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J.Jersák*, J.Stern*

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* Present address: Department of Theoretical Physics, Charles University, Myslikova 7, Prague I, Czechoslovakia

Ерсак И., Штерн Я. E2-3990

Алгебра токов на световом конусе

Предложено обобщение одновременной алгебры токов, содержащее коммутаторы токов на всем свободном конусе.

Препринт Объединенного института ядерных исследований.
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Algebra of Currents on the Light Cone

On the basis of the observation that the fixed-mass sum rules are Fourier transforms of the commutators of currents on a light-like hyperplane a new treatment of the algebra of currents is proposed. It consists in the use of the currents integrated over a light-like hyperplane ("light-like" charges) and thus includes the commutators of currents on the whole light cone. An algebra of the light-like charges is a sufficient assumption for the derivation of fixed-mass sum rules and may be represented in the one-particle space with any momenta. Hence this extension of the usual equal time current algebra allows us to abandon the methods like $p_z \to \infty$ without losing any of its advantages.

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I. Introduction

The assumptions about the equal time commutation relations of currents have shown themselves to be fruitful for the derivation of sum rules. However, in order to derive useful sum rules - e.g. fixed-mass sum rules - one has to perform in any method at least one step requiring some additional assumption. In the method[1] $p \longrightarrow \infty$ one interchanges the order of an integration and the limit. In the dispersion approach[2] certain unsubtracted dispersion relations are used. So some necessary assumption is to be added to the equal time algebra of currents in order to fertilize it. In this paper an attempt is also made to formulate this additional assumption in terms of current commutators. It is shown that any fixed mass sum rule requires some knowledge about the commutators of the currents on the whole light cone.

But after this observation it seems misleading to start the derivation of these sum rules from the equal time current commutators. It would be more reasonable to reformulate algebra of currents in a way including the necessary assumption about the commutators of currents on the whole light cone. In this paper charges defined on a hyperplane touching the light cone (light-like hyperplane) are used for this aim. Such "light-like" charges have been introduced in a previous paper[3] by one of the present authors. Fixed-mass sum rules follow from their algebra without any additional assumption.

Algebraic properties of the light-like charges are evidently undetermined by the equal time algebra. This point is illustrated in Section III within a model based on the Jost-Lehman-Dyson representation of the current commutator.

Two further advantages of using the light-like charges are emphasized in Sec. IV. The assumption that they have no leakage from the space of one-particle states implies neither conservation of the generating current nor its non-locality. Moreover, matrix elements of the light-like charges between any one-particle states contain automatically form-factors at fixed momentum transfer. This suggests the use of light-like charges as the basic tool for a treatement of dynamical symmetries of the whole one-particle space in terms of the local electromagnetic and weak currents. One can for instance postulate the light-like charges to satisfy the usual chiral algebra of currents and consider the saturation of this algebra by certain one-particle states at any momenta. The facility of such a calculation is demonstrated in Section V, where the solution of this problem for general masses, spins and parities is given in terms of representation of the algebra $SU(3) \otimes SU(3)$.

It is pointed out that even if the whole light-like charges do not satisfy the usual $SU(3) \otimes SU(3)$ algebra, a simple symmetry scheme may be realized by their one-particle restrictions.

II. Fixed Mass Sum Rules

Let us consider the absorptive part of the amplitude (for spinless particles) associated with a commutator of two vector or axial currents $\mathcal{J}^i_\alpha(x)$ (symmetrized with respect to the space-time indices α, β)

$$t^{ij}_{\alpha\beta}(p, p', q) = \frac{1}{2}\int d^4x\, e^{iqx}\langle p|[\mathcal{J}^i_\alpha(\tfrac{x}{2}), \mathcal{J}^j_\beta(-\tfrac{x}{2})]|p'\rangle_S = \tag{1}$$

$$= P_\alpha P_\beta\, a^{ij}[(Pq), \Delta^2, (\Delta q), q^2] + \cdots$$

where

$$P = \frac{1}{2}(p + p') , \qquad \Delta = p - p' .$$

As is easily seen from the convolution theorem[4], sum rules for the amplitudes $t^{ij}_{\alpha\beta}(p, p', q)$ are Fourier-components of projections of the current commutator on various hyperplanes $nx = 0$:

