ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2 - 3950

V.G.Kadyshevsky, R.M.Mir-Kasimov,
N.B.Skachkov

THE SCHROEDINGER DIFFERENCE EQUATION FOR TWO RELATIVISTIC PARTICLES IN THE SIMPLEST CASES

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1968

E2 - 3950

V.G.Kadyshevsky, R.M.Mir-Kasimov,
N.B.Skachkov

THE SCHROEDINGER DIFFERENCE EQUATION FOR TWO RELATIVISTIC PARTICLES IN THE SIMPLEST CASES

Submitted to ЯФ

1. The Continuity Equation

This work is the continuation of a series of works on our research on the relativistic two body problem /1-5/. All notations have the same meaning as in the preceding works unless specifically mentioned.

According to /5/, the radial wave function $R_\ell(r,q)$ of a system of two spinless particles with equal mass m in the case of a local quasipotential $V(r;E_q)$ satisfies the equation x)

$$\frac{H_0^{rad}}{2}(2E_q - H_0^{rad})R_\ell(r,q) = V(r;E_q)R_\ell(r,q), \tag{1.1}$$

where

$$H_0^{rad} = 2\,\mathrm{ch}\, i\frac{d}{dr} + \frac{\ell(\ell+1)}{r(r+i)}e^{i\frac{d}{dr}}. \tag{1.2}$$

At real $V(r;E_q)$ we found from (1.1) the following equation for the conjugate wave function $R_\ell^*(r,q)$:

$$\frac{H_0^{rad\,*}}{2}(2E_q - H_0^{rad\,*})R_\ell^*(r,q) = V(r;E_q)R_\ell^*(r,q), \tag{1.3}$$

---

x) We use the system of units where $\hbar = c = m = 1$.

$$H_0^{rad *} = 2ch\, i \frac{d}{dr} + \frac{\ell(\ell+1)}{r(r-i)} e^{-i\frac{d}{dr}} . \tag{1.4}$$

As is easily seen, the following relation is valid for an arbibrary function $F(r)$:

$$H_0^{rad *}\left(\frac{r^{(\ell+1)}}{(-r)^{(\ell+1)}} (-1)^{\ell+1} F(r)\right) = \frac{r^{(\ell+1)}}{(-r)^{(\ell+1)}} (-1)^{\ell+1} H_0^{rad} F(r) \tag{1.5}$$

(for the definition and properties of $r^{(\ell+1)}$ see /5/). Then putting in (1.3)

$$R_\ell^*(r,q) = \frac{r^{(\ell+1)}}{(-r)^{(\ell+1)}} (-1)^{\ell+1} F_\ell(r,q), \tag{1.6}$$

we obtain for $F_\ell(r,q)$ an equation of the same kind as (1.1)

$$\frac{H_0^{rad}}{2} (2E_q - H_0^{rad}) F_\ell(r,q) = V(r;E_q) F_\ell(r,q). \tag{1.7}$$

Our problem now, is to get out of (1.1) and (1.7) the analog of the radial continuity equation. Let us introduce first the wronskian of the functions $R_\ell(r,q)$ and $F_\ell(r,q)$ (cf. (5.1)) from /5/:

$$W(R_\ell, F_\ell) = \begin{vmatrix} R_\ell(r,q) & F_\ell(r,q) \\ \Delta R_\ell(r,q) & \Delta F_\ell(r,q) \end{vmatrix} =$$

$$= i \begin{vmatrix} R_\ell(r,q) & F_\ell(r,q) \\ e^{-i\frac{d}{dr}} R_\ell(r,q) & e^{-i\frac{d}{dr}} F_\ell(r,q) \end{vmatrix} ; \tag{1.8}$$

$$\left(e^{-i\frac{d}{dr}} \equiv 1 - i\Delta\right).$$

Multiplying then, equation (1.1) by $F_\ell(r,q)$, equation (1.7) by $R_\ell(r,q)$, and substracting the first result from the second, we get

$$[e^{-i\frac{d}{dr}} - 1 - \frac{\ell(\ell+1)}{r(r-i)}]\{[e^{-i\frac{d}{dr}} + 1 + \frac{\ell(\ell+1)}{r(r+i)}].$$

$$\cdot\begin{vmatrix} e^{-i\frac{d}{dr}} R_\ell(r,q) & e^{-i\frac{d}{dr}} F_\ell(r,q) \\ e^{i\frac{d}{dr}} R_\ell(r,q) & e^{i\frac{d}{dr}} F_\ell(r,q) \end{vmatrix} + \frac{2E_q}{i} W(R_\ell, F_\ell)\}=0. \qquad (1.9)$$

