ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2 - 4030

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

V.G.Kadyshevsky, M.D.Mateev, R.M.Mir-Kasimov

ON RELATIVISTIC THREE-DIMENSIONAL EQUATIONS FOR THE TWO-BODY PROBLEM IN THE UNEQUAL MASS CASE

1968

E2 - 4030

V.G.Kadyshevsky, M.D.Mateev, R.M.Mir-Kasimov

ON RELATIVISTIC THREE-DIMENSIONAL EQUATIONS FOR THE TWO-BODY PROBLEM IN THE UNEQUAL MASS CASE

I.

In refs.[1-7], three-dimensional equations of Lippmann-Schwinger and Schrödinger types for a system of two relativistic scalar particles with equal masses have been obtained and investigated. It was seen that these equations can be interpreted as generalizations of the corresponding nonrelativistic equations in the spirit of a Lobachevsky geometry realized on the upper sheet of the hyperboloid

$$p_0^2 - \vec{p}^{\,2} = m^2 . \tag{1.1}$$

For example, the obtained relativistic equation, which plays the role of the Lippmann-Schwinger equation in the centre of mass system (c.m.s.) is written in the form:

$$A(\vec{p}, \vec{q}) = -\frac{m}{4\pi} \tilde{V}(\vec{p}, \vec{q}\,; E_q) + \frac{1}{(2\pi)^3} \int \tilde{V}(\vec{p}, \vec{k}\,; E_q) \frac{d\Omega_{\vec{k}}\, A(\vec{k}, \vec{q})}{2E_q - 2E_k + i\epsilon} , \tag{1.2}$$

where

$$E_q = \sqrt{m^2 + \vec{q}^{\,2}}, \quad E_k = \sqrt{m^2 + \vec{k}^{\,2}}, \quad E_p = \sqrt{m^2 + \vec{p}^{\,2}}, \tag{1.3}$$

$$d\Omega_{\vec{k}} = \frac{d\vec{k}}{\sqrt{1 + \frac{\vec{k}^2}{m^2}}} .$$

$\tilde{V}(\vec{p}, \vec{q}; E_q)$ is the "quasipotential" x/, $A(\vec{p}, \vec{q})$ is the relativistic elastic scattering amplitude off the energy shell xx/, while the proper Lippmann-Schwinger equation for the nonrelativistic amplitude $A(\vec{p}, \vec{q})$ in the equal mass case has the form:

$$A(\vec{p}, \vec{q}) = -\frac{m}{4\pi} \tilde{V}(\vec{p}, \vec{q}) + \frac{1}{(2\pi)^3} \int \frac{\tilde{V}(\vec{p}, \vec{k}) d\vec{k}\, A(\vec{k}, \vec{q})}{2E_q - 2E_k + i\epsilon} , \tag{1.5}$$

E_q, E_p and E_k are the nonrelativistic energies:

$$E_q = \frac{\vec{q}^{\,2}}{2m}, \quad E_p = \frac{\vec{p}^{\,2}}{2m}, \quad E_k = \frac{\vec{k}^{\,2}}{2m} \tag{1.6}$$

---

x/ We borrowed this terminology from the quasipotential approach to quantum field theory of Logunov and Tavkhelidze /8/, to which the idea of present formalism is close.

xx/ On the energy shell, $E_p = E_q$, this amplitude is normalized to the differential cross -section:

$$\frac{d\sigma}{d\Omega} = |A(\vec{p}, \vec{q})|^2 \tag{1.4}$$

and $\tilde{V}(\vec{p},\vec{k})$ is the Fourier transform of the interaction potential.

The geometrical analogy between equations (1.2) and (1.5) turned out to be far-reaching and fruitful. It allowed us to approach correctly the concept of relativistic $\vec{r}$ -space and to obtain useful relations and results.

The principal aim of the present article is to obtain a relativistic generalization, as simple as (1.2), of the Lippmann-Schwinger equation in the unequal mass case.

As it is known, to pass in (1.5) to the unequal mass case it is sufficient to replace the parameter m in this equation by twice the reduced mass:

$$m \to 2\mu = \frac{2m_1 m_2}{m_1 + m_2} . \tag{1.7}$$

Doing this, we get the equation:

$$A(\vec{p},\vec{q}) = -\frac{\mu}{4\pi}\tilde{V}(\vec{p},\vec{q}) + \frac{1}{(2\pi)^3}\int \frac{\tilde{V}(\vec{p},\vec{k})\, d\vec{k}\, A(\vec{p}',\vec{q})}{\frac{\vec{q}^{\,2}}{2\mu} - \frac{\vec{k}^{\,2}}{2\mu} + i\epsilon} . \tag{1.8}$$

However, in the relativistic case the aplication of the procedure (1.7) to eq. (1.2) for obtaining the equation we are looking for does not make sense at all. To make this point clear, let us recall how the concept of reduced mass arises in the nonrelativistic theory.

