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UNSTABLE SYSTEM WITH N STATES AND THE PROPAGATOR MATRIX WITH A POLE OF THE N-TH ORDER

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1. Introduction

The relation between the order of the S-matrix pole and the decay law of the unstable particle was first considered in the paper[1]. In[2] it was then shown, that the double pole can be realized by the particle system with two states changing from one state to another, decaying, however, only from a definite one of them. The study of these problems is of some interest now after the experimental finding, that the mass spectrum of the meson A-2 has very probably a higher-pole character[3] (see also[4]).

In this paper we consider an unstable system with N states, similar in its properties to the system studied in paper[2]. Starting from the general quantum-mechanical description (see[5]) we derive in the explicit way the general form of the mass matrix and of the propagator matrix containing the pole of the N-th order.

2. Mass Matrix

Before deriving the general form of the mass matrix, we summarize the result of paper[5] using the slightly changed notation. Consider an unstable system with N states $|s^{\alpha}\rangle$, in which a mutual change from one state to another is possible. This system can decay into a new state $|\phi\rangle$. Then the behaviour of our unstable system is entirely defined by the N equations

$$e^{-iHt}\,|s^{\alpha}\rangle = \sum_{\beta=1}^{N} a_{\alpha\beta}(t)\,|s^{\alpha}\rangle + |\phi_t^{\alpha}\rangle, \qquad (1)$$

where the indexes α, t of the state-vector $|\phi^{\alpha}_{t}>$ mean, that this state depends also on these values. For $t, t' > 0$ the condition

$$<s^{\alpha} | e^{-iHt'} | \phi^{\beta}_{t} > = 0 \tag{2}$$

is supposed to be fulfilled for all α, β. Let also the relations

$$<s^{\alpha} | s^{\beta} > = \delta_{\alpha\beta} \tag{3}$$

be fulfilled.

The amplitude matrix $\mathcal{A}(t)$ with the elements $a_{\alpha\beta}(t)$ fulfils the relation

$$\mathcal{A}(t+t') = \mathcal{A}(t) \cdot \mathcal{A}(t') \tag{4}$$

and it is possible to write it in a general form

$$\mathcal{A}(t) = e^{-\mu t} \sum_{p=0}^{\infty} \frac{1}{p!} \mathcal{F}^{p} t^{p} , \tag{4'}$$

where the fundamental matrix $\mathcal{F}$ is an arbitrary square matrix of the N th order and μ is an arbitrary complex parameter.

As far as $\mathcal{F}^{p} \neq 0$ for an arbitrary p, the matrix $\mathcal{F}$ can be diagonalized and the known mixing of N states, each of which decays exponentially, is obtained. If, however, an integer number q / $q \leq N$ / exists, for which the following conditions are fulfilled

$$\mathcal{F}^{q-1} \neq 0 , \quad \mathcal{F}^{q} = 0 , \tag{5}$$

then an entirely different case takes place; there exists only one state (a linear combination of states $|s^a>$), decaying exponentially. The decay laws of all other orthogonal states are more complicated and depend substantially on the actual form of the matrix $\mathfrak{F}$.

In this latter case the matrix $\mathfrak{F}$ can be written in the general form

$$\mathfrak{F} = \mathfrak{U}\mathfrak{Z}\mathfrak{U}^{-1} ,$$

where all elements of the square matrix $\mathfrak{Z}$ are equal to zero except of the elements $z_{a,a+1}$, which are equal to 1 (or 0). The form of the square matrix $\mathfrak{U}$ is entirely arbitrary. It is easy to show, that the relations

$$D(\mathfrak{F}^p) = Tr(\mathfrak{F}^p) = 0 \tag{6}$$

hold, i.e. both the determinant and the trace equal to zero for any power of $\mathfrak{F}$.

The matrix $\mathfrak{A}(t)$ is then of the form

$$\mathfrak{A}(t) = e^{-i\mu t} \sum_{p=0}^{q-1} \frac{1}{p!} \mathfrak{F}^p t^p . \tag{7}$$

In the special case $N = q = 2$ the matrix $\mathfrak{A}(t)$ becomes

$$\mathfrak{A}(t) = e^{-i\mu t} \begin{pmatrix} 1+\gamma t , & \epsilon \gamma t \\ -\frac{\gamma}{\epsilon} t , & 1-\gamma t \end{pmatrix} \tag{8}$$

where the parameters μ, γ, ϵ are arbitrary complex constants.

Differentiating now the relation (1) and putting $t = 0$ we obtain

$$-iH|s^{\alpha}> = \sum_{\beta=1}^{N} \dot{a}_{\alpha\beta}(0)\,|s^{\beta}> + |\dot{\phi}^{\alpha}_{t=0}> . \qquad (9)$$

Since

$$<s^{\alpha}|\phi^{\beta}_{t}> = 0,$$

we get from equations (9) and (3)

$$<s^{\beta}|H|s^{\alpha}> = i\dot{a}_{\alpha\beta}(0). \qquad (10)$$

If the equation (1) describes the time evolution of the state $|s^{\alpha}>$ corresponding to free unstable particle at rest, the quantities $<s^{\alpha}|H|s^{\beta}>$ are the elements of the mass matrix $\mathfrak{M}$ and we can write

$$\mathfrak{M} = i\,[\frac{\partial}{\partial t}\,\mathfrak{A}^{T}(t)]_{t=0} , \qquad (11)$$

where $\mathfrak{A}^{T}$ means the transposed matrix. With the help of (7) we obtain then the final relation for the mass matrix of our unstable system

$$\mathfrak{M} = \mu I + i\,\mathfrak{F}^{T} \quad ; \qquad (12)$$

I is the unit diagonal matrix.

