Czechosl. J. Phys., 1969, v 19B, №9, p 1067-1075

L-85

ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2 - 4094

Lokajíček. M.

ЛАБОРАТОРИЯ ВЫСОКИХ ЭНЕРГИЙ

REMARKS ON DOUBLE POLES AND NON-EXPONENTIAL DECAYS

1968

E2 - 4094

45-70/3 нр

Lokajíček. M.

REMARKS ON DOUBLE POLES AND NON-EXPONENTIAL DECAYS

Submitted to Czech. J. Phys.

Объединенный институт ядерных исследований БИБЛИОТЕКА

1. *Introduction*

In paper[1] Bell and Goebel have investigated conditions under which in a system of decaying particles deviations from the exponential law can show up, which in the S-matrix would correspond to higher-order poles . They have analyzed in detail a model of an unstable particle with two internal states which can decay from one of these states only and they have demonstrated that the existence of the second-order pole uniquely depends on the relation between the strength of decay and the mutual interaction of both internal states. On the other hand papers[2]and[3] indicate, that the poles of the second order in such a system might exist even under more general conditions. The result of paper[1] is limited by the assumption of small interaction constants, whereas the formalism used in papers [2]and [3], which makes possible to obtain results without such a limitation, did not offer any relation to the strength of decay from the individual states.

In this paper we attempt therefore a new analysis of this problem. We start from the papers[2] and[3], where the general time-dependence of the probability amplitudes of the internal states with a multi-pole behaviour has been shown. We limit ourselves on two internal states, but we extend the formalism so, that it allows to take into account also the strength of interaction of the individual states with the state of decay products.

We shall suppose that the Hamiltonian of the whole system is in general non-hermitian and derive at first the general form of the time dependence for all probability amplitudes. Then we derive the general relations between the various matrix elements of the Hamiltonian. If the requirements of time-reversal invariance and of reality of all interaction parameters are added, it can be shown, that only two real parameters, i.e. the mass parameter M and the decay parameter Γ, are left as free. Thus in a different way, starting from more general assumptions, we come to the same result as in paper/1/.

2. *The Time Dependence of Probability Amplitudes*

Consider a particle existing in two different states $|s^{a}>$ $(a=1,2)$, which change mutually one into the other. Let this particle decay spontaneously and the behaviour of the whole system in the time $t \geq 0$ be described according to general laws of quantum mechanism by the equations

$$e^{-iHt}|s^{a}> = \sum_{\beta=1}^{2} a_{a\beta}(t)|s^{\beta}> + \sum_{\lambda}\int_{r} c_{a}(t,\lambda,r)|\phi(\lambda,r)>dr , \tag{1}$$

$$e^{-iHt}|\phi(\lambda,r)> = d(t,\lambda,r)|\phi(\lambda,r+t)> \qquad r>0, \tag{2}$$

where the parameter λ represents all labels of the state of the decay products and r describes the time evolution of this state. The parameter r has a continuous spectrum; it has the dimension of time and determines the distance of the individual decay products from the centre of mass.

Let us suppose further that the following conditions are fulfilled

$$< s^{\alpha} \mid s^{\beta} > = \delta_{\alpha\beta} , \tag{3a}$$

$$< s^{\alpha} \mid \phi (\lambda , \tau) > = 0 , \tag{3b}$$

$$< \phi (\lambda , \tau) \mid \phi (\lambda' , \tau') > = \delta_{\lambda\lambda'} \delta (\tau - \tau') \tag{3c}$$

and

$$a_{\alpha\beta}(0) = \delta_{\alpha\beta} , \tag{4a}$$

$$c_{\alpha}(t , \lambda , \tau) = 0 \qquad t \leqq \tau . \tag{4b}$$

From the equation (2) it also follows immediately

$$d (0 , \lambda , \tau) = 1 . \tag{4c}$$

With the help of equation (1), (3ab) and (4a) we get the relations

$$a_{\alpha\gamma}(t + t') = \sum_{\beta=1}^{2} a_{\alpha\beta}(t)\, a_{\beta\gamma}(t') \tag{5}$$

and for the system with double-pole behaviour we can write (see[3])

$$a_{11}(t) = e^{-i\mu t} (1 + \gamma t) , \qquad a_{12}(t) = e^{-i\mu t} \gamma \epsilon t ,$$
$$\tag{6}$$
$$- a_{21}(t) = e^{-i\mu t} \frac{\gamma}{\epsilon} t , \qquad a_{22}(t) = e^{-i\mu t} (1 - \gamma t) ,$$

where the parameters μ , γ , ϵ are so far entirely arbitrary constants.