Let

$$E = \frac{\vec{k}_1^{\,2}}{2m_1} + \frac{\vec{k}_2^{\,2}}{2m_2} = \frac{m_1 \vec{v}_1^{\,2}}{2} + \frac{m_2 \vec{v}_2^{\,2}}{2} \tag{1.9}$$

be the total energy of two free nonrelativistic particles in an arbitrary reference system. Let us introduce the vectors of the total momentum

$$\vec{K} = \vec{k}_1 + \vec{k}_2 \tag{1.10}$$

and the relative momentum

$$\vec{k} = \frac{m_2 \vec{k}_1 - m_1 \vec{k}_2}{m_1 + m_2} = \mu (\vec{v}_1 - \vec{v}_2). \tag{1.11}$$

Performing a Galilei transformation

$$\begin{aligned} \vec{v}_1 &= \frac{\vec{k}}{m_1} + \frac{\vec{K}}{m_1 + m_2} \\ \vec{v}_2 &= -\frac{\vec{k}}{m_2} + \frac{\vec{K}}{m_1 + m_2} \end{aligned} \tag{1.12}$$

we shall have instead of (1.9):

$$E = \frac{\vec{K}^2}{2(m_1 + m_2)} + \frac{\vec{k}^2}{2\mu} .$$

In the c.m.s. $\vec{K} = 0$ and

$$E = \frac{\vec{k}^2}{2\mu} . \tag{1.13}$$

The difference of two "one-particle" energies of the type (1.13) gives rise to the energy denominator of the equation (1.8). Let us emphasize, that the momentum of the "effective particle" replacing the two-body system is the relative momentum $\vec{k}$ whose length, because of the formulae:

$$\vec{k}^2 = \mu^2(\vec{v}_1 - \vec{v}_2)^2 \equiv \mu \vec{v}^2 \qquad (1.14)$$

is an invariant under Galilei transformations.

In the relativistic case the 4-momenta of the two free particles can be written in c.m.s. in the form:

$$k_1 = (\sqrt{m_1^2 + \vec{k}^2}\ ,\ \vec{k}\ ),$$
$$k_2 = (\sqrt{m_2^2 + \vec{k}^2}\ ,\ -\vec{k}\ ). \qquad (1.15)$$

The total energy in the same reference system is

$$\sqrt{m_1^2 + \vec{k}^2} + \sqrt{m_2^2 + \vec{k}^2} = \sqrt{s_k}\ . \qquad (1.16)$$

When $m_1 = m_2 = m$ we have:

$$2\sqrt{m_1^2 + \vec{k}^2} = \sqrt{s_k}\ . \qquad (1.17)$$

The energy denominator in equation (1.2) is evidently a difference between expressions of the type (1.17).

Eq. (1.16) and (1.17) imply that independently of whether the masses are equal or not, the total energy of two relativistic particles in c.m.s. can not be regarded as the energy of some effective relativistic particle with momentum $\vec{k}$. However, the fact that in the equal mass case the total energy (1.17) is proportional to the energy of a single particle, allows us to consider the energy denominator of the relativistic Lippmann-Schwinger equation (1.2) as "one-particle" and as a result to reduce the relativistic two-body problem in this case to the problem of motion of a particle with mass m in a quasipotential field. It is clear that if we succeed in writing (1.16) as an expression proportional to the energy of a relativistic particle with mass m' (which reduces to m when $m_1=m_2=m$), then also in the unequal mass case the relativistic two-body problem will become equivalent to the one-particle problem. Moreover, in such a case it would be possible to apply the mathematical methods which we used earlier in the problem of particles with equal masses.

If we denote by $\vec{k}'$ the momentum of the "particle" with mass m', we shall have, in accordance with our assumption:

$$\sqrt{s_k} = \sqrt{m_1^2 + \vec{k}^2} + \sqrt{m_2^2 + \vec{k}^2} = \text{const}\,\sqrt{m'^2 + \vec{k}'^2}. \qquad (1.18)$$

The physical meaning of the quantity $\vec{k}'$, which we shall call the relativistic relative momentum of the system of two particles with unequal masses will be clarified in detail in Section III.

