormalized mass and generalized wave renormalization constants - as characteristics of the light cone singularity. In the last Section some general remarks about the noncanonical singularities are given.

2. The Classes of Analytic Representations of Noncanonical Two Point Functions

In Wightman formalism all two point functions (two point VEV, commutator functions, causal propagator, etc.) are the distribution valued boundary values of an analytic function $G(z^2)$ $(z^2 = z_\mu z^\mu,\ z_\mu = x_\mu + iy_\mu)$ [1-3] holomorphic in a whole complex z^2-plane $(z^2 = s + iu)$ except the points along the positive real axis $(u = 0;\ s \geq 0)$. Such analytic function $G(z^2)$ is characterized by its discontinuity across the cut[x)]

$$\xi(s) = \frac{1}{2\pi i}\{G(s+i0) - G(s-i0)\}. \qquad (2.1)$$

Using the boundary prescriptions for the two point VEV[1] one gets the following formula for the commutator function

$$G(x) = i < 0\,|[\phi(\frac{x}{2}), \phi(-\frac{x}{2})]|\,0 > = 2\pi\epsilon(x_0)\,\xi(x^2). \qquad (2.2)$$

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x) The distributions as boundary values of analytic functions are extensively discussed in[5,8] and [13-15]. The analytic function $G(z^2)$ is called an analytic representation, generating the distribution $\xi(s)$. For a large class of distributions $\xi(s)$ one can write for $G(z^2)$ a Cauchy representation, leading to dispersion relations in coordinate space[16,17].

Because the commutator function is a tempered distribution, we see that

$$\xi(x^2) \subset S'(R_+) \tag{2.3}$$

and $\xi(x^2)$ should be real. This last condition implies that one can write

$$G(z^2) = \tilde{G}(z^2) + F(z^2), \tag{2.4}$$

where $\tilde{G}(z^2)$ has a real discontinuity (2.1) and satisfies the condition

$$\tilde{G}^*(z^2) = \tilde{G}(z^{2*}). \tag{2.5}$$

The function $F(z^2)$ is an entire function in z^2 complex plane. One can say that the function $\tilde{G}(z^2)$ determines the algebraic structure of the theory[x], and $F(z^2)$ depends only on the representation. Finally, using the temperedness assumption for the two point VEV we see that the function $G(s \pm i0)$ should be bounded for large positive as well as negative values of s by a polynomial.

The example of an analytical function, satisfying all requirements mentioned above is provided by the analytic continuation of free field VEV, defined as follows

$$G_0(z^2; m^2) = \frac{m^2}{8\pi i} \frac{H_1^{(1)}(mz)}{mz}. \tag{2.6}$$

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[x] This determination is, of course, a partial one. Only for generalized free field $\tilde{G}(z^2)$ determines the algebraic structure completely.

The discontinuity (2.1) along the positive role axis is described by the function[x)]

$$\xi(s;m^2) = \frac{1}{4\pi^2}\{\delta(s) - \theta(s)\,\frac{m^2}{2}\,\frac{J_1(m\lambda)}{m\lambda}\},\qquad (2.7)$$

where $s = \lambda^2$, and the light-cone behaviour is determined by the singularity of $G_0(z^2;m^2)$ near the point

$$G_0(z^2;m^2) = -\frac{1}{4\pi^2}\,\frac{1}{z^2} \quad + \text{ finite terms } (2.8)$$

We see that the light cone singularity is mass-independent, and because it leads to the canonical commutation relations it will be called a canonical singularity. The interacting fields have, however, different singularities on the light cone. In the following we shall assume that the only singular point for the distribution (2.1) is the light cone, $s = 0$, and that the distribution $\xi(s)$ for large s is described by a real function, satisfying the condition

$$|\xi(s)| < \frac{A}{s^{3/4}}. \qquad (2.9)$$

The first requirement means that the main part of the perturbation propagates without delay along the light cone, and the relation (2.9) determines the asymptotic behaviour of the action with very large delay time[xx)].

