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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

P.S.Isaev, Kh.D.Popov, I.S.Zlatev

METHOD OF CONSTRUCTION OF THE ELASTIC SCATTERING APPROXIMATE AMPLITUDE

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METHOD OF CONSTRUCTION OF THE ELASTIC SCATTERING APPROXIMATE AMPLITUDE

Объединенный институт ядерных исследований
БИБЛИОТЕКА

1. Introduction

The problem of finding a solution satisfying simultaneously analyticity, crossing symmetry and unitarity is one of the unsolved problem of the double dispersion relation method.

The well-known solution of Castillejo, Dalitz and Dyson for scalar mesons has some disadvantages: i) it is not crossing symmetrical with respect to all three channels, ii) poles lying on the physical cut have no clear physical interpretation, iii) CDD model is far from the real case of pion scattering.

In the work recently published /1/, the total amplitude of the elastic scattering of different spinless particles with degenerated masses was considered. The work was aimed to find a formalism by means of which it would be possible to unite crossing symmetry, unitary and analyticity when reconstructing the total scattering amplitude. One tried to find the amplitude as a series, each term of which obey the double dispersion relation. Each term of the series (i.e. each partial wave) was reconstructed by the $\frac{N}{D}$ method. The discontinuity on the left cut was specified by means of certain weight functions, which should obey a number of physical and mathematical requirements. The weight functions were not specified. Such an approach has led to a complicated problem of series convergence which the author has not succeeded in resolving. However, the main shortcoming of the method is, in the authors' opinion, that the branch points in the crossing channels occur in unsuitable places.

While investigating the scattering of real pseudoscalar mesons, at low energies , one obtains as a rule satisfactory agreement with experimental results. This is reached due to the use of a small number of partial waves and to the violation of some combination of crossing symmetry, two-particle unitarity and analyticity or the violation of all three conditions taken together. The double Mandelstam representations are usually used in these investigations. The amplidute is expanded in partial waves. The two-particle unitarity condition enters explicitly the equation for partial waves but the crossing symmetry turns out to be broken.

A large amount of skill would be needed to take the crossing symmetry correctly into account , if this is at all possible, but this would lead to the violation of the two-particle unitarity /3/.

The scattering amplitude can be reconstructed without direct recourse to the double dispersion relations. In ref. /4/ the authors make use of the ordinary expansion of the total amplitude in partial waves, which are calculated by the $\frac{N}{D}$ method. The arbitrariness of the approach s the specification of the expression for the discontinuity on the left cut. In reconstructing the total $\pi\pi$ -scattering amplitude the authors try to keep all the important properties of the scattering problem and the analyticity properties given by the double dispersion relations. The crossing symmetry is fulfilled only approximately. The total amplitude does not obey the unitarity in the u and t -channels.

In the present paper a new method for constructing the elastic $\pi\pi$ -scattering amplitude is used. For simplicity the case of scalar neutral mesons is considered though the method suggested can also be extended to the case of the elastic pseudoscalar π-meson scattering.

The method consists of finding a model function $F(s,u,t)$ which possesses the crossing symmetry property with respect to all three channels, which has the right beginnings of the cuts along s, u, t

corresponding to the presence of two-particle unitarity only and which satisfies the other requirements imposed on the physical scattering amplitude (§2).

Analyticity and crossing symmetry lead to restrictions imposed on the $F(s,u,t)$ function parameters. These restrictions follow from the requirement that there should not be, on the physical sheets, any singularities but the known cuts. One assumes the existence of poles on unphysical sheets and of cuts in the space of two independent complex variables. The regions of the allowed values of the $F(s,u,t)$ function parameters obeying these requirements are calculated in §3. The presence of the poles and the cuts on the unphysical sheeets results in certain sum rules which will also be discussed in §3.

Additional restrictions on the $F(s,u,t)$ function parameters follow from the requirement that the imaginary part of the function $F(s,u,t)$ should satisfy the optical theorem. Use is made of the weakened condition of the optical theorem: it is required that $\mathrm{Im}\, F(s \geq 4, t=0, u=-4\nu) \geq 0$ where ν is the squared momentum of a pion in the s channel in the c.m.s. (§4).

The lowest partial waves of the scattering amplitude for scalar pions are calculated ($\ell=0,2$) and their analytical properties are discussed in §5. Owing to the fact that the expressions for the partial amplitudes are very complicated the validity of the two-particle unitarity is checked only in a restricted interval of the variable values $0 \leq \nu < \frac{1}{4}$ (§6).

Some consequences of the inclusion of pole terms are considered in §7. In the conclusion a series of problems is discussed related to more complicated models of the scattering amplitude (§8).

§2. Choice of the Function $F(s,u,t)$

Let us find the amplitude of the elastic scattering of scalar mesons in the form of a certain function $F(s,u,t)$ which should obey the following requirements:

I. It must have the right beginnings of the cuts with respect to s, u, t ($s \geq 4\mu^2$; $u \geq 4\mu^2$; $t \geq 4\mu^2$; μ is the pion mass. In what follows we assume $\mu = 1$).

II. It must have crossing symmetry property with respect to the replacement $s \rightleftarrows u$, $s \to t$; $t \rightleftarrows u$.

III. It must obey the exact condition of two-particle unitarity in all three channels near the physical threshold of each channel and an approximate condition of the two-particle unitarity on the intercept near the physical threshold (e.g. in the s-channel in the interval $0 \leq \nu < \frac{5}{4}$ where $\nu = \frac{5}{4}$ the threshold of the first inelastic process $\pi + \pi \to 3\pi$).

IV. It must have $\mathrm{Im}\, F(s, u, t) \geq 0$ in the region $s \geq 4$, $u = -4\nu$, $t = 0$.

V. It must have a polynomial increase at infinity in any direction.

VI. The partial waves must have the right threshold behaviour in all the channels.

Here we do not list all the requirements imposed on the physical amplitude of the scalar meson scattering. It is more convenient to discuss some requirements later. The consideration is made in the s-channel in which the role of the energy variable is played by $s = 4(1+\nu)$ and the momentum transfer is described by the variable $t = -2\nu(1-z)$ (see Fig.1) where z is the cosine of the scattering angle.

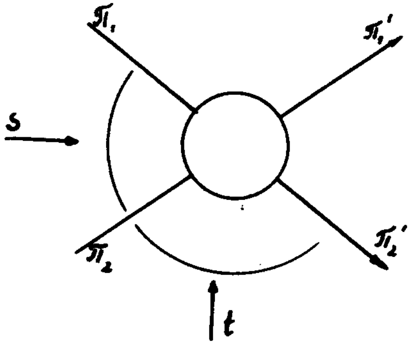

Fig.1

The condition III leads to the requirement that the Riemann surface of the function $F(s, u, t)$ for fixed u and t should be a two-sheet one. Up to now using the dispersion relation method one restricted oneself to the two-particle unitarity and, consequently, did not go beyond the framework of the two-sheet Riemann surface. In this sense, this paper is not an exclusion.