Equation (1.9) in our scheme is the desired continuity equation and the expression in curly brackets has to be considered as the analog of the radial current. If we denote the latter by $J_\ell(r,q)$, (1.9) can be rewritten as

$$\Delta J_\ell(r,q) = \frac{\ell(\ell+1)}{r(r-i)} J_\ell(r,q), \qquad (1.10)$$

Equation (1.10) coincides precisely with equation (5.5) from /5/. Then we may conclude that

$$J_\ell(r,q) = 2i \frac{(-r)^{(\ell+1)}}{r^{(\ell+1)}} (-1)^{\ell+1} C_\ell(r,q), \qquad (1.11)$$

where $C_\ell(r,q)$ is some i-periodic function.

Performing now the inverse substitution of the function $F_\ell(r,q)$ by $R^*_\ell(r,q)$ and using the relations

$$e^{-i\frac{d}{dr}} \frac{(-r)^{(\ell+1)}}{r^{(\ell+1)}} = \omega^*_\ell(r) \frac{(-r)^{(\ell+1)}}{r^{(\ell+1)}}, \qquad (1.12)$$

$$e^{i\frac{d}{dr}} \frac{(-r)^{(\ell+1)}}{r^{(\ell+1)}} = \frac{1}{\omega_\ell(r)} \frac{(-r)^{(\ell+1)}}{r^{(\ell+1)}} ,$$

$$\left(\omega_\ell(r) \equiv 1 + \frac{\ell(\ell+1)}{r(r+i)} \right), \tag{1.12}$$

we obtain

$$\left(e^{-i\frac{d}{dr}} + 1\right) \begin{vmatrix} e^{-i\frac{d}{dr}} R_\ell(r,q) & \omega^*_\ell(r)\, e^{-i\frac{d}{dr}} R^*_\ell(r,q) \\ \omega_\ell(r)\, e^{i\frac{d}{dr}} R_\ell(r,q) & e^{i\frac{d}{dr}} R^*_\ell(r,q) \end{vmatrix} + \tag{1.13}$$

$$+\, 2E_q \begin{vmatrix} R_\ell(r,q) & R^*_\ell(r,q) \\ e^{-i\frac{d}{dr}} R_\ell(r,q) & \omega^*_\ell(r)\, e^{-i\frac{d}{dr}} R^*_\ell(r,q) \end{vmatrix} = 2iC_\ell(r,q).$$

The non-relativistic limit of relation (1.13) is

$$\begin{vmatrix} R_\ell(r,q) & R^*_\ell(r,q) \\ \frac{d}{dr} R_\ell(r,q) & \frac{d}{dr} R^*_\ell(r,q) \end{vmatrix} = \text{const.} \tag{1.14}$$

If the functions $R_\ell(r,q)$ and $F_\ell(r,q)$ in (1.9) are two different expressions for the same solution of equation (1.1), then the radial current constructed from these functions is equal to zero:

$$J_\ell(r,q) = \left[e^{-i\frac{d}{dr}} + 1 + \frac{\ell(\ell+1)}{r(r+i)} \right] \times$$

$$\times \begin{vmatrix} e^{-i\frac{d}{dr}} R_\ell(r,q) & e^{-i\frac{d}{dr}} F_\ell(r,q) \\ e^{i\frac{d}{dr}} R_\ell(r,q) & e^{i\frac{d}{dr}} F_\ell(r,q) \end{vmatrix} + \frac{2E_q}{i} W(R_\ell, F_\ell) = 0. \tag{1.15}$$

In so far as in the non-relativistic limit this relation appears in the continuity condition for the "logarithmic derivative"

$$\frac{dR_\ell}{dr} \Big/ R_\ell = \frac{dF_\ell}{dr} \Big/ F_\ell \tag{1.16}$$

used to connect solutions of the Schroedinger equation for different regions, then (1.5) must be used for the same purpose in the relativistic case.