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x) The analytic representation of $\delta(s)$ is $-\frac{1}{z^2}$ and $\theta(s)$ is generated by $-\ln(-z^2)$ (see/5/ and/13/).

xx) The value 3/4 of the inverse power in (2.9) can be justified by the requirement of positive-definiteness of the metric in the space of physical states.

We shall consider in this paper the following two classes of light-cone singularities

$$\xi_k(s) = \delta^{(k)}(s) \quad k=0,1,2,\dots \tag{2.10a}$$

and

$$\xi_\ell(s) = s_+^{-\ell} \quad \ell = 1,2,3\dots \ . \tag{2.10b}$$

The results can be generalized also to more general cases, particularly to the case when ℓ is a continuous index.

a) $\delta^{(k)}(s)$ $(k=0,1,2\dots)$.

The simplest generalization of the formula (2.6) is to introduce other Hankel functions of the first kind, with the argument mz. It can be easily shown that only such combination of Bessel functions and Nuemann functions satisfy the temperedness assumption for space-like distances (z^2 negative, imz- real and positive). We introduce the following family of analytic functions, which can be used for the description of the analytically continued VEV:

$$G_n(z^2;m^2) = \left(\frac{d}{dz^2}\right)^n G_0(z^2;m^2) \ . \tag{2.11}$$

Using the formula

$$\left(\frac{d}{z\,dz}\right)^n \left\{\frac{H_1^{(1)}(z)}{z}\right\} = (-1)^n \frac{H_{n+1}^{(1)}(z)}{z^{n+1}} \tag{2.12}$$

one gets the following result

$$G_n(z^2;m^2) = \frac{(-1)^n}{4\pi i}\left(\frac{m^2}{2}\right)^{n+1} \frac{H_{n+1}^{1}(mz)}{(mz)^{n+1}} . \tag{2.13}$$

Using the expression for the Hankel function $H_n^{(1)}(mz)$, one gets

$$G_{n-1}(z^2;m^2)=\frac{(-1)^n}{4\pi^2}\left(\frac{m^2}{2}\right)^n\{[2\gamma-\ln 4+\ln(-m^2z^2)]\cdot$$

$$\cdot\frac{J_n(mz)}{(mz)^n}-\frac{1}{\pi}\sum_{k=0}^{n-1}\frac{a_{n;k}}{(mz)^{2(n-k)}}-\frac{1}{\pi}\sum_{k=0}^{\infty}b_{n;k}\,(mz)^{2k}\,, \tag{2.14}$$

where $\gamma=0.577$ (Euler constant),

$$a_{n;k}=2^{n-k}\frac{(n-k-1)!}{k!}$$

$$b_{n;k}=\frac{(-1)^k}{2^{n+2k}}\frac{C_{k+n}+C_n}{k!(k+n)!} \tag{2.15}$$

$$c_s=1+\frac{1}{2}+\dots+\frac{1}{s}\qquad c_0=0$$

and the relation $\ln(-m^2z^2)=2\ln mz+i\pi$ has been used. We obtain the following discontinuity:

$$\xi_{n-1}(s;m^2)=\frac{1}{2\pi i}\{G_{n-1}(s+i0;m^2)-G_{n-1}(s-i0;m^2)\}=$$

$$=\frac{1}{4\pi^2}\sum_{k=0}^{n-1}\frac{(-1)^k}{4^k}\frac{m^{2k}}{k!}\delta^{(n-k-1)}(s)+ \tag{2.16}$$

$$+\left(-\frac{m^2}{2}\right)^n\frac{\theta(s)}{4\pi^2}\cdot\frac{J_n(m\lambda)}{(m\lambda)^n}\,.$$