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} d\lambda\, t^{ij}_{\alpha\beta}(p, p', k + \lambda n) = \int d^4x\, e^{ikx} \langle p | [J^i_\alpha(\tfrac{x}{2}), J^j_\beta(-\tfrac{x}{2})] | p' \rangle \delta(nx) , \qquad (2)$$

Usually a space-like hyperplane ($n^2 > 0$) is used. Then for the evaluation of the sum rule it is necessary to know the equal time commutators only. However, this process can never yield any fixed-mass sum rule, because the mass variables

$$q^2 = n^2\lambda^2 + 2(n k)\lambda + k^2 ,$$
$$\Delta q = \Delta k + (\Delta n)\lambda$$

cannot be constant for any choice of the vector k. To achieve this the limit $p_z \longrightarrow \infty$ is usually performed, which was already shown[3,5] to be equivalent to the limit $n^2 \longrightarrow 0$. The fixed-mass sum rule is then obtained by choosing $kn = \Delta n = 0$.

Relation (2) allows us to write explicitly what is actually needed for the derivation of the fixed-mass sum rule:

$$\frac{1}{\pi}(Pn)\int_{-\infty}^{+\infty} d\nu\, a^{ij}(\nu, \Delta^2, \Delta k, k^2) =$$
$$= n^\alpha n^\beta \int d^4x\, e^{ikx} \langle p | [J^i_\alpha(\tfrac{x}{2}), J^j_\beta(-\tfrac{x}{2})] | p' \rangle \delta(nx) , \qquad (3)$$

where

$$n^2 = 0$$

and the space-like vectors k, Δ satisfy the constraints $kn = \Delta n = 0$. Hence any fixed-mass sum rule is a Fourier-component of the current commutator on a light-like hyperplane. This fact opens a possibility of formulating all assumptions necessary for the derivation of the fixed-mass sum rules in terms of current commutators: one has to add to the usual equal time assumptions some assumption about the commutator on the whole light cone.

The equal time current algebra is often formulated in terms of charges,

$$Q^i(t) = \int d^3x \, J_0^i(\vec{x}, t) \,, \qquad t = 0 \,, \tag{4a}$$

or generally,

$$Q^i(\vec{k}, t) = \int d^3x \, e^{-i\vec{k}\vec{x}} J_0^i(\vec{x}, t) \,, \quad t = 0 \,. \tag{4b}$$

By defining such charges in any frame of reference the integration is always performed over the space-like hyperplane $nx = 0$, $n^2 > 0$ ((4) corresponds to the special case $n \equiv$ (1, 0, 0, 0)). Consequently, the only non-trivial commutator of local currents which the algebra of such charges contains is at the origin of the light cone. In order to include current algebra on the whole light cone, it is convenient to integrate currents over the light-like hyperplane. So we introduce the "light-like" charges:

$$Q^i(n) = \int d^4x \, n^\alpha J_\alpha^i(x) \, \delta(nx) \,, \qquad n^2 = 0 \tag{5a}$$

and

$$Q^i(k, n) = \int d^4x \, e^{ikx} n^\alpha J_\alpha^i(x) \, \delta(nx) \,, \quad n^2 = kn = 0 \,. \tag{5b}$$

If for them the Gell-Mann algebra[6]

$$[Q^i(k_1, n), Q^j(k_2, n)] = i f^{ijk} Q^k(k_1 + k_2, n) \tag{6}$$

is postulated, the fixed-mass sum rules are obtained without any additional assumption:

