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ON THE UPPER LIMIT OF THE RADIUS OF THE ELEMENTARY PARTICLES AND THE LOWER BOUND OF THEIR FORMFACTORS

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ON THE UPPER LIMIT OF THE RADIUS OF THE ELEMENTARY PARTICLES AND THE LOWER BOUND OF THEIR FORMFACTORS

Балуни В., Нгуен Ван Хьеу, Сулейманов В.А. E2-3986

О верхнем пределе радиуса элементарных частиц и нижней границе их формфактора

В работе на основе аналитических свойств формфактора получены верхний предел радиуса элементарных частиц и ограничение снизу на убывание формфактора при отрицательных передачах импульса.

Препринт Объединенного института ядерных исследований.
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Baluni V., Nguyen Van Hieu, Suleymanov V.A. E2-3986

On the Upper Limit of the Radius of the Elementary Particles and the Lower Bound of their Formfactors

On the basis of the analyticity of the formfactor the upper limit of the radius of the elementary particles is obtained. We establish also the lower bound for the formfactor in the region $t < 0$.

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In a series of papers /1-7/ it was shown that from very general assumptions on the analytic properties of the formfactor, we can get various ralations between observable quantities (for π meson) between modulus of the formfactor in the region $t < 0$ and on the cut. In particular some lower bound for the formfactor in the region $t < 0$ was obtained by one of the authors /6/ when it is analytic in the t cut-plane and bounded on the cut. Assuming further, that for the formfactor $F(t)$ in the region $t < 0$ smaller than unity he obtained also the upper limit of the radius of the elementary particles/7/

In the present paper we generalize the results in ref /6,7/ in two directions. First, instead of the Hadamard theorem on three circles used in papers /6,7/ we use the (more general) theorem on two constants in the theory of harmonic measure /8/. This allows us to improve the results obtained in /6,7/. Secondly, instead of boundedness of the formfactor on the cut we make weaker assumption about polynomial growth at $t \to +\infty$.

Throughout this paper we suppose that the formfactor $F(t)$ is analytic in the complex t plane with a cut from $t = 4m^2_\pi$ to $+\infty$. Our results may be formulated in 4 theorems. The proofs of these theorems will be represented very briefly since the detail consideration of the similar problem was given in refs. /6,7/.

<u>Theorem 1</u>. Let $F(t)$ be bounded on the cut and have the modulus smaller than unity in the physical region of the scattering channel

$$|F(t)| \leq \begin{cases} M & \text{if} \quad t \geq 4m_\pi^2 \\ 1 & \text{if} \quad t \leq 0 \, . \end{cases} \tag{1}$$

Then

$$|F'(0)| \leq \begin{cases} \dfrac{M \ln M}{4\pi m_\pi^2} \operatorname{arctg}\left(\dfrac{2\sqrt{M}}{M-1}\right) & \text{if} \quad \ln M \leq \pi \\ \dfrac{1}{8m_\pi^2} \, \dfrac{a}{\Phi(a)} \operatorname{arctg}\left(\dfrac{2\sqrt{\Phi(a)}}{1-\Phi(a)}\right) & \text{if} \quad \ln M \geq \pi , \end{cases} \tag{2}$$

where $a = \frac{2}{\pi} \ln M$ and

$$\Phi(a) = \frac{a^2 - 2 - \sqrt{(a^2-2)^2 - 4}}{2} \exp\left(- a \operatorname{arctg} \frac{2}{\sqrt{a^2-4}}\right). \tag{3}$$

Proof. By means of the conformal mappings

$$z = \frac{1 - \sqrt{1 - t/4m_\pi^2}}{1 + \sqrt{1 - t/4m_\pi^2}} \, , \quad w = \sqrt{z} \tag{4}$$

we transform the complex t plane into the unit circle in the z -plane and then into the right unit semicircle D. In these mappings the cut $t \geq 4m_\pi^2$ and the negative axis are respectively transformed into the right semicircumference a and the segment $[-i, i]$ in the w plane. We put

$$F(t) \equiv f(z) \equiv \phi(w) \, .$$

According to the theorem on two constants we get from condition(1)

$$|\phi(w)| \le M^{\omega(w,a)}, \tag{5}$$

where $\omega(w,a)$ is the harmonic measure of the arc a at the point $w=u+iv$ with respect to the domain D (see, ref. /8/, chapter VI)

$$\omega(w,a) = \frac{2}{\pi} \operatorname{arctg} \frac{2u}{1-|w|^2} . \tag{6}$$