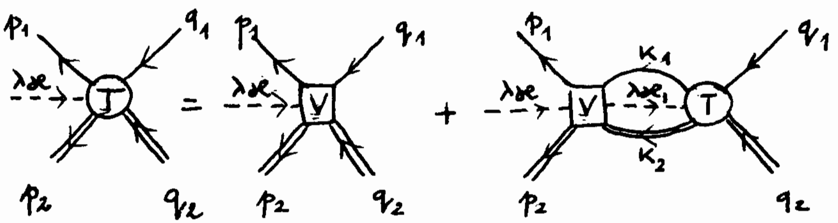
In the next section we shall find the three-dimensional quasipotential type equation for the relativistic scattering amplitude in the case $m_1 \neq m_2$, which later will be the basis for obtaining an exact geometrical analogue of the Lippmann-Schwinger equation (1.8).

II. Equation for the Relativistic Scattering Amplitude $T(\vec{p},\vec{q})$ in the Unequal Mass Case

Let $T(s,t)$ be the invariant elastic scattering amplitude for two scalar particles with masses m_1 amd m_2 connected with the differential cross-section by the relation:

$$\frac{d\sigma}{d\Omega} = \frac{|T(s,t)|^2}{(8\pi)^2 s} . \tag{2.1}$$

According to /1-3/ this amplitude off the energy shell satisfies an integral equation, which can be represented graphycally in the following manner:



(2.2)

The single continuous line corresponds to a particle with mass m_1, the double line to a particle with mass m_2, and the dotted line to a spurious "quasiparticle" with a fourvelocity λ_μ and mass $\varkappa$ or $\varkappa'$.

When $\varkappa = 0$ the solutions of the equation (2.2) $T(\lambda\varkappa, p_1, p_2, q_1, q_2)$ coincide with the amplitude $T(s,t)$. The energy-momentum conservation on the shall $\varkappa = 0$ is governed by four-dimentional δ - functions of the form

$$\delta(\lambda\varkappa_1 + q_1 + q_2 - p_1 - p_2),\ \delta(\lambda\varkappa - \lambda\varkappa' + k_1 + k_2 - p_1 - p_2) \quad \text{etc.}$$

If the vector λ_μ is chosen to be collinear with the total energy-momentum vector of the incoming particles /1,2/

$$\lambda = \frac{q_1 + q_2}{\sqrt{(q_1+q_2)^2}}, \tag{2.3}$$

then because of the energy-momentum conservation this vector will be also collinear with the vectors $(p_1+p_2)/\sqrt{(p_1+p_2)^2}$ and $(k_1+k_2)/\sqrt{(k_1+k_2)^2}$. Therefore, as a result of this choice, the 4-velocity of the considered colliding-particle system will be a conserved quantity:

$$\frac{q_1+q_2}{\sqrt{(q_1+q_2)^2}} = \frac{p_1+p_2}{\sqrt{(p_1+p_2)^2}} = \frac{k_1+k_2}{\sqrt{(k_1+k_2)^2}}. \tag{2.4}$$

Taking into account (2.4) we can write the function $T(\lambda x, p_1, p_2, q_1, q_2)$ as:

$$T(\lambda x, p_1, p_2, q_1, q_2) = T(s_p, t_{pq}, s_q), \tag{2.5}$$

where

$$\begin{aligned} s_p &= (p_1+p_2)^2 \\ s_q &= (q_1+q_2)^2 \\ t_{pq} &= (p_1-q_1)^2 . \end{aligned} \tag{2.6}$$

In a similar manner invariant variables are introduced in the "quasipotential".