Before passing onto the other conditions listed at the beginning of the section we will give a conformal mapping of the surface $s+u+t=4$ onto the surface $\sigma^2+\lambda^2+r^2=-8$ by means of the transformations

$$\sigma=\sqrt{s-4}$$
$$\lambda=\sqrt{u-4} \qquad (1)$$
$$r=\sqrt{t-4}$$

and determine the sign for all the roots in such a manner that in the transition from above to the cuts, with respect to s, u, t in the s, u, t planes respectively, the imaginary parts of the variables σ, λ, r assume positive values. As a result of transformation (1) the two-sheet Riemann surface with respect to the variable s (for fixed u and t) with the cut along the real axis $s \geq 4$ transforms into the one-sheet surface σ without cut. In a similar manner the two-sheet Riemann surface with respect to the variables u and t will turn into the one-sheet surfaces λ and r, respectively. Instead of the function $F(s,u,t)$ we shall consider the function $\Phi(\sigma,\lambda,r)$.

Condition 5, imposed on the function $\Phi(\sigma,\lambda,r)$, now means that it must be meromorphic over the whole space C^2 with respect to any two variables of σ, λ, r. To keep the symmetry with respect to the three variables, which is convenient when checking the crossing symmetry in all three channels, we use all the three variables. The meromorphic function $\Phi(\sigma,\lambda,r)$ essentially depending on the two variables has singularities, i.e. the cuts. The unphysical singularity of the function $\Phi(\sigma,\lambda,r)$ we call a singularity when, in considering the problem in the s-channel for the physical values of t, the imaginary part of the variable σ belonging to any singular point of the function is negative. Owing to the crossing symmetry, a similar requirement is imposed on the problem in the u and t channels.

Thus, the function $\Phi(\sigma,\lambda,r)$ must be a meromorphic function of the three complex variables σ, λ, r symmetrical with respect to the change $\sigma \rightleftarrows \lambda$, $\sigma \rightleftarrows r$, $\lambda \rightleftarrows r$ given on the complex sphere $\sigma^2+\lambda^2+r^2=-8$ and having the singularities in the s -channel , only in the lower half-plane σ of the complex plane σ . We represent it as the ratio of the two symmetrical (with respect to the variables σ, λ, r) polynomials $P(\sigma,\lambda,r)$ and $Q(\sigma,\lambda,r)$:

$$\Phi(\sigma,\lambda,r) = \sum_n \frac{P_n(\sigma,\lambda,r)}{Q_n(\sigma,\lambda,r)} , \qquad (2)$$

where $P_n(\sigma,\lambda,r)$ and $Q_n(\sigma,\lambda,r)$ are the polynomials of the finite degree and n may be either finite or infinite. The representation (2) is the most general representation of the function $\Phi(\sigma,\lambda,r)$ if only the requirement of polynomial boundedness at infinity in any direction is considered.

Now the strict mathematical solution of the whole problem is as follows: is it possible to find, in the class of meromorphic functions, a function of the type (2), which obeys the two-particle unitarity on the intercept, the requirements IV, VI and whose singularities are unphysical?

In principle, it is necessary to answer the question as to whether there exists a solution to the infinite system of nonlinear algebraic equations whose parameters must lie in a given region. The infinite system of nonlinear, algebraic equations arises: i) from the requirement for the two-particle unitarity to be fulfilled for an infinite set of partial amplitudes $\Phi_\ell(\sigma)$ in which the function $\Phi(\sigma,\lambda,r)$ is expanded

$$\Phi(\sigma,\lambda,r) = \sum_{\ell=0}^{\infty} \Phi_\ell(\sigma)\, P_\ell(\lambda,r) , \qquad (3)$$

where $\ell = 0,2,4,\dots;$ $\sigma^2 = 4\nu$; $z = \dfrac{\lambda^2 - r^2}{8+\lambda^2+r^2}$

and ii) from the requirement for the two-particle unitarity to be fulfilled at each point of a finite intercept with respect to $\nu\,(0 \lesssim \nu \lesssim \frac{5}{4})$

In this paper a simpler problem is considered, that is, the case of the finite number of partial waves ($\ell = 0,2$) and the case of the finite number of points on the intercept.

Let us suppose that the amplitude $\Phi(\sigma,\lambda,\tau)$ decreases at infinity $\approx \sigma^{-1}$. As the model function we choose the simplest one

$$\Phi(\sigma, \lambda, \tau) = \frac{g^2}{\sigma^2+3} + \frac{g^2}{\lambda^2+3} + \frac{g^2}{\tau^2+3} + \sum_{n=1}^{3} \frac{A_n}{a_n - i(\sigma+\lambda+\tau)}, \qquad (4)$$

where g^2 is the interaction constant for scalar pions, corresponding to the interaction lagrangian $g\phi^3$ and equal to the residue of the function $\Phi(\sigma,\lambda,\tau)$ at the points $s=u=t=\mu^2=1$ and A_n, a_n are the real number-parameters of the problem. We have taken $n=3$ since the number of equations from which the coefficients A_n and a_n will be determined is six (see §6). In principle, we may increase the number of partial waves, the number of equations for determinating g^2, A_n, a_n as well as the number n. The choice of the model function $\Phi(\sigma,\lambda,\tau)$ only as the sum of poles

$$\Phi(\sigma,\lambda,\tau) = \frac{g^2}{\sigma^2+3} + \frac{g^2}{\tau^2+3} + \frac{g^2}{\lambda^2+3}$$

does not ensure the two particle unitarity in highest partial waves starting with $\ell \geq 2$. This is easily seen from the fact that in the s-channel the only term which has the cut is the first term $\frac{g^2}{\sigma^2+3}$ but it is independent of the angle z.

The function $\Phi(\sigma,\lambda,\tau)$ given by (4) obeys the conditions I, II, III and satisfies the requirement $\mathrm{Im}\, F(s \leq 4,\, u \leq 4,\, t \leq 4) = 0$ which is easily seen from eq. (1) for σ, λ, τ since in the region $s \leq 4$, $t \leq 4$, $u \leq 4$ the quantities σ,λ,τ are imaginary or equal to zero.