In the other words

$$|f(z)| \le \exp\{ a \operatorname{arctg} \frac{2\sqrt{z}}{1-z} \} . \tag{7}$$

Now we apply to $f(z)$ the arguments of the paper /7/. This function does not vanish everywhere in the circle $|z| \le \rho_0$

$$\rho_0 = \frac{\rho}{\max\limits_{|z|=\rho} |f(z)|} . \tag{8}$$

The expression on the r.h.s. of the formula (8) reaches its maximum at the point

$$\rho = \begin{cases} 1 & \text{if } \ell n\, M \le \pi \\ \dfrac{a^2-2-\sqrt{(a^2-2)^2-4}}{2} & \text{if } \ell n\, M \ge \pi . \end{cases} \tag{9}$$

Hence it follows, that $f(z)$ has no zeros in the circle $|z| = z_0$ where

$$z_0 = \begin{cases} [illegible] & [illegible] \le \pi , \\ \phi(z)\ [illegible] & [illegible] \ge \pi . \end{cases}$$

As in the paper /7/, we have

$$|f'(0)| \leq \frac{2}{z_0} \max_{|z|=z_0} \ln |f(z)| \; . \tag{11}$$

From relations (7),(10) and (11) we get

$$|f'(0)| \leq \begin{cases} \dfrac{4}{\pi} M \ln M \operatorname{arctg} \dfrac{2\sqrt{M}}{M-1} & \text{if} \quad \ln M \leq \pi \\ \\ \dfrac{2a}{\Phi(a)} \operatorname{arctg} \dfrac{2\sqrt{\Phi(a)}}{1-\Phi(a)} & \text{if} \quad \ln M \geq \pi \end{cases} \tag{12}$$

and theorem 1 was proved.

Now we generalize the obtained result to the case when $F(t)$ increases polynomially at $t \to +\infty$

$$|F(t)| \leq \text{const}\; t^{N} \; , \quad t \to +\infty$$

In this case $f(z)$ can tend to infinity as $(1+z)^{-2N}$ at $z \to -1$. Instead of $f(z)$ we consider a new function

$$g(z) = f(z)(1+z)^{2N} \; .$$

It is analytic in the unit circle and bounded on its boundary. Furthermore due to condition (1) $|g(z)|<1$ in the region $z<0$. Therefore all arguments of the foregoing proof can be applied to the function $g(z)$. Instead of the condition $|f(z)| \leq M$ on the circumference we use the condition $g(z) \leq M$. Hence it follows

<u>Theorem 2.</u> Let $F(t)$ be smaller than unity in the region $t<0$ and satisfy the following condition on the cut.

$$\left| \frac{2}{1+\sqrt{1-t/4m_\pi^2}} \right|^{2N} |F(t)| \leq M \, , \quad t \geq 4 m_\pi^2 \tag{13}$$

Then

$$|F'(0)+2N| \le \begin{cases} \dfrac{M \ln M}{4\pi m_\pi^2} \operatorname{arctg}\left(\dfrac{2\sqrt{M}}{M-1}\right) & \text{if } \ln M \le \pi \\ \\ \dfrac{1}{8 m_\pi^2} \dfrac{a}{\Phi(a)} \operatorname{arctg}\left(\dfrac{2\sqrt{\Phi(a)}}{1-\Phi(a)}\right) & \text{if } \ln M \ge \pi \ . \end{cases} \tag{14}$$

Now we study lower bounds of the formfactor in the region $t<0$. First we consider the case when $F(t)$ is bounded on the cut.

Theorem 3. Let $F(t)$ be bounded on the cut

$$|F(t)| \le M \ , \quad t \ge 4m_\pi^2 \ . \tag{15}$$

Then

$$\max_{t/4m_\pi^2 < -\beta} |F(t)| \ge \left(\frac{1}{M}\right)^{\psi(\beta)} , \tag{16}$$

where

$$\psi(\beta) = \frac{\operatorname{arctg}\sqrt{\beta}}{\frac{\pi}{2} - \operatorname{arctg}\sqrt{\beta}} \ . \tag{17}$$