Applying the diagram techniques[1-3], the homogeneous term of eq. (2.2) can be written explicitly:

$$\frac{1}{(2\pi)^8}\int dk_1 dk_2\, D^{(+)}(k_1,m_1)D^{(+)}(k_2,m_2)\frac{d\varkappa'}{\varkappa'-i\epsilon}\delta(\lambda\varkappa'+q_1+q_2-k_1-k_2)\times \quad (2.7)$$

$$\times V(\lambda\varkappa, p_1, p_2, k_1, k_2, \lambda\varkappa')T(\lambda\varkappa', k_1, k_2, q_1, q_2),$$

where

$$D^{(+)}(k,m) = \theta(k_0)\delta(k^2-m^2).$$

Furthermore, taking into account (2.3) and (2.4), it is convenient to pass into the c.m.s. where $\vec{q}_1+\vec{q}_2=\vec{k}_1+\vec{k}_2=\vec{p}_1+\vec{p}_2=\vec{\lambda}=0$.
Putting $\vec{q}_1=-\vec{q}_2\equiv\vec{q}$, $\vec{k}_1=-\vec{k}_2\equiv\vec{k}$, $\vec{p}_1=-\vec{p}_2\equiv\vec{p}$,
we obviously shall have (compare with (1.6)):

$$\begin{aligned} s_q &= \left(\sqrt{m_1^2+\vec{q}^{\,2}}+\sqrt{m_2^2+\vec{q}^{\,2}}\right)^2 \\ t_{kq} &= (k_1-q_2)^2 = 2m_1^2-2\left(\sqrt{\vec{k}^{\,2}+m_1^2}\sqrt{\vec{q}^{\,2}+m_1^2}-\vec{k}\vec{q}\right) \\ s_k &= \left(\sqrt{m_1^2+\vec{k}^2}+\sqrt{m_2^2+\vec{k}^{\,2}}\right)^2 \qquad \text{etc.} \end{aligned} \quad (2.8)$$

In agreement with (2.8), the amplitude $T(s_p, t_{pq}, s_q)$ in the c.m.s. can be considered as a function of two three-dimensional vectors $\vec{p}$ and $\vec{q}$:

$$T = T(\vec{p},\vec{q}).$$

The expression (2.7), after passing to the c.m.s and some simple calculations, can be written in the form:

$$\frac{1}{4(2\pi)^{8}}\int\frac{d\vec{k}}{\sqrt{m_1^2+\vec{k}^2}}\;\frac{V(\vec{p},\vec{k};s_q)T(\vec{k},\vec{q})}{\sqrt{m_2^2+\vec{k}^2}\,(\sqrt{s_k}-\sqrt{s_q}-i\epsilon)}\,, \qquad (2.9)$$

where we have denoted the quasipotential $V(\lambda\chi, p_1, p_2, k_1, k_2, \lambda\chi')$ by $V(\vec{p},\vec{k};s_q)$.

We would like to note that the energy denominator in this case is a difference of two expressions of the type (1.16):

$$\frac{1}{\sqrt{s_k}-\sqrt{s_q}-i\epsilon}=\frac{1}{\sqrt{m_1^2+\vec{k}^2}+\sqrt{m_2^2+\vec{k}^2}-\sqrt{m_1^2+\vec{q}^2}-\sqrt{m_2^2+\vec{q}^2}-i\epsilon}\,. \qquad (2.10)$$

If we transform to the variables $\vec{k}'$ and $\vec{q}'$, using (1.18) we get:

$$\frac{1}{\sqrt{s_k}-\sqrt{s_q}-i\epsilon}=\frac{1}{\mathrm{const}\,(\sqrt{m'^2+\vec{k}'^2}-\sqrt{m'^2+\vec{q}'^2}-i\epsilon)}\,. \qquad (2.11)$$

III. The Relativistic Relative Momentum of Two Particles with Different Masses

In this Section we shall investigate the properties of the relative momentum $\vec{k}$ introduced at the end of Section I. When doing this the analogy with the nonrelativistic case will be important.

According to (1.14), in the nonrelativistic theory the length of the relative momentum vector is a function both of the reduced mass μ and of the relative velocity, which is an invariant under the Galilei transformations. Assuming that $\vec{k}'$, plays the role of a relativistic generalization of the vector (1.11), we can also suppose for reasons of correspondence, that $\vec{k}'^2$ is also a function of the reduced mass μ and of the relativistic relative velocity v_{rel} :

$$\vec{k}'^2 = f(\mu, v_{rel}) . \tag{3.1}$$

The form of the function f can be determined from the condition that in the equal mass case, because of (1.16)-(1.18), $\vec{k}'^2 = \vec{k}^2$. Therefore from (3.1) we have:

$$\vec{k}^2 = f(\frac{m}{2}, v_{rel}) . \tag{3.2}$$

On the other hand, as is well known, the relative velocity is connected with the noneuclidean distance s in the relativistic velocity space[x)] and the scalar product of the 4-velocities of the particles by the following relations:

---

x) Let us recall that the relativistic $\vec{v}$-space has a Lobachevsky geometry.