From the definition (5b) one obtains

$$\begin{aligned} &\langle p|[Q^i(k_1, n), Q^j(k_2, n)]|p'\rangle = \\ &= n^\alpha n^\beta \int d^4x\, e^{ikx} \langle p|[J^i_\alpha(\tfrac{x}{2}), J^j_\beta(-\tfrac{x}{2})]|p'\rangle \delta(nx) \times \\ &\times \int d^4y\, e^{i(K+\Delta)y} \delta(ny) \quad , \end{aligned} \tag{7}$$

where $k = \frac{1}{2}(k_1 - k_2)$, $K = k_1 + k_2$. Following the references[3,7] let us denote

$$\int d^4y\, e^{i(K+\Delta)y} \delta(ny) = (2\pi)^3 \delta_n (K+\Delta) . \tag{8}$$

It is easy to see that the presence of this distribution restricts the four-momentum transfer to

$$\Delta_\alpha = -K_\alpha + \frac{m^2 - m'^2 + 2(KP)}{2(Pn)} n_\alpha . \tag{9}$$

This relation together with the constraints $n^2 = k_1 n = k_2 n = 0$ leads to

$$\Delta^2 = K^2, \quad \Delta n = 0, \quad \Delta k = -kK . \tag{10}$$

Owing to this the relation (3) can be used to rewrite (7) as

$$\langle p|[Q^i(k_1,n),\ Q^j(k_2,n)]|p'\rangle =$$
$$= \frac{1}{\pi}(P.n)\int d\nu\, a^{ij}(\nu, K^2, -kK, k^2)(2\pi)^3\delta_n(K+\Delta) . \tag{11}$$

The same factor $(2\pi)^3\delta_n(K+\Delta)$ occurs after sandwiching the right--hand side of the algebra (6). On introducing the form-factors

$$\langle p|J^i_\alpha(0)|p'\rangle = P_\alpha F^i(\Delta^2) + \Delta_\alpha G^i(\Delta^2) \tag{12}$$

we get finally the most general form of the Fubini sum rule

$$\frac{1}{\pi}\int d\nu\, a^{ij}(\nu, K^2, -kK, k^2) = i f^{ijk} F^k(K^2) \tag{13}$$

as a direct consequence of the algebra (6) postulated for the light-like charges.

We should like to emphasize that the algebra (6) is not only sufficient but also a very economical assumption for the derivation of the sum rule (13). For it to be valid this algebra must hold up to some term with zero matrix elements between one-particle states.

III. Model of the Current Commutator

In this section a simple model is given, illustrating the discontinuity between commutators on the space-like and light-like hyperplanes. Let us consider a sort of the Jost-Lehman-Dyson representation for the commutator of two local currents (symmetrized with respect to the indices α, β)

$$[J^i_\alpha(y+\tfrac{x}{2}),\ J^j_\beta(y-\tfrac{x}{2})] = i\int_0^\infty ds\,\Delta(x,s)\,T^{ij}_{\alpha\beta}(x,y,s)\ - \tag{14}$$
$$-\,i\int_0^\infty ds\,\partial_\mu\Delta(x,s)\left\{g_{\alpha\mu}\,V^{ij}_\beta(x,y,s)+g_{\beta\mu}\,V^{ij}_\alpha(x,y,s)-g_{\alpha\beta}\,\bar{V}^{ij}_\mu(x,y,s)\right\}$$

+ (terms with higher derivatives of $\Delta(x,s)$).

Here $\Delta(x,s)$ is the usual commutator distribution

$$\Delta(x,s) = -\frac{1}{2\pi}\,\mathcal{E}(x)\left\{\delta(x^2)-\frac{s}{2}\,\theta(sx^2)\,\frac{J_1(\sqrt{sx^2})}{\sqrt{sx^2}}\right\} \tag{15}$$

and $T^{ij}_{\alpha\beta}$, V^{ij}_α, $\bar{V}^{ij}_\alpha$ are non-local operators with the indicated transformation properties.

The model representation (14) is not the most general one, which can be written starting from generalities like Lorentz invariance, micro-causality and spectral conditions, but it does not contradict them. We shall show that this model is consistent with the equal time current algebra, though not with the Fubini sum rule (13).