Let us investigate the condition (1.15) in more detail at $\ell = 0$ keeping in mind further applications. If the solutions represented by the functions $R_0(r,q)$ and $F_0(r,q)$ are real, then using (1.15) it is easy to show that

$$e^{-i\frac{d}{dr}} J^*_0(r,q) = - J_0(r,q), \tag{1.17}$$

from which follows the equality

$$J_0(r,q) + J^*_0(r,q) = i\Delta J^*_0(r,q). \tag{1.18}$$

At the same time at $\ell = 0$ because of the continuity equation (1.10), we have

$$\Delta J_0(r,q) = 0.$$

Now on the grounds of (1.17) and (1.18), we conclude that

$$J_0(r,q) + J_0^*(r,q) = 0, \tag{1.19}$$

Finally, the condition (1.15) at $\ell = 0$ can be written in the form

$$E_q \begin{vmatrix} R_0(r,q) & F_0(r,q) \\ \frac{1}{i}\,\mathrm{sh}\, i\frac{d}{dr} R_0(r,q) & \frac{1}{i}\,\mathrm{sh}\, i\frac{d}{dr} F_0(r,q) \end{vmatrix} =$$

$$= \left(1 + \mathrm{ch}\, i\frac{d}{dr}\right) \begin{vmatrix} \mathrm{ch}\, i\frac{d}{dr} R_0(r,q) & \mathrm{ch}\, i\frac{d}{dr} F_0(r,q) \\ \frac{1}{i}\,\mathrm{sh}\, i\frac{d}{dr} R_0(r,q) & \frac{1}{i}\,\mathrm{sh}\, i\frac{d}{dr} F_0(r,q) \end{vmatrix}. \tag{1.20}$$

2. The Phase Shifts, Partial Scattering Amplitudes and Cross Sections

In this section we shall concentrate on formulas and relations using the connection between the asymptotic expression for the relativistic wave function $\Phi_q(r)$ and the scattering amplitude $T(\vec{p},\vec{q})$ on the energy shell $E_p = E_q$ x). To find this connection let us note first that off the energy shell, according to /4,5/

$$\Phi_q(\vec{p}) = (2\pi)^3 \delta^{(3)}(\vec{p}(-)\vec{q}) - \frac{1}{2}\,\frac{T(\vec{p},\vec{q})}{2E_p(2E_q - 2E_p + i\epsilon)}, \tag{2.1}$$

from which using the Shapiro transformation /6/ xx), we obtain

---

x) The definitions of the quantities $\Phi_q(r)$ and $T(\vec{p},\vec{q})$ are given in /4-5/.

xx) Let us remember that the Shapiro transformation with the kernel

$$\xi(\vec{q},\vec{r}) = (E_q - \vec{q}\,\vec{n})^{-1-ir} \qquad (\vec{r} = r\,\vec{n})$$

in our formalism plays the role of the Fourier transformation /4-5/.

$$\Phi_q(p) = \xi(\vec{q},\vec{r}) - \frac{1}{2(2\pi)^3} \int \xi(\vec{p},\vec{r})\, d\Omega_p \frac{T(p,q)}{2E_p(2E_q - 2E_p + i\epsilon)} \qquad (2.2)$$

Expanding (2.2) in partial waves (see (1.5), (1.8) and (1.9) in /5/) we get

$$R_\ell(r,q) = s_\ell(r,\chi_q) - \frac{1}{2(4\pi)^2} \int_0^\infty \frac{d\chi_p\, s_\ell(r,\chi_p) T_\ell(\vec{p},\vec{q})}{\operatorname{ch}\chi_p(\operatorname{ch}\chi_q - \operatorname{ch}\chi_p + i\epsilon)} \qquad (2.3)$$

where, as earlier, $E_p \equiv \operatorname{ch}\chi_p$, $E_q = \operatorname{ch}\chi_q$, and the partial amplitude $T_\ell\,(p,q)$ is defined according to

$$T(\vec{p},\vec{q}) = \frac{1}{2pq} \sum_{\ell=0}^{\infty} (2\ell+1) T_\ell(p,q) P_\ell\left(\frac{\vec{p}\vec{q}}{p\,q}\right). \qquad (2.4)$$