The leading light-cone singularity, is mass-independent and equal to $\frac{1}{4\pi^2}\delta^{(n)}(s)$. Putting $m=0$ one gets

$$G_n(z^2;0)=\frac{(-1)^{n+1}}{4\pi^2}\cdot\frac{n!}{(z^2)^{m+1}} \tag{2.17}$$

leading to the result obtained in/5-8/

$$\xi_n(s;0)=\frac{1}{4\pi^2}\delta^{(n)}(s). \tag{2.18}$$

The formula (2.11) can be easily generalized. We introduce

$$G_{n,r}(z^2;m^2)=(z^2)^r\, G_n(z^2;m^2), \tag{2.19}$$

where $-n \le r \le n$. The leading singularity on the light cone for $\xi_{n;r}(s;\ m^2)$ is independent on m^2 and proportional to particularly interesting class one gets by the following choice

$$G_{2k-1;k-1}(z^2;m^2)=-\frac{1}{4\pi i}\left(\frac{m^2}{4}\right)^k \frac{H^{(1)}_{2k}(mz)}{z^2} \qquad (k=1,2,3\ldots) \tag{2.20}$$

with leading singularity on the light-cone described by the limit $m^2 \to 0$

$$G_{2k-1;k-1}(z^2;0)=\frac{1}{4\pi^2}\,\frac{(2k-3)!}{(z^2)^{k+1}}. \tag{2.21}$$

It will be seen in Sect.3 that the functions (2.20) correspond to the case of polynomial behaviour of the Kellen-Lehmann spectral function

b) $s_+^{-\ell}$ ($\ell = 1,2,3\ldots$).

In order to describe the singularities (2.10b) we introduce the following analytic function[x)]

$$\tilde{G}_1(z^2;m^2)=\frac{1}{4\pi i}\,\frac{H^{(1)}_0(mz)}{z^2}. \tag{2.22}$$

Using the formula

$$H^{(1)}_0(mz)=\frac{i}{\pi}\left[\gamma+\frac{1}{2}\ln(-m^2z^2)-\ln 2\right]J_0(mz)+ -\frac{2i}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n}J_{2n}(mz) \tag{2.23}$$

one gets

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x) It can be mentioned that one obtains (2.22) from (2.19) by putting $n=0$, $r=-2$.

$$\tilde{\tilde{\xi}}_1(s;m^2) = \frac{1}{2\pi i}\{\tilde{\tilde{G}}_1(s+i0;m^2) - \tilde{\tilde{G}}_1(s-i0;m^2)\} = \tag{2.24}$$

$$+= \frac{1}{4\pi^2} S_+^{-1} J_0(m\lambda) - \frac{1}{4\pi^2}(\ln m^2 - 2\gamma + \ln 4)\delta(s) \quad .$$

We see that the function (2.19) does not allow to perform the limit $m^2 \to 0$. The logarithmic term, which becomes infinite with vanishing m^2 , occurs in the solutions derivative coupling models in two dimensions[18,19] and four dimensions[20.21]. In order to get only the singularity s_+^{-1} one should subtract from the function (2.19) the following "counterterm"

$$[\ln m^2 + 2(\gamma - \ln 2)]\, G_0(z^2;\mu^2), \tag{2.25}$$

where the mass μ^2 is in general case not related to the mass and particularly can be chosen equal to zero.

In order to get the singularities (2.10b) with $\ell = 1,2,3..$ one should introduce the following analytic functions

$$\tilde{\tilde{G}}_\ell(z^2;m^2) = \left(\frac{d}{dz^2}\right)^{\ell-1} \tilde{\tilde{G}}_1(z^2;m^2). \tag{2.26}$$

Using the formula

$$\left(\frac{d}{z\,dz}\right)^n H_0(z) = (-1)^n \frac{H_n^{(1)}(z)}{z^n} \tag{2.27}$$

one gets

$$\tilde{\tilde{G}}_\ell(z^2;m^2) = \frac{1}{4\pi i}(-1)^{\ell-1} \frac{(\ell-1)!}{(z^2)^\ell} \cdot \sum_{n=0}^{\ell-1} \frac{1}{n!}\left(\frac{mz}{2}\right)^n H_n^{(1)}(mz). \tag{2.28}$$

The formula (2.28) implies the following leading light-cone singularities

$$\tilde{\xi}_{\ell}(s;m^2) = \frac{1}{4\pi^2}(-1)^{\ell-1}(\ell-1)!\, s_+^{-\ell} - \frac{1}{4\pi^2}\{\ln m^2 + 2(\gamma - \ln 2) + c_{\ell-1}\}\delta^{(\ell-1)}(s) + O(s_+^{-\ell+1}). \tag{2.29}$$

Introducing suitable counterterms (compare with (2.25)) one can cancel out all terms with delta functions.