First of all let us project out the relation (14) on a space-like hyperplane $nx=0,\ n^2>0,\ n_0>0$ by multiplying (14) by $\delta(nx)$. Such a product is well-defined provided that $T_{\alpha\beta}$, V_α, $\bar{V}_\alpha$ are not too singular at $x=0$.

Using

$$\left.\begin{aligned}&\delta(nx)\,\Delta(x,s) = 0\,, &(16a)\\ &\delta(nx)\,\partial_\mu\Delta(x,s) = -\frac{n_\mu}{n^2}\,\delta^4(x) &(16b)\end{aligned}\right\}\ n^2>0$$

we can write

$$\left[J^i_\alpha \left(y+\frac{x}{2}\right), J^j_\beta \left(y-\frac{x}{2}\right) \right]_S \delta(nx) =$$

$$= \frac{i}{m^2} \delta^4(x) \int_0^\infty ds \left\{ n_\alpha \tilde{v}^{ij}_\beta(0,y,s) + n_\beta \tilde{v}^{ij}_\alpha(0,y,s) - g_{\alpha\beta} n^\mu \tilde{\bar{v}}^{ij}_\mu(0,y,s) \right\} + \tag{17}$$

$$+ \left(\text{Schwinger terms} \right) .$$

On comparing (17) with the equal time assumptions written on the space-like hyperplane $nx = 0$ as

$$\left[J^i_\alpha \left(y+\frac{x}{2}\right), J^j_\beta \left(y-\frac{x}{2}\right) \right]_S \delta(nx) =$$

$$= \frac{i}{m^2} f^{ijk} \delta^4(x) \left\{ n_\alpha J^k_\beta(y) + n_\beta J^k_\alpha(y) - g_{\alpha\beta} n^\mu J^k_\mu(y) \right\} + \tag{18}$$

$$+ \left(\text{Schwinger terms} \right)$$

we see that the model (14) reproduces (18) if

$$\int_0^\infty ds\, v^{ij}_\alpha(0,y,s) = \int_0^\infty ds\, \bar{v}^{ij}_\alpha(0,y,s) = f^{ijk} J^k_\alpha(y) . \tag{19}$$

However, due to (16a), the equal time commutators impose no constraints on the term [illegible] .

On the other hand, this term contributes to the commutators on the light-like hyperplane and, consequently, to the fixed-mass sum rules. By using (19) and the indentities

$$\left.\begin{array}{l} [illegible] (nx) \Delta(x, s) = \frac{1}{4} \int_{-\infty}^{+\infty} d\lambda\, \epsilon(\lambda) \delta^4(x - \lambda n) \\ [illegible] n^\mu [illegible] s) = \frac{1}{4} \int_{-\infty}^{+\infty} d\lambda\, \epsilon(\lambda) n^\mu [illegible] \delta^4(x - \lambda n) \end{array}\right\} n^2 = 0 \tag{20}$$

the following fixed-mass sum rule is obtained from Eq. [illegible] :

$$\frac{1}{\pi}\int d\nu\, a^{ij}(\nu,\Delta^2,\Delta k,k^2) =$$

$$= i f^{ijk} F^k(\Delta^2) - \frac{i}{4(Pn)} \int_{-\infty}^{+\infty} d\lambda\, \varepsilon(\lambda)\, n^\alpha n^\beta \langle p| \int_0^\infty ds\, T^{ij}_{\alpha\beta}(\lambda n, 0, s)|p'\rangle. \tag{21}$$

Within the same model it can also be shown that providing the term $\left[n^\alpha n^\beta T^{ij}_{\alpha\beta}(x,y,s)\right]_{x^2=0}$ does not contribute, i.e. providing the Fubini sum rule does hold, the light-like charges satisfy the usual algebra (6).