In the scattering problem at large distances, for example in the region

$$r \gg 1/\chi_q \qquad (2.5)$$

the interaction $V(r;E_q)$ has to vanish. Neglecting this term in equation (1.1) and omitting simultaneously the centrifugal term, we arrive at the asymptotic equation

$$\operatorname{ch} i\frac{d}{dr}\left(2\operatorname{ch}\chi_q - 2\operatorname{ch} i\frac{d}{dr}\right) R_\ell^{as}(r,q) = 0. \qquad (2.6)$$

One of the solutions of (2.6) is, as it is easily checked, the expression

$$R_\ell^{as}(r,q) = a_\ell(q) \sin\left(r\chi_q - \frac{\ell\pi}{2} + \delta_\ell\right). \qquad (2.7)$$

The quantities δ_ℓ in analogy with the non-relativistic case, we shall call the phase shifts. Using relation (2.3), it is easy to express the partial amplitudes T_ℓ in terms of the phase shifts δ_ℓ . Actually, in the region (2.5), according to (4.9) of /5/,

$$s_\ell(r, \chi_q) \simeq \sin (r\chi_q - \frac{\ell\pi}{2}).$$

Substituting this expression in (2.3) (and under the integral also)[x)] we obtain

$$R_\ell^{as}(r,q) = \sin(r\chi_q - \frac{\ell\pi}{2}) - \frac{1}{2(4\pi)^2}\int_0^\infty \frac{d\chi_p \sin(r\chi_p - \frac{\ell\pi}{2}) T_\ell(p,q)}{ch\,\chi_p(ch\,\chi_q - ch\,\chi_p + i\epsilon)} \qquad (2.8)$$

Keeping in mind the symbolic equality

$$\frac{1}{2\pi i}\frac{e^{ir(\chi_p-\chi_q)}}{ch\,\chi_p - ch\,\chi_q - i\epsilon} = \begin{cases} \delta(ch\,\chi_p - ch\,\chi_q) & r\to\infty \\ 0 & r\to-\infty \end{cases}$$

we get from (2.8)

$$R_\ell^{as}(r,q) = \sin(r\chi_q - \frac{\ell\pi}{2}) + \frac{1}{\pi}\frac{e^{ir(\chi_q - \frac{\ell\pi}{2})}\,T_\ell(q,q)}{32\,ch\,\chi_q\,sh\,\chi_q}. \qquad (2.9)$$

Comparing (2.9) with (2.7) we can conclude that

$$e^{2i\delta_\ell} = 1 + \frac{i}{8\pi}\frac{T_\ell(q,q)}{sh\,2\chi_q} \equiv s_\ell \qquad (2.10)$$

---

x) It is obvious that here, as in the non-relativistic formalism, we must require the uniform convergence of this integral with respect to r .

$$a_\ell = e^{i\delta_\ell} . \qquad (2.11)$$

Substituting for $T_\ell(q,q)$ from (2.10) into (2.4), we find the expression for the scattering amplitude $T(\vec{p},\vec{q})|_{E_p = E_q}$ in terms of the phase shifts

$$T(\vec{p},\vec{q})|_{E_p = E_q} = 16\pi \,\mathrm{cth}\, \chi_q \sum_{\ell=0}^{\infty} (2\ell+1) \frac{e^{2i\delta_\ell}-1}{2i} P_\ell\left(\frac{\vec{p}\vec{q}}{q^2}\right). \qquad (2.12)$$

In the case of a real quasipotential $V(r;E_q)$ the phase shifts δ_ℓ are real also. Consequently the scattering matrix (2.10) must be unitary. In terms of $T_\ell(q,q)$ the unitarity condition is written in the form

$$\mathrm{Im}\, T_\ell = \frac{1}{16\pi} \frac{|T_\ell|^2}{\mathrm{sh}\, 2\chi_q} . \qquad (2.13)$$

The equality (2.13) is equivalent to the two-particle unitarity condition for the total amplitude $T(\vec{p},\vec{q})|_{E_p = E_q}$ considered by us earlier (see (3.32) /4/). Indeed, putting (2.4) into this expansion, we get the relation (2.13) for the partial amplitudes T_ℓ .