If we require the relations (1)-(3) to be invariant under the time reversal we get

$$\mathcal{M} = \mathcal{M}^T$$

and therefore
$$\mathcal{F} = \mathcal{F}^T .$$

On the other hand neither the matrix $\mathcal{M}$ nor the hamiltonian H are hermitian.

In the special case $N = q = 2$ one can write

$$\mathcal{M} = \begin{pmatrix} M - i(\frac{\Gamma}{2} - \gamma), & -i\frac{\gamma}{\epsilon} \\ i\epsilon\gamma , & M - i(\frac{\Gamma}{2} + \gamma) \end{pmatrix} \tag{13}$$

wnere

$$\mu = M - i\frac{\Gamma}{2} . \tag{14}$$

In the case of time reversal invariance it must hold $\epsilon = \pm i$.

3. *Propagator Matrix*

When the unstable system with N states works as an intermediate state in some collision process, the mass in the corresponding propagator has to be substituted by the mass matrix. The

$$\mathcal{P} = (k^2 I - \mathcal{M}^2)^{-1} , \tag{15}$$

where k^2 is the square of the four-momentum. Then the propagator matrix $\mathcal{P}$ can be written in the form

$$\mathcal{P} = D^{-1}(k^2 I - \mathcal{M}^2)\ \bar{\mathcal{N}}(k^2 I - \mathcal{M}^2) , \tag{16}$$

where the elements of the matrix $\bar{\mathfrak{N}}(\theta)$ are the corresponding minors of the matrix θ.

From the properties of matrix $\mathfrak{F}$ it results that the equation

$$D(k^2 I - \mathfrak{M}^2) = 0 \tag{17}$$

has a N-fold root

$$k^2 = \mu^2 . \tag{17'}$$

We can show it in the following manner. When we introduce the matrix (see eq./12/)

$$\mathfrak{R} = 2i\mu\,\mathfrak{F}^T - \mathfrak{F}^{T^2} = -\mu^2 I + \mathfrak{M}^2 , \tag{18}$$

we easily find with the help of relation (6) that

$$D(\mathfrak{R}) = 0;$$

differentiating now the left-hand side of equation (17), one finds with the help of relations (18) and (6) that

$$\frac{1}{N!}\frac{\partial^{N-1}}{\partial(k^2)^{N-1}} D(k^2 I - \mathfrak{M}^2) = k^2 - \frac{1}{N}\mathrm{Tr}(\mathfrak{M}^2) = k^2 - \mu^2 ,$$

hence the proof is finished.

We can write therefore

$$D(k^2 I - \mathfrak{M}^2) = (k+\mu)^N (k-\mu)^N . \tag{19}$$

Using (14) and supposing $M \gg \Gamma$, we obtain in the case of the positive real values of the $k \approx M$ the expression

$$D(k^2 I - \mathfrak{M}^2) = (2M)^N (k-\mu)^N . \tag{20}$$

We now derive the explicit form of the matrix $\bar{\mathfrak{N}}(k^2 I - \mathfrak{M}^2)$. This matrix is defined by the relation

$$[(k^2-\mu^2)I - \mathfrak{R}]\bar{\mathfrak{N}} = (k^2-\mu^2)^N I. \tag{21}$$

One can write

$$\bar{\mathfrak{N}} = \sum_{j=0}^{N-1} (k^2-\mu^2)^j \, \mathfrak{N}_j , \tag{22}$$

where the matrices $\mathfrak{N}_j$ $(j=0,\dots,N-1)$ no more depend on the value k^2. From the relation (21) we get the recurrent formula

$$\mathfrak{N}_j = \mathfrak{R}\mathfrak{N}_{j+1} \cdot$$

which together with condition

$$\mathfrak{N}_{N-1} = I$$

gives the general formula

$$\mathfrak{N}_j = \mathfrak{R}^{N-j-1} . \tag{23}$$

It holds then

$$\mathcal{P} = \sum_{j=1}^{N} (k^2-\mu^2)^{-j} \, \mathfrak{R}^{j-1} \tag{24}$$

and if we use the same approximation as in the relation (20) we obtain the final expression

$$\mathcal{P} = (k-\mu)^{-N} \sum_{\ell=0}^{N-1} (k-M)^{\ell} \, \mathfrak{N}'_{\ell} , \tag{25}$$

where

$$\mathfrak{N}'_{\ell} = \sum_{j=\ell}^{N-1} \binom{j}{\ell} (2M)^{j-N} \left(i\frac{\Gamma}{2}\right)^{j-\ell} \mathfrak{R}^{\cdot N-j-1} . \qquad (25')$$

In this way we obtain in the corresponding S matrix the pole of the N-th order without imposing any other limitation on the matrix $\mathfrak{F}$. The propagator matrix contains only the singular terms; it contains no regular term depending on the k^2, which is usually obtained when we consider the interaction with other particles. If we assume that the interaction fulfils the condition (2) for all $t, t' > 0$ and that all bound states of the decay products are identical with the internal states of our unstable particle, we can consider the matrix (24), resp. (25) as the general form of the propagator matrix of the physical particle.

In this paper a detailed form of the interaction with decay states has not been considered. As it follows from paper/6/, where an appropriate mathematical analysis of the model of an unstable particle with two internal states has been carried out, the exact form of the interaction with the decay products involves further conditions which bind together the elements of the matrix $\mathfrak{F}$ with the real parameter Γ. One can hope that with the help of matrix $\mathfrak{F}$ it will be possible to describe the behaviour of strongly interacting particles and their resonances in a mathematically consistent manner.

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