$$\frac{1}{\sqrt{1-\vec{v}^2_{rel}}} = \operatorname{ch} s = u_1 u_2 = u_{10} u_{20} - \vec{u}_1 \vec{u}_2 , \qquad (3.3)$$

where in the general case [x)]

$$u_1 = \frac{k_1}{m_1} = \left(\frac{1}{\sqrt{1-\vec{v}_1^{\,2}}} , \frac{\vec{v}_1}{\sqrt{1-\vec{v}_1^{\,2}}} \right),$$

$$(3.4)$$

$$u_2 = \frac{k_2}{m_2} = \left(\frac{1}{\sqrt{1-\vec{v}_2^{\,2}}} , \frac{\vec{v}_2}{\sqrt{1-\vec{v}_2^{\,2}}} \right).$$

In the c.m.s., when $m_1 = m_2$, with the help of (3.3) and (3.4) it is easy to see that

$$\vec{k}^2 = m^2 \operatorname{sh} \frac{s}{2}$$

from where, taking into account (3.1)-(3.4), we get:

$$\vec{k}'^2 = 4\mu^2 \operatorname{sh}^2 \frac{s}{2} = -\mu^2 (u_1 - u_2)^2 =$$

$$(3.5)$$

$$= -\left(\frac{m_2 k_1 - m_1 k_2}{m_1 + m_2} \right)^2 .$$

---

x) In the c.m.s., using the Einstein addition law for the collinear vectors $\vec{v}_1 = \vec{k}/\sqrt{m_1^2 + \vec{k}^2}$ and $\vec{v}_2 = -\vec{k}/\sqrt{m_2^2 + \vec{k}^2}$, one can define the relative velocity vector $\vec{v}_{rel}$:

$$\vec{v}_{rel} = (\vec{v}_1 - \vec{v}_2)/1 - \vec{v}_1 \vec{v}_2 = \frac{2\sqrt{s_k}}{s_k - m_1^2 - m_2^2} \vec{k} \qquad (3.3')$$

Therefore, $-\vec{k}'^2$ is the square of a space-like 4-vector $\frac{m_2 k_1 - m_1 k_2}{m_1 + m_2} = k$, which in the c.m.s. becomes:

$$k = \left(\frac{m_2 \sqrt{m_1^2 + \vec{k}^2} - m_1 \sqrt{m_2^2 + \vec{k}^2}}{m_1 + m_2} , \vec{k} \right). \tag{3.6}$$

In the nonrelativistic limit $s \ll 1$, (3.5) goes into (1.14).

Combining relations (3.3) and (3.5) the following formula can be obtained

$$\frac{\vec{k}'^2}{2\mu} = \frac{\mu}{\sqrt{1 - \vec{v}^2_{rel}}} - \mu , \tag{3.7}$$

which says that in the relativistic case the kinetic energy of the relative motion can be expressed with the help of the relative momentum $\vec{k}'$ and the reduced mass μ in a "nonrelativistic" way.

Substituting , further, (3.5) in (1.18) we easily find:

$$\sqrt{s_k} = \sqrt{m_1 + \vec{k}^2} + \sqrt{m_2 + \vec{k}^2} = \frac{m_1 + m_2}{\sqrt{m_1 m_2}} \sqrt{m_1 m_2 + \vec{k}'^2} . \tag{3.8}$$

Therefore, the mass m' of the "effective particle" in the relations (1.18) and (2.11) is the geometric mean of the masses m_1 and m_2 :

$$m' = \sqrt{m_1 m_2} .$$

At the same time, it is easy to see that in the nonrelativistic limit (2.11) goes exactly into the denominator of the eq. (1.8), depending on the reduced mass:

$$\frac{1}{\frac{m_1 + m_2}{\sqrt{m_1 m_2}} \left(\sqrt{m_1 m_2 + \vec{k}'^2} - \sqrt{m_1 m_2 + \vec{q}'^2} - i\epsilon \right)} \to \frac{1}{\frac{\vec{k}'^2}{2\mu} - \frac{\vec{q}'^2}{2\mu} - i\epsilon} .$$