Now let us write down the expression for the total elastic scattering cross-section in terms of the phase-shifts δ_ℓ . Since (see (3.26) in /4/)

$$\frac{d\sigma}{d\omega} = \frac{|T(\vec{p},\vec{q})|^2}{(8\pi)^2 \cdot 4E_q^2} \Big|_{E_p = E_q}$$

upon substituting here $T(\vec{p},\vec{q})$ from (2.12) and performing the integration over solid angle we find easily that

$$\sigma = \frac{4\pi}{\mathrm{sh}^2 \chi_q} \sum_{\ell=0}^{\infty} \frac{|e^{2i\delta_\ell}-1|}{4} (2\ell+1). \qquad (2.14)$$

For real δ_ℓ, formula (2.14) coincides exactly with the corresponding non-relativistic expression

$$\sigma = \frac{4\pi}{q^2} \sum_{\ell=0}^{\infty} \sin^2 \delta_\ell (2\ell+1). \tag{2.15}$$

We conclude with the treatment of the general formalism connected with our approach to the relativistic two-body problem. In the following two sections the solutions of a number of problems with local quasipotentials of simple kinds, will be given with the aim of illustrating the method developed. It is important to remember that according to /4/, the choice of a local quasipotential means that we are taking into consideration, to some degree, the existence of a spectral representation in momemtum transfer for the relativistic scattering amplitude. In other words, we are taking into account the locality and causality conditions in the sense of quantum field theory.

3. The Problem of a Spherical "Potential Box"

Let the quasipotential $V(r;E_q)$ in equation (1.1) have the form

$$\begin{aligned} V(r;E_q) &= 0 \quad \text{for} \quad r \leq a, \\ V(r;E_q) &= \infty \quad \text{for} \quad r > a . \end{aligned} \tag{3.1}$$

The energy levels and wave functions of the particle in this "potential box" must be found. Taking into account the existence of the factor $1/r$ in the expansion (1.5) of /5/ and the character of the quasipotential (3.1) the boundary conditions for the function $R_\ell(r,q)$ must be formulated in the following way

$$\begin{aligned} R_\ell(0,q) &= 0, \\ R_\ell(a,q) &= 0 . \end{aligned} \tag{3.2}$$

In the region $r \leq a$ equation (1.1) coincides with the free equation

$$\frac{H_0^{rad}}{2}(2E_q - H_0^{rad})R_\ell(r,q)=0. \qquad (3.3)$$

The solution of (3.3) which satisfies the first of conditions (3.2) and has the right non-relativistic limit is the function $s_\ell(r,\chi_q)$ defined by formula (1.9) of /5/ x).

$$R_\ell(r,q)=s_\ell(r,q)=\sqrt{\frac{\pi}{2}\,\mathrm{sh}\,\chi_q}\,(-1)^{\ell+1}(-r)^{(\ell+1)}P^{-1/2-\ell}_{-1/2+ir}(\mathrm{ch}\,\chi_q). \qquad (3.4)$$

Obviously

$$s_\ell(a,\chi_q)=0 \qquad (3.5)$$

i.e. the second condition of (3.2) can be used to determine the energy levels in the given field.

Let us consider the simplest case $\ell = 0$. Since according to (3.4)

$$s_0(a,\chi_q)=\sin a\chi_q ,$$

from (3.5) it follows that the following equality must exist

$$a\chi_q=\pi n, \quad n=0,1,2,\ldots \qquad (3.6)$$

Then

$$2E_q=2\,\mathrm{ch}\,\chi_q=2\,\mathrm{ch}\,\frac{\pi n}{a}. \qquad (3.7)$$

In usual units, (3.7) takes the form

---

x) We have neglected in (3.4) the arbitrary i-periodic coefficient because it is not essential now.

$$2E_q = 2mc^2 \operatorname{ch} \frac{\pi n \hbar}{a m c} \qquad (3.8)$$

The transition to the non-relativistic limit leads to the usual quantum-mechanical formula for the energy levels of a particle in a "potential box".