Another way of introducing the light-cone singularities of type (2.10b) is to multiply the functions $G_n(z^2;m^2)$ by the function $\ln(-z^2)$. . Let us consider for example

$$G_0^{\ln}(z^2;m^2) = \ln(-z^2)G_0(z^2;m^2). \tag{2.30}$$

The discontinuity of (2.30) is given by the formula

$$\xi_0^{\ln}(s;m^2) = \frac{1}{4\pi^2}\{s_+^{-1} - \ln s_+ \frac{m^2}{2}\frac{J_1(m\lambda)}{m\lambda}\}. \tag{2.31}$$

The differentiation of (2.30) with respect to z^2 leads to analytic functions with the discontinuity $\xi(s)$ having singularities only of the type (2.10b).

3. The Analytic Representations and Nonintegrable Kallen-Lehmann Spectral Functions

The two point functions are usually described by means of the spectral function $\rho(\kappa^2)$ which represents a Lorentz-invariant four-dimensional Fourier transform of the distribution $\xi(x^2)$. In this Section we shall find the analytic representations, corresponding to some standard choices of nonintegrable spectral functions.

Let us write the Kallen-Lehmann spectral representation in complex coordinate space

$$G(z^2) = \int_0^\infty \rho(\kappa^2) G_0(z^2;\kappa^2) d\kappa^2 . \tag{3.1}$$

We shall consider (3.1) for z^2 off the real axis, i.e. for $z = \lambda + i\eta$, where z lies in upper half plane $(\eta > 0)$. The function $G_0(z^2;\kappa^2)$ behaves for large κ like $e^{-\kappa\eta}$ and the integral exists for all locally integrable $\rho(\kappa^2) \subset S'(R_+)$.

Now we shall introduce the following operator, acting on complex variables z_μ

$$\frac{\partial}{\partial z_\mu} \frac{\partial}{\partial z^\mu} f(z^2) \overset{df}{=} \Box_z f(z^2) = -4\left(\frac{d}{dz^2}\right)^2 [z f(z^2)] . \tag{3.2}$$

Using the identity

$$z\left(\frac{d}{z\,dz}\right)^2 z H_1^{(1)}(mz) + m^2 H_1^{(1)}(mz) = 0 \tag{3.3}$$

one gets

$$(\Box_z - \kappa^2) G_0(z^2;\kappa^2) = 0 \tag{3.4}$$

and

$$\Box_z^n G(z^2) = \int_0^\infty (\kappa^2)^\ell \rho(\kappa^2) G_0(z^2;\kappa^2) d\kappa^2 . \tag{3.5}$$

Using the formula (3.5) one can always relate the spectral function $\rho(\kappa^2)$ which is locally integrable and belongs to $S(R_+)$ with a function having a Hankel transform on the real axis. We introduce in general for $\eta > 0$

$$\tilde{g}(z) = \int_0^\infty g(\kappa) H_1^{(1)}(\kappa z)(\kappa z)^{1/2} dx . \tag{3.6}$$

If $g(\kappa) \subset L_1(0,\infty)$ the transform (3.6) necessarily exists also if $\eta = 0$ [22]. Using (2.6) and (3.6) one can write (3.1) as follows

$$G(z^2) = \frac{1}{4\pi i\, z^{3/2}} \tilde{g}(z), \tag{3.7}$$

where

$$g(\kappa) = \rho(\kappa^2)\,\kappa^{3/2} . \tag{3.8}$$

Our method of determining the analytic representation for nonintegrable spectral functions will be based on the following two steps: α) We take from the tables of integral transforms[x)] the Hankel transform (3.6) for

$$g_{reg}(\kappa) = \rho_{reg}(\kappa^2)\,\kappa^{3/2} = \frac{\rho(\kappa^2)\,\kappa^{3/2}}{(\kappa^2)^n}, \tag{3.9}$$

where is chosen sufficiently large,.