In particular if $F(t)$ decreases monotonely with increasing $|t|$ in the region $t<0$, then we have

$$|F(t)| \ge \left(\frac{1}{M}\right)^{\psi\left(\frac{t}{4m_\pi^2}\right)} \tag{18}$$

In this case $F(t)$ can decrease by e times in the interval $(0, -t_e)$ only if

$$t_e \geq \left[\mathrm{tg} \frac{2}{\pi(\ln M + 1)} \right]^2 4 m_\pi^2 \quad . \tag{19}$$

Proof. By means of the conformal mappings

$$\zeta = \frac{\sqrt{\beta+1} - \sqrt{1 - t/4m_\pi^2}}{\sqrt{\beta+1} + \sqrt{1 - t/4m_\pi^2}} , \quad \eta = \sqrt{\zeta} \tag{20}$$

we transform the complex t -plane into the unit circle $|\zeta| \leq 1$ and further into the right unit semicircle D . The segment $[-i, i]$ in the η -plane is the image of the infinite interval $t \leq -4m_\pi^2 \beta$. We put $h(\eta) = F(t)$. Applying the theorem on two constants, we get

$$|F(0)| = |h(\eta_0)| \leq m \left(\frac{M}{m} \right)^{\omega(\eta_0, a)} , \tag{21}$$

where

$$\eta_0 = \sqrt{\zeta_0} = \left[\frac{\sqrt{\beta+1} - 1}{\sqrt{\beta+1} + 1} \right]^{1/2} , \tag{22}$$

$$m = \max_{\eta \in [-i, i]} |h(\eta)| = \max_{t/4m_\pi^2 < -\beta} |F(t)| . \tag{23}$$

Since $F(0) = 1$, we obtain

$$m \geq \left(\frac{1}{M} \right)^{\frac{\omega(\eta_0, a)}{1 - \omega(\eta_0, a)}} . \tag{24}$$

It is easy to check that

$$\omega(\eta_0, a) = \frac{2}{\pi} \operatorname{arctg} \sqrt{\beta} \tag{25}$$

hence it follows theorem 3.

Consider now the case when $F(t)$ increases polynomially at $t \to +\infty$. By the arguments similar to the proof of the theorem 2 one can prove the following theorem.

<u>Theorem 4.</u> Let there exist such positive numbers M and N that the condition (13) holds. Then

$$\max_{t/4 m_\pi^2 \leq -\beta} |F(t)| \geq \xi_N(\beta) \left(\frac{1}{M}\right)^{\psi(\beta)}, \tag{26}$$

where $\psi(\beta)$ is defined by formula (17) and

$$\xi_N(\beta) = \left(\frac{2\sqrt{\beta+1}}{\sqrt{\beta+1}+1}\right)^{\frac{2N}{1-\frac{2}{\pi}\operatorname{arctg}\sqrt{\beta}}}. \tag{27}$$

We note, that the results obtained in the case, when $F(t)$ is bounded on the cut are the improvements of the results in refs./6,7/. For example, for $\ell n M \leq 2$ formula (2) gives

$$|F'(0)| \leq \frac{M \ell n M}{8 m_\pi^2} \cdot \frac{2}{\pi} \operatorname{arctg} \frac{2\sqrt{M}}{M-1} < \frac{M \ell n M}{8 m_\pi^2}. \tag{28}$$

From the result of the paper /7/ we have

$$|F'(0)| \leq \frac{M \ell n M}{2 m_\pi^3}. \tag{29}$$

For $M \to \infty$ from formula (2) it follows that

$$|F'(0)| \leq \frac{4}{\pi^2} \frac{e(\ell n M)^2}{4m_\pi^2} . \tag{30}$$

The limit obtained in paper /7/ is

$$|F'(0)| \leq \frac{e(\ell n M)^2}{4m_\pi^2} . \tag{31}$$

We consider a simple example of the formfactor satisfying all conditions of the theorem 1

$$F(t) = e^{1-\sqrt{1-t/4m_\pi^2}} .$$

In this case $M = e$ and formula (2) gives

$$|F'(0)| < \frac{e}{8m_\pi^2} \frac{2}{\pi} \operatorname{arc\,tg} \frac{2\sqrt{e}}{e-1} \approx \frac{1}{8m_\pi^2} \cdot 1{,}9 ,$$

while from the expression $F(t)$ it follows

$$F'(0) = \frac{1}{8m_\pi^2} .$$

This example shows, that the formula (28) can not be considerably improved.

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