$$W = 2E_q - 2mc^2 = \frac{\pi^2 n^2 \hbar^2}{m a^2} \qquad (3.9)$$

4. The Particle in a "Potential Well"

Let us consider now the problem of determining the discrete energy spectrum of a particle with angular momentum $\ell = 0$, placed in a spherical "potential well" of finite depth

$$\begin{aligned} V(r) &= -V_0 \quad \text{for} \quad r \leq a, \\ V(r) &= 0 \quad \text{for} \quad r > a. \end{aligned} \qquad (4.1)$$

As is well-known, potentials of the kind (4.1) are used in composite models of elementary particles /7-9/. In such an approach m has to be considered as the mass of the initial particles (quarks for example), $2E_q$ as the mass of the composite particle (π-meson for example), and the width of the well a as the size of the composite particle.

In the region $r \leq a$ equation (1.1) for $\ell = 0$ is written as

$$\operatorname{ch} i \frac{d}{dr}\left(2\cos\chi_q - 2\operatorname{ch} i \frac{d}{dr}\right) R_0^I(r,q) = -V_0 R_0^I(r,q) \qquad (4.2)$$

where the quantity $\cos\chi_q$ is connected with the total energy $2E_q$ and the bound energy $-|W|$ by the relation

$$2E_q = 2\cos\chi_q = 2 - |W|. \qquad (4.3)$$

To satisfy the boundary condition

$$R_0^I(0,q) = 0$$

(cf. preceding problem), we look for a solution in the form

$$R_0^{I}(r,q) = A^{I}(r)\sin\alpha r \qquad (4.4)$$

where $A^{I}(r)$ is an arbitrary i-periodic function. Substituting (4.4) into (4.2), it is easy to see that it is a solution of the given equation if

$$\begin{aligned} \mathrm{ch}\,\alpha &= 1/2\cos\chi_q + 1/2\sqrt{\cos^2\chi_q + 2V_0} = \\ &= 1/2 - \frac{|W|}{4} + 1/2\sqrt{\left(1 - \frac{|W|}{2}\right)^2 + 2V_0}\,. \end{aligned} \qquad (4.5)$$

In the region $r > a$ equation (1.1) takes the form

$$\mathrm{ch}\, i\frac{d}{dr}\left(2\cos\chi_q - 2\,\mathrm{ch}\, i\frac{d}{dr}\right)R_0^{II}(r,q) = 0, \qquad (4.6)$$

One of the solutions of (4.6) is proportional to a decreasing exponential $e^{-2\chi_q}$. This solution which we must choose from the physical point of view, is as the radial wave function describing a particle with energy (4.3) in the region $r > a$. So,

$$R_0^{II}(r,q) = A^{II}(r)\, e^{-r\chi_q} \qquad (4.7)$$

where $A^{II}(r)$ is an unknown i-periodic function.

The function $R_0^{I}(r,q)$ and $R_0^{II}(r,q)$ given by formulas (4.4) and (4.7) represent the same solution of equation (1.1). Therefore they must satisfy relation (1.20) playing the role of the condition of continuity of the "logarithmic derivative" of the wave function, as was noted earlier. Putting in (1.20)

$$R_0(r,q) = R_0^{I}(r,q)$$

$$F_0(r,q) = R_0^{II}(r,q)$$

and taking the point $r = a$, we receive after cancelling the "constants" A^{I} and A^{II} the following equation for the definition of the energy levels

$$\operatorname{ctg} \alpha_a = \frac{\sin \chi_q [\operatorname{ch} \alpha (1 + \operatorname{ch} \alpha \cos \chi_q) - \cos \chi_q (1 - \sin \alpha \operatorname{sh} \alpha)]}{\operatorname{sh} \alpha \operatorname{ch} \alpha + \cos \chi_q (\cos \chi_q \operatorname{ch} \alpha + 1)(\sin \alpha - \operatorname{sh} \alpha)}. \qquad (4.8)$$

When solving the usual Schroedinger equation with the potential (4.1) the energy levels are found from the equation /10/

$$\operatorname{ctg} a \sqrt{m (V_o - |W|)} = - \sqrt{\frac{|W|}{V_o - |W|}} \qquad (4.9)$$

It is easy to see that (4.9) is the non-relativistic limit of (4.8).