Keeping in mind the analogy with the nonrelativistic formula (1.11), we choose the direction of the vector $\vec{k}'$ to be collinear to the direction of the relative velocity vector $\vec{v}_{rel}$, defined in (3.3'). Then, taking into account (3.5), $\vec{k}'$ can be written:

$$\vec{k}' = \frac{\vec{k}}{|\vec{k}|}\sqrt{\vec{k}^2 - \frac{1}{(m_1+m_2)^2}\left(m_2\sqrt{m_1^2+\vec{k}^2} - m_1\sqrt{m_2^2+\vec{k}^2}\right)^2}\,, \qquad (3.9)$$

or in four-dimensional form, in an arbitrary reference system:

$$k'_\mu = (k'_0, \vec{k}') = \sqrt{\frac{k^2}{k_\perp^2}}\,(k_\perp)_\mu\,, \qquad (3.10)$$

where

$$k = \frac{m_2 k_1 - m_1 k_2}{m_1 + m_2}\,,$$

$$k_\perp = k - \lambda(\lambda \cdot k) = \frac{k_1 - k_2}{2} - \frac{m_1^2 - m_2^2}{2\sqrt{s_k}}\,\lambda\,,$$

$$(\lambda k_\perp) = 0\,, \qquad \lambda = \frac{k_1 + k_2}{\sqrt{s_k}}\,.$$

As one can see from (3.9)

$$k'^2 = k^2$$

in all references systems. In the c.m.s., where $\lambda = (1,\vec{0})$ we have $k'_0 = 0$ and the eq. (3.11) goes into (3.5), and (3.10) into (3.9). The vector $k_\perp$, which appears here, has been introduced earlier and is usually called the Wightmann-Görding relative momentum /9,10/.

A number of simple relations for the momentum $\vec{k}'$, appears if we pass to spherical coordinates on the hyperboloids $k_1^2 = m_1^2$, $k_2^2 = m_2^2$ and $k'^2 = m'^2$:

$$|\vec{k}_1| = m_1 \operatorname{sh} \chi_1, \qquad k_{10} = m_1 \operatorname{ch} \chi_1$$
$$|\vec{k}_2| = m_2 \operatorname{sh} \chi_2, \qquad k_{20} = m_2 \operatorname{ch} \chi_2 \tag{3.12}$$
$$|\vec{k}'| = m' \operatorname{sh} \chi', \qquad k_0' = m' \operatorname{ch} \chi' .$$

Substituting (3.12) into (3.8) and taking into account (1.15), (3.3) and (3.4), we have:

$$m_1 \operatorname{ch} \chi_1 + m_2 \operatorname{ch} \chi_2 = (m_1 + m_2) \operatorname{ch} \chi'$$
$$|\vec{k}'| = 2\mu \operatorname{sh} \frac{\chi_1 + \chi_2}{2} . \tag{3.13}$$

When $m_1 = m_2$, evidently $\chi_1 = \chi_2 = \chi'$.

IV. The Equation for the Scattering Amplitude in "One-Particle" Form

Let us return to eq. (2.2) and make in it a change of variables of the type (3.9). Then the volume element transforms into:

$$\frac{d\vec{k}}{\sqrt{m_1^2 + \vec{k}^2}\sqrt{m_2^2 + \vec{k}^2}} = \frac{1}{2\sqrt{m_1 m_2}} \frac{d\vec{k}'}{\sqrt{1 + \frac{\vec{k}'^2}{m_1 m_2}}} \cdot f(\vec{k}', m_1, m_2), \tag{4.1}$$

where

$$f(\vec{k}', m_1, m_2) = \frac{\sqrt{\vec{k}'^2 + 4\mu^2}}{\vec{k}'^2 + m_1 m_2} = \frac{\sqrt{4\mu^2 \mathrm{sh}^2\frac{s}{2} + m_1 m_2}}{4\mu^2 \mathrm{ch}^2\frac{s}{2}},$$

$$\mathrm{ch}\, s = u_1 u_2, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}. \tag{4.2}$$

Then putting

$$A(\vec{p},\vec{q}) = \sqrt{f(\vec{p}, m_1, m_2)}\; T(\vec{p},\vec{q}) \sqrt{f(\vec{q}, m_1, m_2)}, \tag{4.3}$$

$$\tilde{V}(\vec{p},\vec{q}) = -\frac{1}{8\mu}\sqrt{f(\vec{p}, m_1, m_2)}\; V(\vec{p},\vec{q}) \sqrt{f(\vec{q}, m_1, m_2)},$$

$$d\Omega_{\vec{k}'} = \frac{d\vec{k}'}{\sqrt{1 + \frac{\vec{k}'^2}{m_1 m_2}}}, \quad E_{q'} = \sqrt{m_1 m_2 + \vec{q}'^2}, \quad E_{k'} = \sqrt{m_1 m_2 + \vec{k}'^2}$$