By similar procedure which led to relations (5) and (6) we can get also the general form of time dependence of probability amplitudes d and c_a. From the relation(2) we obtain

$$d(t+t',\lambda,r) = d(t,\lambda,r)\, d(t',\lambda,r+t). \qquad (7)$$

And if we write

$$d(t,\lambda,r) = e^{-i\xi(t,\lambda,r)} \qquad (8)$$

we get after interchanging t and t'

$$\xi(t,\lambda,r) - \xi(t,\lambda,r+t') = \xi(t',\lambda,r) - \xi(t',\lambda,r+t).$$

It can be proved that this equation may hold for arbitrary t, t', r only if both sides of it are identically equal to zero; $\xi(t,\lambda,r)$ is independent of r. Then according to equation (7) it holds

$$\xi(t+t',\lambda) = \xi(t,\lambda) + \xi(t',\lambda)$$

and therefore

$$d(t,\lambda,r) = e^{-i\xi t}, \qquad (9)$$

where ξ may depend only on the parameter λ. Differentiating eq. (2) according to t we obtain with the help of (9)

$$H|\phi(\lambda,r)\rangle = \xi|\phi(\lambda,r)\rangle \qquad (r>0); \qquad (10)$$

and we may replace the λ by a pair of parameter (ξ,Λ), where Λ labels all degenerate states belonging to the same value of ξ.

We derive now the time dependence of the integral amplitudes of the decay states. Using the equation (1) and with the help of eq. (5) we obtain

$$\sum_{\lambda}\int_{r} c_a(t+t',\lambda,r)|\phi(\lambda,r)\rangle\, dr = \sum_{\beta,\lambda}\int_{r} a_{a\beta}(t)\, c_{\beta}(t',\lambda,r)|\phi(\lambda,r)\rangle\, dr + \sum_{\lambda}\int_{r} c_a(t,\lambda,r)\, e^{-iHt'}|\phi(\lambda,r)\rangle\, dr. \qquad (11)$$

Multiplying eq.(11) from the left by

$$\sum_{\Lambda' r'} \int <\phi(\lambda', r')| dr'$$

we get with the help of (2), (3c) and (9)

$$C_\alpha(t+t') = \sum_\beta a_{\alpha\beta}(t) C_\beta(t') + e^{-i\xi t'} C_\alpha(t), \tag{12}$$

where we introduced the integral amplitude

$$C_\alpha(t) = \sum_{\Lambda r} \int c_\alpha(t, \lambda, r) dr. \tag{13}$$

Substituting into both equations (12) (i.e. for $\alpha = 1,2$) according to equations (6) we obtain by linear combination

$$e^{-i\mu t} f(t') + e^{-i\xi t'} f(t) = f(t+t'),$$

where

$$f(t) = C_1(t) + \epsilon\, C_2(t).$$

Interchanging now the variables t and t' we have easily the relation

$$C_1(t) + \epsilon C_2(t) = K(e^{-i\xi t} - e^{-i\mu t}). \tag{14}$$

Repeating the above procedure with equations (12) (substracting this time the equations) and using relations (6) and (14) we arrive to a similar expression

$$C_1(t) - \epsilon C_2(t) = -2K e^{-i\mu t} \gamma t + L(e^{-i\mu t} - e^{-i\xi t}). \tag{15}$$

From the equations (14) and (15) we then obtain the final relations

$$\begin{aligned} C_1(t) &= e^{-i\mu t}\{-K\gamma t - \frac{1}{2}(K-L)[1-e^{i(\mu-\xi)t}]\}, \\ C_2(t) &= \frac{1}{\epsilon} e^{-i\mu t}\{K\gamma t - \frac{1}{2}(K+L)[1-e^{i(\mu-\xi)t}]\}, \end{aligned} \tag{16}$$

where K and L represent two so far arbitrary complex functions of ξ .