According to (5b) one has

$$[Q^i(k_1,n),\, Q^j(k_2,n)] =$$

$$= \int d^4y\, e^{iKy}\delta(ny)\int d^4x\, e^{ikx}\left[n^\alpha J^i_\alpha(x+\tfrac{x}{2}),\, n^\beta J^j_\beta(y-\tfrac{x}{2})\right]\delta(nx), \tag{22}$$

where again $K = k_1 + k_2,\ k = \frac{1}{2}(k_1 - k_2)$. By using the relations (14), (19) and $kn = 0$ this commutator can be written as

$$[Q^i(k_1,n),\, Q^j(k_2,n)] =$$

$$= -\frac{1}{2}\int e^{iKy}\delta(ny)\, d^4y \int_{-\infty}^{+\infty} d\lambda\, \varepsilon(\lambda)\frac{d}{d\lambda}\int_0^\infty ds\, n^\alpha\, V^{ij}_\alpha(\lambda n, y, s) = \tag{23}$$

$$= i f^{ijk} Q^k(k_1+k_2,\, n).$$

We thus see that, at least in the model defined by the representation (14), the requirement of validity of the Fubini sum rules is equivalent to the assumption that the light-like charges $Q^i(k,n)$ satisfy the algebra (6).

IV. Matrix Elements of the Light-Like Charges

For completeness a brief review of the properties of the matrix elements of light-like charges is given in this section. Such properties have been studied in a previous paper [7].

Any matrix element of operators $Q^i(K,n)$ can be written as

$$\langle p|Q^i(K,n)|p'\rangle = (2\pi)^3 \langle p|n^\alpha J^i_\alpha(0)|p'\rangle \delta_n(p-p'+K), \tag{24}$$

where $\delta_n(p-p'+K)$ (defined by the formula (8)) restricts the momentum transfer $\Delta = p-p'$ by the relation (9). Because of the condition $Kn = 0$ the momentum transfer squared is fixed in (24) as $\Delta^2 = K^2$ for any choice of the vectors p, p'. Consequently, $Q^i(K,n)$ annihilates the vacuum provided that massless particles are absent. This means that the Coleman theorem [8] cannot be applied to the light-like charges and they may have no leakage in spite of the locality and non-conservation of the generating currents. Let us point out that these properties are achieved for the usual charges (4) only in the sub-space with $p_z \longrightarrow \infty$.

In the following the simplest case $K \equiv 0$ is considered. The charges $Q^i(n)$ can be shown to commute with the projection of the spin onto the direction

$$\vec{l} = \frac{\vec{P}}{E+m}\left[1+\frac{m}{pn}n_0\right] - \frac{m}{pn}\vec{n}, \quad \vec{l}^{\,2} = 1 . \tag{25}$$

Because of this, instead of characterizing the one-particle states by their helicities, the projection of spin onto the direction (25) will be used and denoted by λ. The matrix elements of operators $Q^i(n)$ between one-particle states can then be written as

$$\langle a, p_a, \lambda|V^i(n)|b, p_b, \lambda'\rangle = \delta_{\lambda\lambda'} F^i_{ab}(\lambda) \frac{pn}{\sqrt{E_a E_b}} \delta_n(p_a - p_b)$$
$$\langle a, p_a, \lambda|A^i(n)|b, p_b, \lambda'\rangle = \delta_{\lambda\lambda'} G^i_{ab}(\lambda) \frac{pn}{\sqrt{E_a E_b}} \delta_n(p_a - p_b). \tag{26}$$

Here the symbols $\mathcal{V}^{i}(n)$, $\mathcal{A}^{i}(n)$ distinguish the cases of $Q^{i}(n)$ generated by vector and axial-vector currents, respectively. Indices a, b label different particles and $\mathcal{F}_{ab}(\lambda)$, $\mathcal{G}_{ab}(\lambda)$ are a sort of invariant form-factors at zero momentum-transfer squared.