and taking into account (2.9) and (3.8) we have:

$$A(\vec{p}', \vec{q}') = -\frac{\mu}{2\pi}\tilde{V}(\vec{p}', \vec{q}'; E_{q'}) + \tag{4.5}$$

$$+ \frac{1}{(2\pi)^3}\, \frac{\sqrt{m_1 m_2}}{m_1 + m_2} \int \frac{\tilde{V}(\vec{p}', \vec{k}', s_q)\, d\Omega_{k'}\, A(\vec{k}', \vec{q}')}{E_{q'} - E_{k'} + i\epsilon}.$$

The eq. (4.5) can be considered as a relativistic generalization of the Lippmann-Schwinger equation (1.8), in the spirit of the Lobachevsky geometry realized on the upper sheet of the hyperboloid $p'^2 = m_1 m_2$. The difference between the coefficients in front of the integral terms is not essential because, as one immediately sees, (1.8) can be written in the form (4.5):

$$A(\vec{p},\vec{q}) = -\frac{\mu}{2\pi}\tilde{V}(\vec{p},\vec{q}) + \frac{1}{(2\pi)^3}\frac{\sqrt{m_1 m_2}}{m_1+m_2}\int \frac{\tilde{V}(\vec{p},\vec{k})d\vec{k}\, A(\vec{k},\vec{q})}{\frac{\vec{q}^2}{2\sqrt{m_1 m_2}} - \frac{\vec{k}^2}{2\sqrt{m_1 m_2}} - i\epsilon}. \tag{4.6}$$

On the energy shell $E_{p'} = E_{q'}$ the amplitude $A(\vec{p}',\vec{q}')$, because of (2.1) and (4.3), satisfies the following normalization condition:

$$\frac{d\sigma}{d\Omega} = \frac{1}{f}|A(\vec{p}',\vec{q}')|^2 = \frac{4\mu^2 \operatorname{ch}^2\frac{s}{2}}{\sqrt{4\mu^2 \operatorname{sh}^2\frac{s}{2} + m_1 m_2}}|A(p',q')|^2. \tag{4.7}$$

Now let us briefly discuss the formalism connected with the Schrödinger equation. The wave function of the system we define as (compare with/4/):

$$\Psi_{q'}(\vec{p}') = (2\pi)^3\,\delta(\vec{p}'(-)\vec{q}') - \frac{2\pi}{\sqrt{m_1 m_2}}\,\frac{1}{\sqrt{m_1 m_2 + \vec{q}^2} - \sqrt{m_1 m_2 + \vec{p}'^2}}\,A(\vec{p}',\vec{q}'), \tag{4.8}$$

where $\delta(\vec{p}\,'(-)\vec{q}\,') = \sqrt{1+\vec{p}\,'^2/m_1m_2}\,\delta(\vec{p}\,'-\vec{q}\,')$. With the help of (4.8) and (4.5) we easily get

$$\left(\sqrt{m_1m_2+\vec{q}\,'^2}-\sqrt{m_1m_2+\vec{p}\,'^2}\right)\Psi_{q'}(\vec{p}\,') =$$

$$= \frac{1}{(2\pi)^3}\,\frac{\sqrt{m_1m_2}}{m_1+m_2}\int \tilde{V}(\vec{p}\,',\vec{k}\,')\,d\Omega_{\vec{k}'}\,\Psi_{q'}(\vec{k}\,'). \tag{4.9}$$

Equation (4.9) is evidently relativistic analogue of the Schrödinger equation

$$\left(\frac{\vec{q}^{\,2}}{2\mu}-\frac{\vec{p}^{\,2}}{2\mu}\right)\Psi_q(\vec{p}) = \frac{1}{(2\pi)^3}\int \tilde{V}(\vec{p},\vec{k})\,d\vec{k}\,\Psi_q(\vec{k})\,. \tag{4.10}$$

Let us further define the relativistic wave function in the $\vec{r}$-representation:

$$\Psi_{q'}(\vec{r}\,') = \frac{1}{(2\pi)^3}\int \xi(\vec{p}\,',\vec{r}\,')\,\Psi_{q'}(\vec{p}\,')\,d\Omega_{p'}\,, \tag{4.11}$$

where

$$\xi(p',r) = \left(\frac{\sqrt{m_1m_2+\vec{p}\,'^2}-\vec{p}\,'\cdot\vec{n}}{\sqrt{m_1m_2}}\right)^{-1-ir\sqrt{m_1m_2}}, \quad \vec{r}=r\vec{n},\ \vec{n}^2=1, \tag{4.12}$$

is the kernel of the Shapiro-transformation /11/ x).