The quantity $c_a(t)$ means the amplitude of the probability that the system, which at time $t=0$ is in the state $|s^a>$, will be at time $t>0$ in some of the states of the decay products. Therefore

$$|C_a(t)|^2 \leqq 1$$

must hold for every t. Writing now

$$\mu = M - i\frac{\Gamma}{2} \tag{17}$$

with M and Γ real and eliminating the trivial case $\Gamma=0$, we obtain

$$\mathrm{Im}(\xi-\mu) = \frac{\Gamma}{2} > 0. \tag{18}$$

Let us introduce the total-energy distribution of decay products by relations

$$D_1^2(\xi) = |C_1(\infty)|^2 = \frac{1}{4}|K-L|^2,$$
$$D_2^2(\xi) = |C_2(\infty)|^2 = \frac{1}{4|\epsilon|^2}|K+L|^2. \tag{19a}$$

The quantities D_1 and D_2 are real functions of ξ, fulfilling the relations

$$D_1(\xi) \geqq 0, \qquad D_2(\xi) \geqq 0, \tag{19b}$$

$$\int D_1^2 d\xi = \int D_2^2 d\xi = 1. \tag{19c}$$

In the case $\Gamma=0$ it holds identically

$$K \equiv L \equiv D_1 \equiv D_2 \equiv 0 \tag{20a}$$

and from the condition

$$|a_{a1}|^2 + |a_{a2}|^2 \equiv 1$$

also

$$\gamma = 0. \tag{20b}$$

Instead of (19c) there holds now

$$\int D_1^2 \, d\xi = \int D_1^2 d\xi = 0 . \tag{20c}$$

From equations (18) and (17) it follows

$$\xi = M + x \;, \tag{21}$$

where x is also real.

3. *Matrix Elements of the Hamiltonian*

Let us return now to the basic equation (1). Having differentiated it according to t and using the equation(1) once more we obtain

$$\sum_\beta a_{\alpha\beta}(t) H | s^\beta > + \sum_\lambda \int_r c_\alpha(t,\lambda,r) H | \phi(\lambda,r) > dr = i \sum_\beta \dot{a}_{\alpha\beta}(t) | s^\beta > + i \sum_\lambda \int_r \dot{c}_\alpha(t,\lambda,r) | \phi(\lambda,r) > dr \tag{22}$$

Multiplying this equation from the left by $<\phi(\lambda', r')|$ and summing and integrating over Λ' and r', we get with the help of (3ab), (10) and (13)

$$\sum_\beta a_{\alpha\beta}(t) g_\beta(\xi) + \xi C_\alpha(t) = i \dot{C}_\alpha(t), \tag{23}$$

where we have introduced interaction parameters

$$g_\alpha(\xi) = \sum_\Lambda \int_r < \phi(\lambda, r) | H | s^\alpha > dr . \tag{24}$$

From equations (23) we obtain with the help of (6) and (16).

$$\begin{aligned} g_1 &= -i K\gamma + \frac{1}{2}(K-L)(\xi-\mu), \\ g_2 &= \frac{1}{\epsilon}\left[i K\gamma + \frac{1}{2}(K+L)(\xi-\mu) \right]. \end{aligned} \tag{25}$$

Multiplying the equation (22) by $<s^\gamma|$ we obtain

$$\sum_\beta a_{\alpha\beta}(t) G_{\gamma\beta} + \int_\xi X_\alpha(t) \bar{g}_\gamma(\xi) \rho(\xi) d\xi = i \dot{a}_{\alpha\gamma}(t), \tag{26}$$

where it was introduced

$$<s^{\alpha}|H|s^{\beta}> = G_{\alpha\beta} \ , \tag{27}$$

$$<s^{\alpha}|H|\phi(\lambda,\tau)> = \bar{g}_{\alpha}(\xi)\,\delta(\tau), \tag{28}$$