β) We use the formula (3.5).

We see, therefore, that every spectral function with $g(\kappa)$ having the real Hankel transform (3.6) generates the family of analytic representations for all two-point functions with the spectral functions of the form $g(\kappa)(\kappa^2)^n$.

We shall consider below two such families:

a) $\rho(\kappa^2) = \theta(\kappa^2 - m^2)(\kappa^2)^k \qquad (k = -1, 0, 1, \dots)$.

Let us consider firstly $k = -1$. From (3.8) follows that one should find the Hankel transform (3.6) with $g(\kappa) = \theta(\kappa - m)\kappa^{-1/2}$. One gets[/23/]

$$\Delta_{-1}(z^2; m^2) = \frac{1}{4\pi i\, z^{3/2}} \int_m^\infty \kappa^{-1/2} H_1^{(1)}(\kappa z)(\kappa z)^{1/2}\, d\kappa = \tag{3.10}$$

$$= \frac{1}{4\pi i\, z^2} H_0^{(1)}(mz).$$

x) See, for example,/23/.

We see, therefore, that the analytic function (2.22) describes the two-point functions characterized by logarithmically divergent wave renormalization constant. Because

$$\Delta_{-1}(z^2;m^2)=\int_{m^2}^{\infty}\frac{d\kappa^2}{\kappa^2}\,G_0(z^2;\kappa^2) \tag{3.11}$$

it is clear the origin of the term $\ln m^2$ in (2.24), describing the infrared divergence of the wave renormalization constant.

The results for $k=0,1$,... one gets using the relation (3.5). We have

$$\Delta_k(z^2;m^2)=\int_{m^2}^{\infty}d\kappa^2\,(\kappa^2)^k\cdot G_0(z^2;\kappa^2)=\Box_2^{k+1}\Delta_{-1}(z^2;m^2). \tag{3.12}$$

Using the formula $\Box_z=-4z^2\left(\frac{d}{dz^2}\right)^2-8\frac{d}{dz^2}$ one gets for the most interesting cases $k=0$ and $k=1$

$$\Delta_0(z^2;m^2)=-\frac{m^2}{4\pi i z^2}H_2^{(1)}(mz) \tag{3.13a}$$

and

$$\Delta_1(z^2;m^2)=-\frac{m^2}{4\pi i\, z^2}H_4^{(1)}(mz)=\frac{m^3}{\pi i\, z^3}H_3^{(1)}(mz)\ . \tag{3.13b}$$

We see that for $k=-1$ and $k=0$ we obtained up to some constant factor the analytic functions (2.20). For $k\geq 1$ one gets also some additional terms proportional to the functions $G_{k+1;0}(mz)$, $G_{k+2;1}(mz)$ ... $G_{k+l+1;l}(mz)$... $G_{2k;k-1}(mz)$.
All these functions contribute to the leading light cone singularity, which can be obtained if $m\to 0$. For example, for $k=1$ we get, using the formula

$$H_n^{(1)}(z)\underset{z\to 0}{=}-\frac{i}{\pi}\left(\frac{1}{2}z\right)^{-n}(n-1)!$$

that

$$\Delta_1(z^2;0) = \frac{8}{\pi^2}\cdot\frac{1}{z^4} \tag{3.14}$$

and both terms in (3.13b) contribute to the result (3.14)

b) $\rho(\kappa^2) = \theta(\kappa^2 - m^2)(\kappa^2)^k \ell n \kappa^2 \quad k = -1, 0, 1 \ldots$.