We may add to the expression $\Phi(\sigma,\lambda,\tau)$ the sum of pole terms of the kind

$$\sum_{m=1}^{N} \Big[\frac{G_m}{a_m - ib_m - i\sigma} + \frac{G_m^*}{a_m + ib_m - i\sigma} + \frac{G_m}{a_m - ib_m - i\lambda} + \frac{G_m^*}{a_m + ib_m - i\lambda} + \tag{4a}$$

$$+ \frac{G_m}{a_m - ib_m - i\tau} + \frac{G_m^*}{a_m + ib_m - i\tau} \Big] , \qquad a > 0 \quad .$$

which keep the right analytic properties of $\Phi(\sigma,\lambda,\tau)$ and are interpreted as the contribut on to the amplitude $\Phi(\sigma,\lambda,\tau)$ from the resonance states of the $\pi\pi$ -system. The poles (4a) in the plane σ for fixed λ , τ give the contribution to the amplitude $\Phi(\sigma,\lambda,\tau)$ which is similar to that from the Breit-Wigner poles (at the point $\sigma \approx \sqrt{a^2+b^2}$)

$$\frac{1}{a-ib-i\sigma} + \frac{1}{a+ib-i\sigma} = \frac{2(a-i\sigma)}{a^2+b^2-\sigma^2-2a\sigma i} \approx \frac{2(a-i\sigma)}{2\sqrt{a^2+b^2}\,[\sqrt{a^2+b^2}-\sigma-ia]} \, .$$

The contributions of the poles of this kind to the scattering amplitude will be discussed in more detail in §7.

3. Determination of the Region of Allowed a_n

The region of allowed values of the parameters a_n is determined from the requ rement that the singularities of the demominators $a_n - i(\sigma+\lambda+\tau) = 0$ (see eq. (4)) should lie in the region $\mathrm{Im}\,\sigma < 0$ for physical values of the momentum transfers t (i.e. for $\tau_i \geq 2$).

The singularities of the function $\Phi(\sigma,\lambda,\tau)$ lie at the points obeying the system of equations

1. $a_n - i(\sigma+\lambda+\tau) = 0$
2. $\sigma^2 + \lambda^2 + \tau^2 = 8$

In what follows we shall omit the index n of the parameter a_n.
In the general case, when $\Phi(\sigma, \lambda, \tau)$ is the ratio of the polynomials(2) and it is necessary to find the parameters for which all the singularities of the function $\Phi(\sigma, \lambda, \tau)$ are in the lower half-plane of the complex variable σ, the problem reduces to the Raus-Hurwitz problem. However, in the case of eqs. (5) the problem can be solved by the direct solution of the problem (5). Let $\sigma = \sigma_r + i\sigma_i$, $\lambda = \lambda_r + i\lambda_i$, $\tau = \tau_r + i\tau_i$. Eliminating from (5) the variables λ_i and λ_r we arrive at the system

$$
\begin{aligned}
&1.\ \sigma_i(2\sigma_r + \tau_r) = -\tau_i(\sigma_r + 2\tau_r) - a(\sigma_r + \tau_r) \\
&2.\ \sigma_i^2 + \tau_i^2 + \sigma_i\tau_i + a(\sigma_i + \tau_i) = \sigma_r^2 + \tau_r^2 + \sigma_r\tau_r - \frac{8 - a^2}{2}.
\end{aligned}
\qquad (6)
$$

Eqs. (6) are symmetrical with respect to the replacement $\sigma_i \rightleftarrows \tau_i$, $\sigma_r \rightleftarrows \tau_r$. Eliminating τ_i (or σ_i) from the first equation of (6) we get an equation of the second degree with respect to σ_i (or τ_i) the solution of which is

$$
\sigma_i^{1,2} = -\frac{a}{3} \pm \sqrt{\frac{1}{3}\left[\frac{8(\sigma_r + 2\tau_r)^2 + a^2\sigma_r^2}{2(\sigma_r^2 + \sigma_r\tau_r + \tau_r^2)} + (\sigma_r + 2\tau_r)^2\right] - \frac{2a^2}{9}} \qquad (7)
$$

$$
\left(\text{or} \quad \tau_i^{1,2} = -\frac{a}{3} \mp \sqrt{\frac{1}{3}\left[\frac{8(\tau_r + 2\sigma_r)^2 + a^2\tau_r^2}{2(\sigma_r^2 + \sigma_r\tau_r + \tau_r^2)} + (\tau_r + 2\sigma_r)^2\right] - \frac{2a^2}{9}}\ \right)
$$

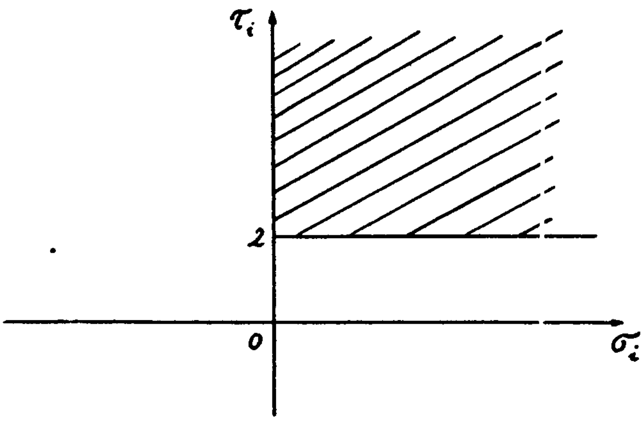

Fig. 2.

In this case to the root $\sigma_1 = -\frac{a}{3} + \sqrt{\dots}$ there corresponds the root $r_1 = -\frac{a}{3} - \sqrt{\dots}$. In Fig. 2 the "forbidden" area for the roots (7) is shaded. Thus, it is possible to write out two series of inequalities which the roots (7) must satisfy:

I system

1. $\sigma_1^{(1)} < 0$
2. $r_1^{(1)} < 0$

(8a)

II system

1. $\sigma_1^{(2)} < 0$
2. $r_1^{(2)} < 0$

(8b)

The inequalities (8a) and (8b) are analysed under the condition that all the σ_r values should be allowed and the r_r values should form at least a narrow band near $r_r = 0$. (The region of the r_r values near zero introduces the largest restrictions on the parameter a). It should be noted that both in the first and second inequalities it is sufficient to fulfil one of the inequalities for the singularity to be unphysical.

It is seen from eqs. (7) that if $a > 0$ then the singularities will not lie on the physical sheets since either $r_1^{(1)} < 0$ or $\sigma_1^{(2)} < 0$ (the complex values of the roots σ_1 and r_1 mean the absence of the intersection of the plane $a - i(\sigma + \lambda + r) = 0$ with the sphere $\sigma^2 + \lambda^2 + r^2 = -8$). The negative a values satisfy not one of the sets of inequalities. Thus, for $a > 0$ for the function given by (4) the dispersion relations in the variable s in the region $t < 0$ can be written.