Let us introduce the n + 1 = $min(s_a - \frac{1}{2}, s_b - \frac{1}{2}) + 1$ independent form-factors $f_{\alpha}^{ab}(0)$ and $g_{\alpha}^{ab}(0)$, $(\alpha = 0, 1, \ldots n)$ associated with the vertices

$$(-1)^{\alpha}\, \overline{w}^{\lambda}_{\mu_1 \ldots \mu_n}(\vec{p}_a)\, \gamma\, \Gamma^{\mu_1 \nu_1} \ldots \Gamma^{\mu_\alpha \nu_\alpha}\, w_{\nu_1 \ldots \nu_n}(\vec{p}_b) \times$$
$$\times\, \Delta^{\mu_{\alpha+1}} \ldots \Delta^{\mu_n} \Delta^{\nu_{\alpha+1}} \ldots \Delta^{\nu_n}, \qquad (27)$$

where $\gamma = \hat{n}$ or $\hat{n}\gamma_5$ and $\Gamma^{\mu\nu} = g^{\mu\nu} + \frac{2}{(m_a + m_b)^2}\Delta^{\mu}\Delta^{\nu}$.
$w^{\lambda}_{\mu_1 \ldots \mu_n}$ is the Rarita-Schwinger amplitude describing the particle with the spin $s = m + \frac{1}{2}$. Then the form-factors $\mathcal{F}^{i}_{ab}(\lambda)$, $\mathcal{G}^{i}_{ab}(\lambda)$ are related to this parametrization of the matrix elements of $Q^{i}(n)$ by the formulae

$$\mathcal{F}_{ab}(\lambda) = \frac{1}{2}(-\mathrm{sign}\,\lambda)^{s_a - s_b}\rho(s, s', \lambda)\sum_{\alpha=0}^{s-\frac{1}{2}} f_{\alpha}^{ab}(0)\Big[\Big(1 + \frac{\lambda}{\alpha + \frac{1}{2}}\Big)(-1)^{s_a - s_b} + \Big(1 - \frac{\lambda}{\alpha + \frac{1}{2}}\Big)\pi_{ab}\Big]\rho^{2}(\alpha + \tfrac{1}{2}, s, \lambda)$$
$$\mathcal{G}_{ab}(\lambda) = \frac{1}{2}(-\mathrm{sign}\,\lambda)^{s_a - s_b}\rho(s, s', \lambda)\sum_{\alpha=0}^{s-\frac{1}{2}} g_{\alpha}^{ab}(0)\Big[\Big(1 + \frac{\lambda}{\alpha + \frac{1}{2}}\Big)(-1)^{s_a - s_b} - \Big(1 - \frac{\lambda}{\alpha + \frac{1}{2}}\Big)\pi_{ab}\Big]\rho^{2}(\alpha + \tfrac{1}{2}, s, \lambda) \qquad (28)$$

Here $s = min(s_a, s_b)$; $s' = max(s_a, s_b)$; π_{ab} is the relative parity of the particles a and b

$$
\rho(s,s';\lambda) = \begin{cases} \prod\limits_{i=s+\frac{1}{2}}^{s'-\frac{1}{2}} \left[\dfrac{(i+\frac{1}{2})^2 - \lambda^2}{i(2i+1)} \right]^{\frac{1}{2}} & \text{for } s' > s \\ 1 & \text{for } s' = s \end{cases} \tag{29}
$$

According to (28) the form-factors $\mathcal{F}^i_{ab}(\lambda)$, $\mathcal{G}^i_{ab}(\lambda)$ satisfy the relation

$$
\mathcal{F}^i_{ab}(-\lambda) = \Pi_{ab}\, \mathcal{F}^i_{ab}(\lambda) \,, \quad \mathcal{G}^i_{ab}(-\lambda) = -\Pi_{ab}\, \mathcal{G}^i_{ab}(\lambda) \,. \tag{30}
$$

V. Saturation of an Algebra of the Light-Like Charges

The light-like charges $Q^i(n)$ can form an algebra. In this section attention is paid to the representation of such an algebra in the space of one-particle states with any momenta.