Firstly we shall consider the case $k = -1$. One gets after somewhat tricky calculations the following estimate at $z = 0$ x):

$$\Delta_{-1}^{\ell og}(z^2;m^2) = \frac{1}{4\pi i z^{3/2}}\int_m^\infty \frac{\ell n \kappa^2}{\kappa^{1/2}} H_1^{(1)}(\kappa z)(\kappa z)^{1/2} d\kappa =$$

$$= \frac{3}{8\pi^2}\frac{\ell n^2 z^2}{z^2} + 0\left(\frac{\ell n z^2}{z^2}\right) \tag{3.15}$$

and the discontinuity (2.1) has a following leading singularity

$$\xi_{-1}^{\ell og}(s;m^2) = \frac{3}{8\pi}\ell n s \; s_+^{-1} + 0(s_+^{-1}) . \tag{3.16}$$

Applying the operator (3.2) one gets

$$\Delta_k^{\ell og}(z^2;m^2) = \int_{m_2}^\infty (\kappa^2)^k \ell n \kappa^2 G_0(z^2;\kappa^2) d\kappa^2 =$$

$$= \frac{(-1)^k\, 3\cdot 4^{k-1}\cdot (k!)^2 (k+1)}{\pi^2 (z^2)^{k+2}} \ell n z^2 + 0\left(\frac{1}{z^{2(k+2)}}\right) \tag{3.17}$$

---

x) A complete formula for $\Delta_{-1}^{\ell og}$ will be given in the second part of this paper.

and one can check easily that the leading singularity is of a type (2.10b).

Finally, it should be stressed that even when

$$\int \rho(\kappa^2)\kappa^{1/2} d\kappa^2 < \infty \qquad (3.18)$$

what assures that the Hankel transform (3.6) exists for $\eta = 0$, the function $G_0(z;\kappa^2)$ cannot be expanded under the integral into the powers series, because the coefficients of consecutive powers will not exist. If we, nevertheless, use such method, it is wasy to see that the differentiation (3.5) will not produce any noncanonical singularities. Indeed, the terms occuring in power expansion of $G_0(z^2;m^2)$ are $\frac{1}{z^2}$, $\ln z^2 (z^2)^k$ $(k=0,1..)$ and $(z^2)^k$. Because

$$\begin{aligned}
\Box_z \frac{1}{z^2} &= 0 \\
\Box_z (z^2)^k &= -4k(k+1)(z^2)^{k-1} \\
\Box_z \ln z^2 (z^2)^k &= -4\{k(z^2)^{k-1} + 2(k+1)(z^2)^k + k(k+1)\ln z^2 (z^2)^{k-1}\}
\end{aligned} \qquad (3.19)$$

the differentiation (3.5) will again reproduce only the terms occuring in $G_0(z^2;m^2)$. We see, therefore, that noncanonical terms can be easily lost if we use unjustified mathematics.

4. Remarks about the Renormalization Procedure

It has been mentioned in Introduction that the wave renormalization constant measure the modification of free field singularities.One introduces the cutt-off dependent wave renormalization constant $z_3^{-1}(\Lambda^2)$, where

$$z_3^{-1}(\Lambda^2) = \int_0^{\Lambda^2} \rho(\kappa^2) d\kappa^2 \tag{4.1}$$

and one studies the limit

$$z_3^{-1} = \lim_{\Lambda^2 \to \infty} z_3^{-1}(\Lambda^2). \tag{4.2}$$

Different types of infinities correspond to different types of noncanonical singularities.

The wave renormalization constant can be, however, defined by means of the analytic representation $G(z^2)$ as follows

$$z_3^{-1} = \lim_{z^2 \to 0} \frac{G(z^2)}{G_0(z^2)}, \tag{4.3}$$

where the function $G_0(z^2)$ can be characterized by any mass. Similarly, one can introduce the unrenormalized mass parameter m_0^2 by means of the following limit

$$m_0^2 = \lim_{z^2 \to 0} \frac{\Box G(z^2)}{G_0(z^2)} . \tag{4.4}$$

It is easly to see that for the free field $G(z^2) = G_0(z^2; m^2)$ one gets $m_0^2 = m^2$, and in general case

$$m_0^2 = \lim_{\Lambda^2 \to \infty} \int_0^{\Lambda^2} \kappa^2 \rho(\kappa^2) d\kappa^2 . \tag{4.5}$$

These two characteristics of light-cone singularity comes out from the comparison with the free field case. It is possible, however, to introduce generalized wave-renormalization constants $z_{3;n}^{-1}$ describing the light-cone singularity compared with the singularities of the analytic functions $G_n(z^2; m^2)$ for $z^2 \to 0$.