The elastic scattering amplitude represented as a sum of the expression (4) -(4a) leads to that the desired amplitude will have in the plane for fixed physical values of t the following singularities:

a) Poles lying on the imaginary axis σ and corresponding to the bound states of the system of two π -mesons;

b) Poles lying in the lower half-plane σ symmetrical with respect to the imaginary axis and corresponding to the resonance states of the system created by two pions.

c) Projections of the cuts in the lower half-plane in the form of a pole point the beginning of which may be connected with the thresholds of the following reactions

$$\pi + \pi \to \pi^* + \pi \quad \text{or} \quad \pi + \pi \to \pi^* + \pi^* ,$$

where π^* is the resonance states of a pion.

If the amplitude decreases at infinity sufficiently rapidly, then applying in the plane σ (for a fixed τ) the Caushy theorem to the function $\Phi(\sigma,\lambda,\tau)$ we can obtain the following sum rule:

$$\sum_i g_i^2 + \sum_j \mathrm{Re}\, G_j + \sum_k \mathrm{Res}\, \Phi_k(\sigma_{inel}) = 0 \qquad (1a)$$

where $\sum_k \mathrm{Res}\, \Phi_k(\sigma_{in})$ is the sum of the $\Phi(\sigma, \tau_{fix}, \lambda_{fix})$ function residues at the pole points corresponding to the projections of the cuts on the lower half-plane σ for a fixed momentum transfer . These residues must be connected with the inelastic two-particle unitarity of the processes $\pi+\pi \to \pi^*+\pi$ or $\pi+\pi \to \pi^*+\pi^*$ and so on at the points of the cut projections.

4. Restrictions Imposed on the Parameters of the Model Function $\Phi(\sigma,\lambda,\tau)$ by the Condition $\mathrm{Im}\, F(s \geq 4, t=0, u=-4\nu) \geq 0$

From eq. (4) for the function $\Phi(\sigma,\lambda,\tau)$ it follows that

$$\frac{1}{2\nu^{1/2}} \mathrm{Im}\, F(s \geq 4, t=0, u=-4\nu) = f_0(y, a_i, A_i) =$$

$$= A_1[(a_2+2+2y)^2+4y^2-4][(a_3+2+2y)^2+4y^2-4]+A_2[(a_1+2+2y)^2+4y^2-4]\times \qquad (11)$$

$$\times[(a_3+2+2y)^2+4y^2-4] + A_3[(a_1+2+2y)^2+4y^2-4][(a_2+2+2y)^2+4y^2-4];$$

$$y=\sqrt{1+\nu}\ .$$

We demand that the optic theorem should be fulfilled at least in the weakened form

$$\mathrm{Im}\, F(s \geq 4,\ t=0,\ u=-4\nu) \geq 0$$

in the interval $0 \leq \nu \leq \infty$ (the quantity ν may be chosen to be finite but sufficiently large). This requirement imposes some restrictions on the coefficients A_i , a_i . The restrictions follow the Sturm theorem about the determination of the number of real roots.

It is obvious that the interval of the y values which does not contradict all the other requirements, imposed on the function $\Phi(\sigma,\lambda,r)$, depends on the choice of the function $\Phi(\sigma,\lambda,r)$. For a successful choice of $\Phi(\sigma,\lambda,r)$ this interval may turn out to be equal to the desired one:

$$0 \leq \nu \leq \infty$$

Let us construct the Sturm series

$$f_0(y, a_i, A_i) = 64 y^4 F_4 + 32 y^3 F_3 + 8 y^2 F_2 + 4 y F_1 + F_0 ,$$

$$f_1(y, a_i, A_i) = \frac{1}{4} f_0' = 64 y^3 F_4 + 24 y^2 F_3 + 4 y F_2 + F_1 ,$$

$$f_2(y, a_i, A_i) = y^2 [3 F_3^2 - 4 F_2 F_4] + y \left[\frac{F_2 F_3 - 6 F_1 F_4}{2} \right] + \frac{F_1 F_3 - 8 F_0 F_4}{8} ,$$

$$f_3 = 4 y V + W$$

$$f_4 = \frac{W}{8V} (F_2 F_3 - 6 F_1 F_4) - \frac{W^2}{16 V^2} (3 F_3^2 - 4 F_2 F_4) - \frac{F_1 F_3 - 8 F_0 F_4}{8} ,$$

where

$$V = (F_2 F_3 - 6 F_1 F_4)(9 F_3^3 - 16 F_2 F_3 F_4 + 24 F_1 F_4^2) + 2 F_4 (F_1 F_3 - 8 F_0 F_4)(3 F_3^2 - 4 F_2 F_4) - F_2 (3 F_3^2 - 4 F_2 F_4)^2 ,$$

$$W = (F_1 F_3 - 8 F_0 F_4)(9 F_3^3 - 16 F_2 F_3 F_4 + 24 F_1 F_4^2) - F_1 (3 F_3^2 - 4 F_2 F_4)^2 ,$$

$$F_4 = A_1 + A_2 + A_3 ,$$

$$F_3 = A_1(a_2 + a_3 + 4) + A_2(a_1 + a_3 + 4) + A_3(a_1 + a_2 + 4) ,$$

$$F_2 = A_1[a_2(a_2+4) + a_3(a_3+4) + 2(a_2+2)(a_3+2)] + A_2[a_1(a_1+4) + a_3(a_3+4) +$$

$$+ 2(a_1+2)(a_3+2)] + A_3[a_1(a_1+4) + a_2(a_2+4) + 2(a_1+2)(a_2+2)] ,$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (12)$$

$$F_1 = A_1[a_2(a_2+4)(a_3+2) + a_3(a_3+4)(a_2+2)] + A_2[(a_1+4)(a_3+2)a_1 +$$

$$+ a_3(a_3+4)(a_1+2)] + A_3[a_1(a_1+4)(a_2+2) + a_2(a_2+4)(a_1+2)] ,$$

$$F_0 = A_1 a_2 a_3 (a_2+4)(a_3+4) + A_2 a_1 a_3 (a_1+4)(a_3+4) + A_3 a_1 a_2 (a_1+4)(a_2+4) .$$

Let us make the table of the sings for the functions $f_0, f_1 \ldots f_4$ at the points $y = 1$ and $y_0 = \infty$ (see Table I).