For this purpose it is convenient to introduce the restrictions of operators $Q^i(n)$ to the sub-space of one-particle states:

$$
\hat{Q}^i(n) = \sum_\lambda \sum_{ab}{}' \int dp_a\, dp_b\, \langle a, p_a, \lambda | Q^i(n) | b, p_b, \lambda \rangle \overset{+}{\beta}{}^a_\lambda(\vec{p}_a)\, \beta^b_\lambda(\vec{p}_b) \,, \tag{31}
$$

where λ is the projection of spin onto the direction (25) and the sum $\sum'$ is extended over particles with $s_a, s_b \geq |\lambda|$. $\beta^a_\lambda(\vec{p}_a)$ is the annihilation operator of the particle a. Due to the relations (26) and (30) we can write the corresponding operators $\hat{V}^i(n)$ and $\hat{A}^i(n)$ as

$$\hat{V}^i(n) = \sum_{\lambda>0} Sp[\tilde{\mathcal{F}}^i(\lambda) W_\lambda(n) + \Pi\tilde{\mathcal{F}}^i(\lambda)\Pi W_{-\lambda}(n)]$$

$$\hat{A}^i(n) = \sum_{\lambda>0} Sp[\tilde{\mathcal{G}}^i(\lambda) W_\lambda(n) - \Pi\tilde{\mathcal{G}}^i(\lambda)\Pi W_{-\lambda}(n)] \tag{32}$$

$\mathcal{F}^i(\lambda)$, $\mathcal{G}^i(\lambda)$ are matrices with the elements $\mathcal{F}^i_{ab}(\lambda)$, $\mathcal{G}^i_{ab}(\lambda)$ given by the formulae (28) - where a, b denote particles with the spins $\geqslant \lambda$, Π is a diagonal matrix with parities on the diagonal and tilda denotes the transposition. $W_\lambda(n)$ is a matrix with the elements being operators

$$W^{ab}_\lambda(n) = \int dp_a\, dp_b \frac{p_a n}{\sqrt{E_a E_b}} \delta_n(p_a - p_b) \overset{+}{\beta}{}^a_\lambda(\vec{p}_a) \beta^b_\lambda(\vec{p}_b) . \tag{33}$$

They satisfy the algebra

$$[W^{ab}_\lambda(n), W^{cd}_{\lambda'}(n)] = \delta_{\lambda\lambda'} \left\{ \delta_{bc} W^{ad}_\lambda(n) - \delta_{ad} W^{cb}_\lambda(n) \right\}. \tag{34}$$

The saturation problem can now be easily solved in terms of the matrices $\mathcal{F}^i(\lambda)$ and $\mathcal{G}^i(\lambda)$. Let for instance the light-like charges (5a) satisfy the usual algebra $SU(3) \otimes SU(3)$. (According to Section II. this implies the validity of the Fubini sum rule $\frac{1}{\pi}\int d\nu\, a^{ij}(\nu,0,0,0) = if^{ijk}F^k(0)$.) Further let us assume that this algebra is saturated by a given set S of baryons. Then the same algebra $SU(3) \otimes SU(3)$ has to be satisfied by the operators (32), where $\mathcal{F}^i(\lambda)$ and $\mathcal{G}^i(\lambda)$ are now matrices defined in the space of baryons from the set S with the spins $s \geqslant \lambda$ ($\lambda = 1/2, 3/2$ .... maximal spin in S).

Putting the operators (32) into the commutation relations of the considered algebra and using (34) it can easily be seen that the same algebra must be satisfied by the matrices $\mathcal{F}^i(\lambda)$ and $\mathcal{G}^i(\lambda)$ i.e.

$$\begin{aligned} [\mathcal{F}^i(\lambda), \mathcal{F}^j(\lambda)] &= if^{ijk}\mathcal{F}^k(\lambda) , \\ [\mathcal{F}^i(\lambda), \mathcal{G}^j(\lambda)] &= if^{ijk}\mathcal{G}^k(\lambda) , \\ [\mathcal{G}^i(\lambda), \mathcal{G}^j(\lambda)] &= if^{ijk}\mathcal{F}^k(\lambda) \end{aligned} \tag{35}$$

for any λ admissible by spins of particles from the set S. In contrary, any solution of the matrix algebra (35) leads to operators $\hat{V}^i(n)$ and $\hat{A}^i(n)$ satisfying the algebra $SU(3) \otimes SU(3)$. Consequently, representing the algebra of light-like charges in the space of one-particle states leads to the relations between form-factors at zero momentum transfer, given by the equations (28), where $\mathcal{F}^i(\lambda)$, $\mathcal{G}^i(\lambda)$ constitute matrix representations of the algebra $SU(3) \otimes SU(3)$.