$$\sum_{\Lambda} c_{\alpha}(t,\lambda,0) = X_{\alpha}(t). \tag{29}$$

In eq.(26) the summation over the variable ξ has been replaced by integration; $\rho(\xi)$ is the density of terms. In deriving the relation (26) we have used eq.(3ab) and the relation

$$<s^{\gamma}|H|\phi(\lambda,\tau)> = 0 \qquad (\tau>0),$$

which follows from the relations (10) and (3b). At the same time we have supposed that the quantity $\bar{g}_{\alpha}$ depends only on the parameter ξ (and not on Λ). The explicit τ-dependence of the matrix elements in eq. (28) represents also the first assumption, which limits the application of the equations (1-4).

In order to solve equations (26) we have to mention some additional properties of the coefficients $c_{\alpha}(t,\lambda,\tau)$ in the equation (1). With regard to equations (2) and (9) there holds the relation

$$c_{\alpha}(t,\lambda,\tau) = e^{-i\xi\tau'} c_{\alpha}(t-\tau',\lambda,\tau-\tau') \qquad (\tau'<\tau),$$

which in the limit $\tau'=\tau$ may be rewritten as

$$c_{\alpha}(t,\lambda,\tau) = e^{-i\xi\tau} c_{\alpha}(t-\tau,\lambda,0). \tag{30}$$

Then with the help of equations (13), (4b) and (30) we can write

$$C_{\alpha}(t) = \int_0^t \sum_{\Lambda} e^{-i\xi\tau} c_{\alpha}(t-\tau,\lambda,0)\,d\tau$$

or
$$C_\alpha(t)=e^{-i\xi t}\int_0^t\sum_\Lambda e^{i\xi y}\,c_\alpha(y,\lambda,0)\,dy,$$

Differentiating this equation according to t and using (29) we get
$$\Upsilon_\alpha(t)=\frac{\partial\, C_\alpha(t)}{\partial t}+i\xi C_\alpha(t). \qquad (31)$$

Solving now the equation system (26) we obtain with the help of (6), (31) and (16)
$$G_{\alpha\beta}=\mu\delta_{\alpha\beta}+iF_{\alpha\beta}+i\int\bar{g}_\alpha g_\beta\,\rho(\xi)\,d\xi, \qquad (32)$$

where
$$F_{\alpha\beta}=\begin{pmatrix}\gamma, & -\frac{\gamma}{\epsilon}\\ \epsilon\gamma, & -\gamma\end{pmatrix} \qquad (32a)$$

and the quantities g_α are defined by (25).

The equations (32) together with the (25) represent the general relations between the quantities $G_{\alpha\beta}$, $g_\alpha(\xi)$ and $\bar{g}_\alpha(\xi)$, i.e. between the matrix elements of the non-hermitian hamiltonian H. If we now demand the whole description to be invariant under the time reversal, we obtain
$$G_{12}=G_{21}, \quad g_\alpha(\xi)=\sum_\Lambda\bar{g}_\alpha(\xi)=\bar{g}_\alpha(\xi)\,\rho'(\xi), \qquad (33)$$

where $\rho'(\xi)$ is the number of all degrees of freedom for a given ξ. From the (32) and (33) it follows immediately
$$\epsilon=\pm i\;; \qquad (34)$$

the matrix F defined by the (32a) is symmetric (see also /3/). The other complex parameters, i.e. $\mu, \nu, K(\xi)$ and $L(\xi)$, are so far quite arbitrary.

4. The Strength of Interaction and the Energy Distribution

The behaviour of the system is of course to a large degree dependent on the strength of communication of the individual internal states with the state of decay products. If we assume, that the state of decay products is identical for both internal states, and that only the absolute value of the interaction parameters g_a may be different; we may write

$$g_1(x) = \eta \; g_2(x), \qquad (35)$$

where η is a real constant. Further it is justified to suppose that the interactions are either real or imaginary. We assume similarly as in paper/1/, that it holds

$$\mathrm{Im}\, g_a = 0, \qquad (36a)$$

$$\mathrm{Im}\, G_{12} = 0. \qquad (36b)$$

On the other hand we have to admit that at least some of the parameters G_{aa} is complex. The relations (35) and (36) represent the second principal limitation, which we have used.