Table I

y	f_0	f_1	f_2	f_3	f_4	Number of sign changes (N)
1	+	$(\pm)$	$(\pm)$	$(\pm)$	$f_4(1) = f_4(\infty)$	
$y_0 = \infty$	+	+	$(\pm)$	$(\pm)$	$f_4(1) = f_4(\infty)$	

The insertion of $y = 1$ and $y = \infty$ in the functions (12) leads to the following expressions

$$
\begin{cases}
f_0(1) = 64F_4 + 32F_3 + 8F_2 + 4F_1 + F_0 > 0 \\
f_0(\infty) = F_4 > 0 \, ,
\end{cases}
$$

$$
\begin{cases}
f_1(1) = 64F_4 + 24F_3 + 4F_2 + F_1 \gtrless 0 \\
f_1(\infty) = F_4 > 0 \, ,
\end{cases}
$$

$$
\begin{cases}
f_2(1) = 3F_3^2 - 4F_2F_4 + \dfrac{F_2F_3 - 6F_1F_4}{2} + \dfrac{F_1F_3 - 8F_0F_4}{8} \gtrless 0 \\
f_2(\infty) = 3F_3^2 - 4F_2F_4 \gtrless 0 \, ,
\end{cases}
\tag{13}
$$

$$
\begin{cases}
f_3(1) = 4V + W \gtrless 0 \\
f_3(\infty) = V \gtrless 0 \, ,
\end{cases}
$$

$$
f_4(1) = f_4(\infty) \gtrless 0 \, .
$$

According to the Sturm theorem the function $f_0(y, a_i, A_i)$ has no zeros in the interval $1 \leq y \leq \infty$ if the number of the sign changes N in the first line of the Table I is equal to that in the second line. The maximum possible number of changes is three. Totally Table I gives 34 different sets of inequalities. Each of them gives its own region of the allowed values of the parameters a_i, A_i. In this paper these regions are not considered. Table I is simply checked for the A_1, a_1 values which will be found from the conditions of the two-particle unitarity (§6) for partials s and d waves.

5. Analytical Properties of the Partial Amplitudes

The partial amplitudes $\Phi_\ell(\sigma) = F_\ell(\nu)$ are determined in the usual manner

$$F_\ell(\nu) = \frac{1}{2}\int_{-1}^{+1}\Phi(\sigma,\lambda,\tau)\, P_\ell(z)\,dz \ .$$

In the present model we can look for what singularities arise in the partial waves and in what way, if in the total amplitude on the physical sheets the physical cuts (with respect to s, u, t) and the poles corresponding to the bound states of the $\pi\pi$ -system are given.

s-wave

$$F_0(\nu) = \frac{g^2}{3+4\nu} - \frac{g^2}{\nu} Q_0\left(\frac{1+2\nu}{2\nu}\right) +$$

$$+ \sum_{n=1}^{3} \frac{A_n}{\nu}\Big\{\sqrt{1+\nu} - 1 + \frac{2i\nu^{1/2} - a_n}{4}\arcsin\frac{\nu}{\nu+2} + \qquad (14)$$

$$+ \frac{a_n^2 - 8 - 8\nu - 4ia_n\nu^{1/2}}{8\sqrt{16+12\nu - a_n^2 + 4ia_n\nu^{1/2}}}\Big[\ln\frac{(4+2\nu)^2 - (x_2 - x_1\sqrt{1+\nu})^2}{(4+2\nu)^2 - (x_1 - x_2\sqrt{1+\nu})^2} - 2\ln\frac{(2\sqrt{1+\nu}-x_1)(2-x_2)}{(2\sqrt{1+\nu}-x_2)(2-x_1)}\Big]\Big\},$$

where Q_0 of the Legendre function of the second rank,

$$x_{1,2} = \frac{1}{2}\left(2i\nu^{1/2} - a_n \pm \sqrt{16+12\nu - a_n^2 + 4ia_n\nu^{1/2}}\right) .$$

The analysis of the expressions in eq.(14) shows that the s -wave has the following singularities:

1. Pole at the point $\nu = -\frac{3}{4}$ corresponding to the usual pole term;

2. Logarithmic branch point $\nu = -\frac{1}{4}$ corresponding to the beginning of the cuts from the poles in crossing channels with respect to the variables u and t at $z = \pm 1$;

3. Root cut in the interval $0 \leq \nu \leq \infty$ corresponding to the two-particle unitarity in the s -channel.

4. Root cut in the interval $-\infty \leq \nu \leq -1$, corresponding to the beginning of the physical cuts in crossing channels with respect to the variables u and t at $z = \pm 1$ (see Fig.3).

5. Logarithmic singularity at the point $\nu = -2$ connected with the branching of the cuts of u and t channels (see Fig.3)

6. Root singularities lying on the unphysical sheet and determined from the equation $16 + 12\nu - a_n^2 + 4ia_n\nu^{1/2} = 0$, where ν assumes the complex values. The first five singularities are well known. The sixth singularity was not investigated earlier.

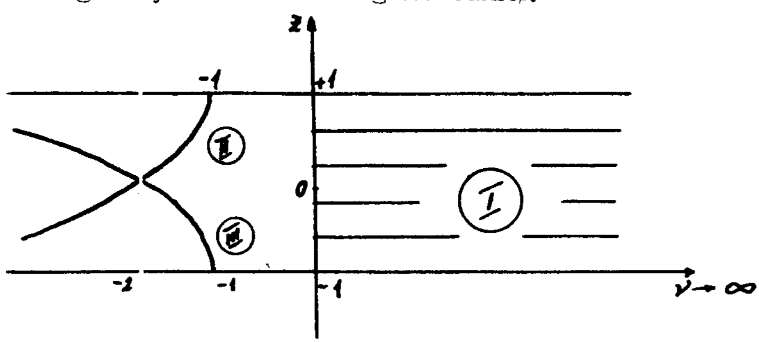

Fig. 3

Fig.3 Cuts given by the Mandelstam representations in the reaction $\pi + \pi \to \pi + \pi$ in the s -channel (on the plane ν, z):

I - cut from s -channel

II - cut from u -channel

III -cut from t -channel

It can be interpreted as a cut arising from the two-particle inelastic unitarity in the production of resonance mesons: $\pi+\pi\to\pi^*+\pi$ or $\pi+\pi\to\pi^*+\pi^*$. The effect of this cut on the $\pi+\pi\to\pi+\pi$ scattering amplitude can lead to various anomalies in the cross section (resonance, curve knee points etc).

d-wave

$$F_2(\nu)=\sum_{i=1}^{3}\frac{A_i}{8\nu}\left[\frac{3(4+2\nu)^2}{4\nu^2}-1\right]F_0(\nu,a_i,A_i)-\frac{g^2}{\nu}Q_2\left(\frac{1+2\nu}{2\nu}\right)-$$

$$-\sum_{i=1}^{3}\frac{3A_i(2+\nu)}{8\nu^3}\{2(\sqrt{1+\nu}-1)(9+8\nu-a_i^2)+2(\sqrt{1+\nu}-1)4ia_i\nu^{1/2}-$$

$$-\frac{a_i^2-8-8\nu-4ia_i\nu^{1/2}}{2(x_1-x_2)}\left(x_1^2\ln\frac{2\sqrt{1+\nu}-x_1}{2-x_1}-x_2^2\ln\frac{2\sqrt{1+\nu}-x_2}{2-x_2}\right)-\frac{8}{3}(\sqrt{(1+\nu)^3}-1)-$$