---

x) The orthogonality and normalization conditions for the ξ-functions have the form:

$$\frac{1}{(2\pi)^3}\int \xi(\vec{r},\vec{p}\,')\,\xi^*(\vec{\rho},\vec{p}\,')\,d\Omega_{\vec{p}'} = \delta^{(3)}(\vec{r}-\vec{\rho})$$

$$\frac{1}{(2\pi)^3}\int \xi(\vec{r},\vec{p}\,')\,\xi^*(\vec{r},\vec{q}\,')\,d\vec{r} = \delta^{(3)}(\vec{p}\,'(-)\vec{q}\,')$$

Applying the Shapiro transformation to (4.9) a relativistic analogue of the Schrödinger equation in $\vec{r}$ -space can be obatined:

$$(E_{q'} - H_0)\Psi_{q'}(\vec{r}) = \int V(\vec{r}, \vec{\rho})\, \Psi_{q'}(\vec{\rho})\, d\vec{\rho} \ , \qquad (4.13)$$

where $V(\vec{r}, \vec{\rho})$ is introduced through the relation:

$$V(\vec{p}, \vec{q}) = \int \xi^*(\vec{p}, \vec{r})\, V(\vec{r}, \vec{\rho})\, \xi(\vec{q}, \vec{\rho})\, d\vec{r}\, d\vec{\rho} \qquad (4.14)$$

and

$$H_0 = 2\sqrt{m_1 m_2}\, ch\left(i \frac{1}{\sqrt{m_1 m_2}} \frac{\partial}{\partial r}\right) + \frac{2i}{r}\, sh\left(i \frac{1}{\sqrt{m_1 m_2}} \frac{\partial}{\partial r}\right) - \frac{\Delta_{\theta,\phi}}{\sqrt{m_1 m_2}\, r^2}\, e^{i \frac{1}{\sqrt{m_1 m_2}} \frac{\partial}{\partial r}} \qquad (4.15)$$

The comparison of (4.13)-(4.15) with the similar relations, corresponding to the equal mass case /4/, allows us to conclude that all the results /4-7/ can be directly generalized in the above stated formalism.

The authors are sincerely grateful to M.Freeman, R.M.Muradyan, N.B.Skachkov, A.N.Tavkhelidze and I.T.Todorov for useful discussions.

References

1. V.G.Kadyshevsky. ITF Preprint No 7, Kiev (1967).
2. V.G.Kadyshevsky, Nuclear Physics, B6, 125 (1968).
3. V.G.Kadyshevsky, M.D.Mateev. Nuovo Cimento, 55A, 275 (1967).
4. V.G.Kadyshvesky, R.M.Mir-Kasimov and N.B.Skachkov. Nuovo Cimento, 55A, 233 (1968).
5. V.G.Kadyshevsky, R.M.Mir-Kasimov, N.B.Skachkov. Preprint E2-3949, Dubna (1968).
6. V.G.Kadyshevsky, R.M.Mir-Kasimov, N.B.Skachkov. Preprint E2-3950, Dubna (1968).
7. M.Freeman, V.G.Kadyshevsky, R.M.Mir-Kasimov. Preprint E2-3948, Dubna (1968).
8. A.A.Logunov, A.N.Tavkhelidze. Nuovo Cimento, 29 , 380 (1963) . For a further list of references see A.N.Tavkhelidze, Lectures on Quasipotential Method in Field Theory , Tata Institute of Fundamental Research, Bombay (1964).
9. A.S.Wightman. "Lectures on Invariance in Relativistic Quantum Mechanics" (Les Houches 1960) in the book Dispersion Relations and Elementary Particles , (Paris, 1960).
10. I.M.Namislovski. Phys. Rev., 160, 1522 (1967).
11. I.S.Shapiro. DAN Sov. Acad. of Sciences 106, 647 (1956). Zurn. Eksp. Teor. Fiz. 43, 1727 (1962). English Translation: Soviet Phys. JETP, 16, 1219 (1963). Phys. Lett., 1, 253 (1962).

Received by Publishing Department
on August 9, 1968.