Then from the (25), (36), (36a), (21) and (19a) we obtain

$$K(x) = \frac{g_2(\eta \pm i)}{x + i\frac{\Gamma}{2}},$$

$$L(x) = K(x)\left(\frac{1 \pm i\eta}{1 \mp i\eta} - \frac{2i\gamma}{x + i\frac{\Gamma}{2}}\right), \qquad (37)$$

$$g_2(x) = \pm D_1(x)\,\frac{x^2 + \frac{\Gamma^2}{4}}{Z_1(x)}$$

and

$$D_2(x) = D_1(x)\,\frac{Z_2(x)}{Z_1(x)},$$

where the real non-negative functions $Z_1(x)$ and $Z_2(x)$ are defined by

$$Z_1^2(x) = \eta^2\left(x^2 + \frac{\Gamma^2}{4}\right) + (\gamma_1^2 + \gamma_2^2)(1+\eta^2) - 2x\eta(\pm\gamma_1 + \eta\gamma_2) - \Gamma\eta(\pm\gamma_2 - \eta\gamma_1),$$

(37a)

$$Z_2^2(x) = x^2 + \frac{\Gamma^2}{4} + (\gamma_1^2 + \gamma_2^2)(1+\eta^2) + 2x(\gamma_2 \mp \eta\gamma_1) - \Gamma(\gamma_1 \pm \eta\gamma_2),$$

and for the real parameters γ_1 and γ_2 it holds

$$\gamma = \gamma_1 + i\gamma_2 \,. \tag{37b}$$

From (32), (33) and (36b) we get also

$$\gamma_2 = \pm\eta \int g_2^2 \frac{\rho(\xi)}{\rho'(\xi)}\, d\xi \,. \tag{38}$$

The complex functions $K(x)$ and $L(x)$ (and also $D_2(x)$) are now fully determined by the real non-negative function $D_1(x)$ and by the real parameter η .

Let us put now

$$\eta = 1 \,. \tag{39}$$

One can easily conclude that in this case the behaviour of both internal states must be identical, i.e. it must hold

$$D_1(x) = D_2(x) \rightarrow Z_1(x) = Z_2(x) \,. \tag{39a}$$

From the (39), (39a) and (37a) we get immediately

$$\gamma_1 = \gamma_2 = 0 \tag{39b}$$

and from the (38) and (37) also

$$D_1 = D_2 = K = L = 0 \,; \tag{39c}$$

hence

$$\Gamma = 0. \tag{39d}$$

The same conditions hold also for $\eta = -1$, since this choise is equivalent to the interchange $\epsilon \to -\epsilon$ at $\eta = 1$.

Now all the equations depend continuously on the parameter η with the only exception of $\eta = 0$. Since it is impossible to go over continuously from the condition (20c) to the condition (19c), one can conclude that the relations (39b-d) hold for all values of $\eta \neq 0$.

In the case

$$\eta = 0 \tag{40}$$

we obtain from eq. (38)

$$\gamma_2 = 0 \qquad (\gamma = \gamma_1). \tag{41}$$

Then with the help of (40) and (37) we get for the parameters g_α and $G_{\alpha\beta}$

$$\begin{aligned} &g_1 = 0, \qquad g_2 = \pm \frac{D_1}{\gamma}(x^2 + \frac{\Gamma^2}{4}), \\ &G_{12} = \pm\gamma, \qquad G_{11} = M + i(\gamma - \frac{\Gamma}{2}), \\ &G_{22} = M - i(\gamma + \frac{\Gamma}{2}) + i \int g_2^2 \rho(\xi)\, d\xi, \end{aligned} \tag{42}$$

where we have included $\rho'(\xi)$ into $\rho(\xi)$. The independent real parameters are now M , Γ , γ . From the (40),(41), (37'), (37a) and (19c) we get only

$$\int x^2 D_1^2(x)\, dx = \frac{\Gamma}{2}(2\gamma - \frac{\Gamma}{2}) > 0 \tag{43}$$

and hence the following condition must be fulfilled

$$\gamma > \frac{\Gamma}{4}. \tag{43'}$$

It remains still to find how to choose the real non-negative function $D_1(x)$ fulfilling the condition (19c).