$$-(4+2\nu)(x_1+x_2)\arcsin\frac{\nu}{2+\nu}+(x_1^2+x_1x_2+x_2^2)(2\sqrt{1+\nu}-2)+(x_1^3+x_1^2x_2+x_1x_2^2+x_2^3)\times$$

$$\times\arcsin\frac{\nu}{2+\nu}+\frac{x_1^3x_2}{x_1-x_2}\ln\frac{(2-x_1)(4+2\nu+x_2-x_1\sqrt{1+\nu})}{(2\sqrt{1+\nu}-x_1)(4+2\nu-x_1+x_2\sqrt{1+\nu})}-\tag{15}$$

$$-\frac{x_2^3x_1}{x_1-x_2}\ln\frac{(2-x_2)(4+2\nu+x_1-x_2\sqrt{1+\nu})}{(2\sqrt{1+\nu}-x_2)(4+2\nu-x_2+x_1\sqrt{1+\nu})}\}+$$

$$+\sum_{i=1}^{3}\frac{3A_i}{32\nu^3}\{\frac{2^5}{5}(\sqrt{(1+\nu)^5}-1)-(a_i^2-8-8\nu-4ia_i\nu^{1/2})\left(\frac{4}{3}\sqrt{(1+\nu)^3}-\frac{4}{3}+\nu(x_1+x_2)+\right.$$

$$\left.+(\sqrt{1+\nu}-1)(x_1^2+x_1x_2+x_2^2)\right)-\frac{a_i^2-8-8\nu-4ia\nu^{1/2}}{2(x_1-x_2)}\left(x_1^4\ln\frac{2\sqrt{1+\nu}-x_1}{2-x_1}-x_2\ln\frac{2\sqrt{1+\nu}-x_2}{2-x_2}\right)+$$

$$+8\left(\frac{4(1+\nu)}{5}+\frac{8(2+\nu)}{15}\right)-8\sqrt{(1+\nu)^3}\left(\frac{4}{5}+\frac{8(2+\nu)}{15}\right)+(x_1^2+x_1x_2+x_2^2)8(1-\sqrt{(1+\nu)^3})-$$

$$-(x_1+x_2)[2\nu\sqrt{1+\nu} + (4+4\nu - \frac{(8+4\nu)^2}{4})\sqrt{1+\nu} + \frac{(8+4\nu)^2}{8}\arcsin\frac{\nu}{2+\nu}] -$$

$$-\frac{x_1^2+x_1x_2+x_2^2}{2}(8+4\nu)\arcsin\frac{\nu}{2+\nu} + \frac{x_1^5-x_2^5}{x_1-x_2}(2\sqrt{1+\nu}-2) + \frac{x_1^6-x_2^6}{x_1-x_2}\arcsin\frac{\nu}{2+\nu} +$$

$$+\frac{x_1^5x_2}{x_1-x_2}\ln\frac{(2-x_1)(x_2+4+2\nu-x_1\sqrt{1+\nu})}{(2\sqrt{1+\nu}-x_1)(4+2\nu-x_1+x_2\sqrt{1+\nu})} - \qquad (15)$$

$$-\frac{x_2^5x_1}{x_1-x_2}\ln\frac{(2-x_2)(4+2\nu+x_1-x_2\sqrt{1+\nu})}{(2\sqrt{1+\nu}-x_2)(4+2\nu-x_2+x_1\sqrt{1+\nu})}\}$$

Summing the second $\sum_{i=1}^{3}\ldots$ and the third $\sum_{i=1}^{3}\ldots$ sums from (15) it is seen that no additional singularities, compared to those which were in the s -wave, will arise since the logarithmic singularities are cancelled in just the same way as in the s -wave which is seen from the following:

$$\frac{3A}{8\nu^3}\left(-\frac{(2+\nu)x_1^3x_2}{x_1-x_2}\ln(1)+\frac{x_1^5x_2}{4(x_1-x_2)}\ln(1)\right) = -\frac{3A\;x_1^3x_2}{32\nu^3(x_1-x_2)}\ln(1)$$

$$\frac{3A}{8\nu^3}\left(\frac{(2+\nu)x_2^3x_1}{x_1-x_2}\ln(2)-\frac{x_2^5x_2}{4(x_1-x_2)}\ln(2)\right) = \frac{3A\;x_1^3x_2}{32\nu^3(x_1-x_2)}\ln(2) \quad .$$

In these expressions the logarithms have the common multiplier. The ratio under the logarithm sign has no singularities.

6. Two-Particle Unitarity

In §3 and §4 we have determined the range in which the parameters change

1. $a_n > 0$,

2. a_n and A_n must satisfy the conditions (13) taking into account the Table I of the change of signs.

The equations which will give the exact values for the parameters a_n , A_n and g^2 follow from the two-particle unitarity. The expressions for the s and d -waves (14) and (15) are very cumbersome. In singling out the real and imaginary part we would meet algebraic difficulties, therefore for the sake of simplicity we write these formulas as a series in ν which is valid in the interval $0 \leq \nu < \frac{1}{4}$. To this end we firstly expand the expression

$$\frac{A_i}{a_i + \sqrt{4+2\nu+2\nu z} + \sqrt{4+2\nu-2\nu z} - 2i\nu^{1/2}}$$

and then integrate them with the Legendre polynomials. We obtain

$$F_0(\nu) = \overline{\mathrm{Re}\, F_0(\nu)} + \frac{g^2}{3+4\nu} - \frac{g^2}{2\nu} \ln(1+4\nu) + i\,\nu^{1/2} \,\mathrm{Im}\, F_0(\nu),$$

where

$$\overline{\mathrm{Re}\, F_0(\nu)} = \sum_{i=1}^{3} \frac{A_i}{(a_i+4)^2} \{ (a+4) + \nu [1 - 2\frac{a_i+6}{a_i+4}] -$$

$$- \nu^2 [\frac{1}{12} + \frac{6+a_i}{2(a_i+4)} - \frac{a_i}{4(a_i+4)} - \frac{4(a_i+6)^2}{(a_i+4)^3}] + \tag{16}$$

$$+ \nu^3 [\frac{1}{16} - \frac{2+3a_i}{24(a_i+4)} - \frac{9+a_i}{24(a_i+4)^2} - \frac{a_i(a_i+6)}{(a_i+4)^3} + \frac{4(a_i+6)^2}{(a_i+4)^4} - \frac{8(a_i+6)^3}{(a_i+4)^5}] -$$

$$- \nu^4 \left[- \frac{21}{640} + \frac{19}{120(a_i+4)} + \frac{35+12a_i}{48(a_i+4)^2} - \frac{27a_i^2+160a_i+96}{48(a_i+4)^3} + \frac{2(a_i-3)(a_i+6)}{3(a_i+4)^4} - \right.$$