We define x)

x) Particularly, $z_{3;0}^{-1} \equiv z_3^{-1}$.

$$Z^{-1}_{3;n} = \lim_{z^2\to 0} \frac{G(z^2)}{G_n(z^2)} \qquad (n=0,1\dots). \qquad (4.6)$$

It follows from the postulate of positive metric in the space of physical states, that $z^{-1}_{3;0} \geq 1$. If $z^{-1}_{3;0} = \infty$ it can be found however, always such n that $z^{-1}_{3;n} < \infty$.

5. Conclusions

In this paper we have introduced new class of basic two-point functions, describing noncanonical two-body forces, more singular in static approximation than the Yukawa term. This modification of $1/r$ singularity is caused by the exchange of infinite number of quanta with very large momenta. In usual appraoch such process leads to ultraviolet divergencies and the necessary of infinite renormalization. In our approach we introduce some objects, characterized by the continuous mass spectrum, and formed out of infinite number of quanta. These objects x) are chosen in such a way that the "one particle exchange" approximates in a correct way the short distance singularity for complete two-body forces.

The presence of noncanonical forces modifies the interaction at very small distance in such a way that the notion of charge and mass for these distances are not valid. Indeed, the charge and mass can be defined only under the assumption that the interaction has (in static approximation) the Yukawa form. One defines the unrenormalized, parameters as follows

x) In /24/ the free field with polynomial spectral functions have been called the "inverse multipole field".

$$e_0^2 = \lim_{r \to 0} (V(r)\, r) \tag{5.1}$$

$$m_0^2 = \lim_{r \to 0} \frac{\partial^2}{\partial r^2} (V(r)\, r). \tag{5.2}$$

The formulas (5.1),(5.2) give infinite results because the Yukawa law for very small (we call them submicroscopic)[x] distance is modified.

If we consider two-body forces one can always split them into two parts: with $1/r$ singularity (canonical terms) and with the singularity stronger than $1/r$ (non-canonical terms). The submicroscopic distances are defined by the requirement that the effects of noncanonical terms cannot be neglected. The validity of perturbations expansion is strictly connected with the effects of noncanonical terms of above classification and cannot be used for submicroscopic distances. Using first orders of the perturbation theory one can guess, however, that the submicroscopic distances in QED are indeed beyond the range of physical measurements. One can calculate/17,25/ that the non-canonical terms, occuring in the second order of perturbation theory in QED can be neglected if

$$\frac{a}{3\pi} \ln \frac{\Lambda}{M_e} \ll 1, \tag{5.3}$$

where M_e denotes the electron mass, and $\Lambda = \frac{1}{a}$ describes the cut-off parameter corresponding to the penetration distance a . Using the value $a = \frac{e^2}{4\pi} \approx \frac{1}{137}$ one obtains

$$\ln \frac{\Lambda}{M_e} \ll 1000 \tag{5.4}$$

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x) See /17/, Sect. 5.

We see easily from (5.4) that it is not possible to detect scattering experiments; the modification of the Coulomb law singularity, and particularly, the Pauli-Villars regularization procedure, removing noncanonical terms, can be used[x)]. To the contrary, it is easy to check that the estimate for strong interactions leads to the range of submicroscopic distance overlapping with the values of scattering parameters in present high energy experiments. We see, therefore, that the convertional perturbation expansion cannot be used, and some other approximations, suing perhaps the propagators introduced in this paper, should be developed.

The author is indebted to Professors R.Haag, G.Kallen, and R.F.Streater for helpful discussions at the V-th Winter School of Theoretical Physics in Karpacz.

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x) For the demonstration how the regularization procedure removes noncanonical terms see/26,27/.

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Received by Publishing Department
on November 12, 1968.