The present formalism may, however, have a good physical meaning even if the light-like charges do not satisfy the chiral algebra of currents or if this algebra (for dynamical reasons) cannot be saturated by a set of one-particle states. Independently of the algebraic properties of the whole light-like charges $Q^i(n)$ their one-particle restrictions (32) may satisfy a simple algebra, implying the same algebra to be satisfied by the matrices $\mathcal{F}^i(\lambda)$ and $\mathcal{G}^i(\lambda)$. Let us emphasize that this possibility is offered just because of the use of the light-like charges: for them the creation of massive pairs from the vacuum is suppressed, and consequently, their one-particle restrictions are again generated by local quantities. For instance, the quark model suggests [7] that the operators $\hat{V}^i(n)$, $\hat{A}^i(n)$ (and thus the matrices $\mathcal{F}^i(\lambda)$, $\mathcal{G}^i(\lambda)$) satisfy the algebra $SU(3) \otimes SU(3)$ up to a renormalisation of the weak currents for quarks, which manifests itself like a phenomenological renormalisation of the Cabibbo angle and leads effectively to a two angle theory with $\theta_A \neq \theta_V$.

The algebra of form-factors at zero momentum transfer (35) is also obtained when the algebra of the usual "space-like" charges (4a) and its saturation by the set S in the space of one-particle states with $p_z \longrightarrow \infty$ are assumed [9]. So there is a certain equivalence between these assumptions and saturation of the algebra of light-like charges. However, working with the light-like charges no restriction to the subspace with $p_z = \infty$ is necessary: the algebra (35) is obtained using the states of any momenta.

V. Conclusion

A new treatement of the algebra of currents has been proposed. It includes commutators of currents on the whole light cone, which has been shown to be necessary for derivation of the fixed-mass sum rules. The method allows us to formulate assumptions, usually used in working with the current algebra, in a more compact algebraic way. Then the standard results are obtained and the main advantage of the light-like charges lies in the simplification of all procedures. This is seen distinctly in the saturation of the algebra of charges $\mathcal{Q}^i(n)$ and in their use for formulation of internal symmetries, which are not confined to the sub-space with infinite momentum. Also the generalized light-like charges $Q^i(k, n)$ have properties promissing analogous advantages of representation of their algebra in the whole one-particle space.

Inclusion of the commutators of currents on the light cone is an extension of the algebra of currents and its non-triviality has been illustrated by a simple model. Thus the question arises: What is the physical basis of postulating some algebra of light--like charges? Some hope lies in the fact that the current commutators on the light cone are closely related to the asymptotic behaviour of the corresponding weak amplitudes [10], which can be evaluated using Regge pole model and pole dominance (like PCAC) in terms of measurable quantities.

In the case when the whole light-like charges prove not to fulfil a desirable algebra or if the saturation hypothesis is uncorrect, it is still possible to work with the one-particle restrictions of operators $Q^i(k, n)$. They are generated by local currents and this, together with their Lorentz properties and physical meaning make them a suitable basis for postulation of some algebra. The one-particle restrictions of $Q^i(n)$ might then generate an internal symmetry (and thus some algebra of form-factors) independently of the whole charges.

If the light-like charges $Q^i(n)$ (or their one-particle restrictions) are used for generation of an internal symmetry, then their particular form allows us to couple this symmetry with the Poincaré one into some infinite-dimensional Lie algebra, suffering from ambiguity much less than is usually expected [7].

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