$$\left. - \frac{3a_i(a_i+6)^2}{(a_i+4)^5} + \frac{8(a_i+6)^3}{(a_i+4)^6} + - \frac{16(a_i+6)^4}{(a_i+4)^7} \right] \Big\} ;$$

$$\mathrm{Im}\, F_0(\nu) = \sum_{i=1}^{3} \frac{2A_i}{(a_i+4)^2} \Big\{ 1 - \nu \frac{2(a_i+6)}{(a_i+4)^2} + \nu^2 \left[\frac{1}{12(a_i+4)} + \frac{a_i}{4(a_i+4)^2} + \frac{4(a_i+6)^2}{(a_i+4)^3} \right] -$$

$$- \frac{\nu^3}{3} \left(\frac{8+6a}{16(a_i+4)^2} + \frac{a_i+6}{(a_i+4)^3} + \frac{3a_i(a_i+6)}{(a_i+4)^4} + \frac{24(a_i+6)^3}{(a_i+4)^6} \right) + \qquad (16)$$

$$+ \frac{\nu^4}{3(a_i+4)^2} \left[\frac{26+15a_i}{30} + \frac{a_i}{8(a_i+4)} + \frac{3a_i^2+4(a_i+6)(8+3a_i)+12a_i(a_i+6)}{16(a_i+4)^2} + \right.$$

$$\left. + \frac{3(a_i+6)}{(a_i+4)^3} + \frac{9a_i(a_i+6)^2}{(a_i+4)^4} + \frac{48(a_i+6)^4}{(a_i+4)^6} \right] \Big\}$$

$$F_2(\nu) = \overline{\mathrm{Re}\, F_2(\nu)} - \frac{g^2}{\nu} Q_2\left(\frac{1+2\nu}{2\nu} \right) + i\, \nu^{1/2}\, \mathrm{Im}\, F_2(\nu).$$

$$(17)$$

$$\overline{\mathrm{Re}\, F_2(\nu)} = \sum_{i=1}^{3} \frac{A_i \nu^2}{60(a_i+4)^2} + O(\nu^3)$$

$$\mathrm{Im}\,F_2(\nu) = \sum_{i=1}^{3} \frac{A_i\,\nu^2}{(a_i+4)^2}\Big\{ \frac{1}{15(a_i+4)} - \nu\Big[\frac{8+3a_i}{60(a_i+4)^2} + \frac{(a_i+6)}{15(a_i+4)^3}\Big] +$$

$$+ \nu^2\Big[\frac{8+3a_i}{84(a_i+4)^2} + \frac{a_i}{30(a_i+4)^3} + \frac{(8+3a_i)(a_i+6)}{15(a_i+4)^4} + \frac{12(a_i+6)^2}{15(a_i+4)^5}\Big]\Big\} + O(\nu^5),$$

The expressions for the pole terms in eqs. (16) and (17) are written down without expansions. (The expansion will be given in §7). In the expansion $\overline{\mathrm{Re}\,F_2(\nu)}$ all terms containing a power higher than the second one are omitted since the unitarity will be checked within the accuracy up to $\approx \nu^4$. Since $\mathrm{Im}\,F_2(\nu)$ near the threshold must be $\approx \nu^4$ it is necessary to require that in the imaginary part $\mathrm{Im}\,F_2(\nu)$ the coefficients for ν^2 and ν^3 dependent on A_i and a_i should be strictly zero (see eq. (18a)). The strict fulfilment of the two-particle unitarity at the threshold (near $\nu = 0$) leads to another two equations imposed on A_i, a_i (see eq. (18b).) Another series of two-particle unitarity conditions in the s - and d -waves at different points $0 < \nu \le \frac{5}{4}$ can be added to the mentioned two conditions (18b). In this work for the methodical purpose we consider the simplest approximation therefore we add to (18b) only two equations which correspond to the unitarity for the s -and d -waves at the point $0 < \tilde{\nu} < \frac{1}{4}$ (equation (18c))

$$1)\quad \sum_{i=1}^{3} \frac{A_i}{(a_i+4)^3} = 0 \qquad (18a)$$

$$2)\quad \sum_{i=1}^{3} \frac{A_i}{(a_i+4)^4}\Big[8+3a_i + \frac{16(a_i+6)}{a_i+4}\Big] = 0$$

$$3)\quad \operatorname{Im} F_0(\nu) = \left| \overline{\operatorname{Re} F_0(\nu)} - \frac{5g^2}{3} \right|^2$$

(18b)

$$4)\quad \operatorname{Im} F_2(\nu) = \left| \overline{\operatorname{Re} F_2(\nu)} - \frac{16}{15} g^2 \right|^2$$

$$5)\quad \operatorname{Im} F_0(\tilde{\nu}) = \left| \overline{\operatorname{Re} F_0(\tilde{\nu})} + \frac{g^2}{3+4\tilde{\nu}} - \frac{g^2}{2\tilde{\nu}} \ln(1+4\tilde{\nu}) \right|^2 + \tilde{\nu} \left| \operatorname{Im} F(\tilde{\nu}) \right|^2$$

(18c)

$$6)\quad \operatorname{Im} F_2(\tilde{\nu}) = \left| \overline{\operatorname{Re} F_2(\tilde{\nu})} - \frac{g^2}{\tilde{\nu}} Q_2\left(\frac{1+2\tilde{\nu}}{2\tilde{\nu}} \right) \right|^2$$

The six equations (18) form a system of nonlinear algebraic equations used for the determination of the exact values of the seven coefficients g^2, a_1, a_2, a_3, A_1, A_2, A_3. In the considered problem g^2 is a free parameter. The system (18) was solved here as follows. Three numbers a_1, a_2, a_3 were given. Then the system consisting of four equations (18a) and (18b) was solved exactly. Using the values of A_1, A_2, A_3 and g^2 eqs. (18c) were checked. It was assumed that in eqs. (18c) small deflections are possible (about 3-5 percent). For given a_1, a_2 and a_3 the requirement for g^2 to be positive select the unique solution. As an illustration we give two solution of this kind

1. $a_1 = 2;\ a_2 = 4;\ a_3 = 6;\ A_1 = 0{,}545;\ A_2 = -3{,}684;\ A_3 = 4{,}673;\ g^2 = 0{,}00291$

(19)

2. $a_1 = 0{,}1;\ a_2 = 0{,}5;\ a_3 = 1;\ A_1 = 11{,}19;\ A_2 = -30{,}03;\ A_3 = 20{,}89;\ g^2 = 0{,}0256$.

As should be expected small changes in the location of the beginning of the cuts a_1 effect a little parameters A_i and g^2 i.e. the solutions of the system (18) are stable. The addition of the parameters a_4, A_4 also changes a little the parameters A_i and g^2.