To do this we shall suppose that the parameter g_2 is a constant, i.e. independent on x . This assumption is equivalent to the condition $\Gamma \ll M$. We obtain then

$$D_1(x) = \frac{g_2 \gamma}{x^2 + \frac{\Gamma^2}{4}}, \qquad (44)$$

and from the (37'), (19c) and (43) also

$$\gamma = \frac{\Gamma}{2}, \qquad (45)$$

which is identical with the condition derived in paper /1/

For $D_2(x)$ we get

$$D_2(x) = \frac{2|x|}{\Gamma} D_1(x) . \qquad (46)$$

From eq. (19c) it follows that the parameter g_2 is a function of Γ and m_0 (m_0 being a minimal admissible value of the parameter ξ). Supposing

$$\Gamma \ll M - m_0$$

we obtain from (19c)

$$g_2 = \pm \sqrt{\frac{\Gamma}{\pi}}$$

and hence also

$$G_{22} \doteq M - i\Gamma(1 - \frac{1}{\pi} \int \rho(\xi)\, d\xi) .$$

We may therefore summarize the results as follows: As the consequence of our limiting conditions, only one internal state can directly communicate with the state of decay products. The decay properties are then determined by one real parameter Γ which is the only parameter (apart of M , of course) left as free. If the produced internal state communicates directly with the final state, the probability amplitude and the energy distribution corresponding to this internal state are given by

$$|a_{22}(t)| = e^{-\frac{\Gamma}{2}}\left(1-\frac{\Gamma}{2}t\right),$$

$$D_2^2(\xi)= \frac{\Gamma}{\pi}\ \frac{(\xi-M)^2}{\left[(\xi-M)^2+\frac{\Gamma^2}{4}\right]^2}.$$

If the produced state can communicate with the state of decay products only indirectly (i.e. through the second state), these quantities are

$$|a_{11}(t)| = e^{-\frac{\Gamma}{2}}\left(1+\frac{\Gamma}{2}t\right),$$

$$D_1^2(\xi) = \frac{\Gamma^3}{4\pi}\ \frac{1}{\left[(\xi-M)^2+\frac{\Gamma^2}{4}\right]^2}.$$

We can easily convince ourselves that both expressions for the energy distribution are entirely equivalent to those which can be obtained from the matrix elements P_{12} and P_{22} of the propagator matrix derived in paper/3/, if we use the relation (45) and suppose also $\Gamma \ll M$.

5. *Concluding Remarks*

As it was already emphasized our results have been obtained under two main limiting conditions: 1) the $\delta(r)$--dependence of the parameters g_a , 2) the validity of the relations (35) and (36) . The exact meaning of the first approximation is not entirely clear; we may say that it is applicable without any doubt, if the decay proceeds with great energy excees, i.e. when the decay products are moving with great velocities from the place of their origin. Some results, e.g. the equations (25) and the relations derived from them (see (37), (37')) also), hold, of course, without this limitation. And it seems that the $\delta(r)$ -dependence in the relation (28) does not mean practically any restriction of the extent of applicability of equations (1- 4).

As for the second limiting condition we have to remark that the relation (35) cannot hold when each internal state decays in different way (i.e. into different decay products). Also it remains questionable whether we have to insist on the necessity of condition (36b).

The main limitation consist, of course, in our restriction to only two internal states of the unstable system. The purpose of our work is namely to explore a new mathematical approach to the quantum-mechanical description of unstable particle rather than to check the results of paper /1/.

To Prof. V. Votruba and Dr. J. Pernegr I am deaply indebted for useful discussions and valuable criticism.

References

1. *J.S.Bell, C.J.Goebel. Phys. Rev.,* <u>*138*</u>*, B1198-201(1965).*
2. *J.Jersak. Preprint E2-3470, Dubna (1967).*
3. *M.Lokajíček. Preprint E2-4093, Dubna (1968).*

Received by Publishing Department
on October 4, 1968.