7. Contributions of Pole Terms

In the considered model there may be contributions of pole terms of the two types:

a) contributions from the pole corresponding to the bound states of a system consisting of two pions;

b) contributions from the poles corresponding to the resonance states of the $\pi\pi$ -system (they are given by eqs. (4a)). The contributions of the type a) are taken into account in eqs. (4), (14), (15), (18). In the physical domain the contributions of these pole terms do not lead to the cross section behaviour of the resonance type. In this sense the pole of the second type are of much greater interest. Consider, e.g. the construction of one such pole in the s -wave

$$\phi_0^{(pole)}(\nu) = \frac{2(a^2+b^2+4\nu)aG_r - 2(a^2+b^2-4\nu)G_1}{(a^2-b^2+4\nu)^2+4a^2b^2} + \frac{4}{\nu}G_r(\sqrt{1+\nu}-1)$$

$$+\frac{2}{\nu}G_r\left[a\ln\frac{r_0}{r_1}+b(\alpha_1-\alpha_0)\right]+\frac{2}{\nu}G_1\left[b\ln\frac{r_0}{r_1}+a(\alpha_0-\alpha_1)\right]+ \tag{21}$$

$$+4i\nu^{1/2}\frac{G_r(a^2-b^2+4\nu)-2abG_1}{(a^2-b^2+4\nu)^2+4a^2b^2},$$

where $G_r+iG_1=G$ is the residue of the function $\Phi(\sigma,\lambda,\tau)$ at the point $\sigma=-b-ia$;

$$r_0=\sqrt{(a+2)^2+b^2}; \qquad \alpha_0=\operatorname{arc\,tg}\frac{b}{a+2}$$

$$r_1=\sqrt{(a\pm 2\sqrt{1+\nu})^2+b^2}; \qquad \alpha_1=\operatorname{arc\,tg}\frac{b}{a+2\sqrt{1+\nu}}.$$

If $b^2>a^2$, which corresponds to that the mass of the resonance state is larger than the width, then the s -wave will have a maximum in the croos section. If the width of the resonance is small then the peak will be large and sharp, if the resonance width is comparable in magnitude with the resonance state mass then the peak will be low. In Fig.4 the shape of the peak corresponding to the partial cross section (s -wave) is given as an example for the case when $a = 0.1$, $b = 2$; $G_1 = 0$.

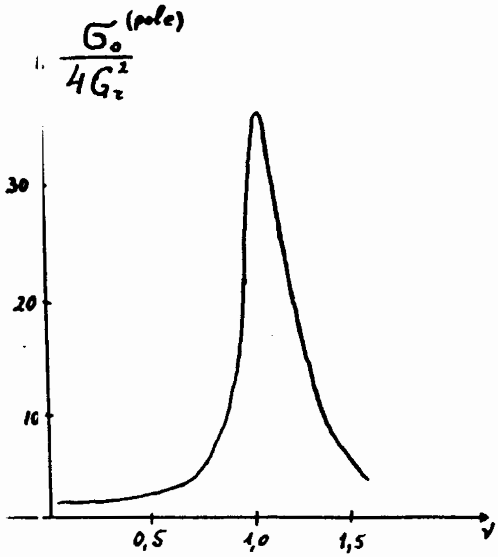

Fig.4. The shape of the cross section $\frac{\sigma_0^{(pole)}}{4G_r^2}$

8. Conclusion

In this paper we considered only one example of the model function $\Phi(\sigma,\lambda,r)$ taken in the form of (4). It was shown that using even such a simple form it is possible to satisfy the exact two-particle unitarity near the threshold $(\nu \to 0)$ and the approximate condition of unitarity on the cut $(0 \lesssim \nu \lesssim \nu_0)$. Thus, the suggested method for constructing the elastic scattering amplitude as compared to other methods ensures the fulfilment of a larger number of requirements imposed on the physical amplitude to be fulfilled. It also provides the existence of the dispersion relations with respect to s for $t < 0$. To illustrate the essence of the considered method we consider the domain of small ν such that $|a_i| > |\sigma + \lambda + r|$. In this case each term of the sum in eq. (4) can be expanded in the series

$$\sum_{k=0}^{\infty} B_k (\sigma + \lambda + r)^k .$$

This series is a harmonic polynomial which depends on the three variables σ, λ, r given on the sphere $\sigma^2 + \lambda^2 + r^2 = -8$. If the coefficients B_k were arbitrary numbers then there would be a complete system of functions (harmonic polynomials) by means of which it would be possible, within any degree of accuracy, to approximate any function analytical on the sphere for given values of ν. However, the coefficients B_k are combined in a definite manner from the coefficients A_i, a_i therefore the given series can approximate the desired function only within a certain degree of accuracy.

In this paper a more general problem is considered: on the complex sphere $\sigma^2 + \lambda^2 + r^2 = -8$ by means of a small set of fraction-linear meromorphic functions

$$\sum_{i=1}^{8} \frac{A_i}{a_i - i(\sigma + \lambda + r)}$$

we approximate another meromorphic function, the desired scattering amplitude on a certain intercept $0 \leq \nu \leq \text{const}$ for a restricted number of partial waves. It is obvious that, in principle, such a problem can be solved within a certain accuracy.

When the number of partial waves increases, a more complicated version of the function $\Phi(\sigma, \lambda, r)$ with a larger number of terms in the sum $\sum_i \ldots$, i.e. with a larger number of parameters should be taken.

From the given work it is easily seen that all the dynamics of the scattering processes (resonances and other anomalies in the cross section) are defined by the singularities of the model function $\Phi(\sigma,\lambda,\tau)$ lying on the unphysical sheet. If all the singularities of the function $\Phi(\sigma,\lambda,\tau)$ are eliminated then it should be represented as a symmetrical polynomial of the variables σ, λ, τ. If it is assumed that the amplitude at infinity should not be higher than the first or second degree then the considered polynomial is so simple that using it, it is impossible to describe satisfactorily the complicated behaviour of the scattering cross section. In the real case of scattering of pseudoscalar pions on pions it is interesting to look for the appearance of the known resonances in the $\pi\pi$ -system (e.g. of ρ meson in the state $T=I=1$). In this case we should take into account the contribution from inelastic processes. With the aim of taking into account approximately inelastic processes we can, instead of the transformations (I) introduce more complicated ones, for instance

$$\sigma = \pm\sqrt{s-4\pm\sqrt{s-16}}$$

$$\lambda = \pm\sqrt{u-4\pm\sqrt{u-16}}$$

$$\tau = \pm\sqrt{t-4\pm\sqrt{t-16}}$$

or

$$\sigma = \pm\sqrt{s-4\pm\sqrt{s-16\pm\sqrt{s-36}}} \qquad \text{etc.}$$

Finally it is interesting to know if there exists a set of the Feynmann diagrams which would lead to the model function (4).

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