Let us investigate the formula (4.8) in two important cases. First, let us find the minimum depth of the well V_0^{min} at which the first discrete energy level $E_q = 2 \; (w = 0)$ arises. Putting in (4.8)

$$\cos \chi_q = 1 \quad , \quad \sin \chi_q = 0 \; , \qquad (4.10)$$

$$\operatorname{ch} \alpha = 1/2 + 1/2 \sqrt{1 + 2 V_0^{min}} \equiv \operatorname{ch} \alpha_0 \; , \qquad (4.11)$$

we get

$$\operatorname{ctg} \alpha_o a = 0 \, , \qquad (4.12)$$

or, taking into account (4.11),

$$\operatorname{ch} \frac{\pi}{2a} = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 2 V_0^{min}} \qquad (4.13)$$

From here, for the quantity V_0^{min} we obtain in usual units

$$V_0^{min} = 2 m c^2 \operatorname{ch} \frac{\pi \hbar}{2 a m c} \left(\operatorname{ch} \frac{\pi \hbar}{2 a m c} - 1 \right). \qquad (4.14)$$

The non-relatiмistic limit of (4.14) has the form

$$V_0^{min} = \frac{\pi^2 \hbar^2}{4 a^2 m} , \tag{4.15}$$

Consequently, if the radius of the well a is small, then its minimum depth V_0^{min} at which the particle is still confined to the well in the relativistic theory is much more that in the non-relativistic approach.

Let us consider the case when

$$V_0 = 2 , \tag{4.16}$$

and the energy $2E_q$ is small relative to $m = 1$

$$2E_q = 2 - |W| \equiv \mu \ll 1 . \tag{4.17}$$

If we use the language of the composite models (see the beginning of this section) then condition (4.17) means, for example, that the mass of the π -meson has to be much less than the mass of quark.

Now, neglecting in (4.8) all powers of μ higher than the first, we can represent this formula in the form

$$\operatorname{ctg} a \sqrt{\frac{\mu}{2}} = - \sqrt{\frac{2}{\mu}} , \tag{4.18}$$

from which in usual units there arises the following approximate value for a – the "radius" of a π -meson

$$a \simeq \frac{\pi \sqrt{2} \hbar}{c \sqrt{m \mu}} \tag{4.19}$$

Putting conventionally

$$m = 5\ \mathrm{GeV}/c^2 \simeq 35\,\mu$$

we find from (4.19)

$$a \simeq \frac{\hbar}{\mu c} .$$

Let us estimate the order of magnitude of the average modulus of the momentum of the quark in the well. It follows from the uncertainty principle that

$$p\,a \sim \hbar . \tag{4.20}$$

Substituting here instead of the radius a its value from (4.19), we get

$$\frac{p^2}{m^2 c^2} \sim \frac{\mu}{m} \tag{4.21}$$

Consequently, it is possible to consider the quarks in the well as non-relativistic particles, if the mass of the bound state is small enough in comparison with the quark mass (condition (4.17)). So, the relativistic formulation of the two body problem, developed by us, allows us to approach the foundation of one of the main hypotheses assumed in the quark model, namely the hypotheses about the nonrelativistic character of the motion of the quarks inside the composite particle.

We consider then the problem of the scattering of a slow particle $\left(\frac{|q|}{m} = \operatorname{sh}\chi_q \ll 1\right)$ with angular momentum $\ell = 0$ by the potential well (4.1). The Schroedinger equation in regions $r \le a$ and $r > a$ has the form

$$\operatorname{ch} i\frac{d}{dr}\left(2\operatorname{ch}\chi_q - 2\operatorname{ch} i\frac{d}{dr}\right) R_0^{I}(r,q) = -V_0\, R_0^{I}(r,q) \quad (r \le a) \tag{4.22}$$

$$\operatorname{ch} i\frac{d}{dr}\left(2\operatorname{ch}\chi_q - 2\operatorname{ch} i\frac{d}{dr}\right) R_0^{II}(r,q) = 0 \quad (r > a). \tag{4.23}$$

The solution of (4.22) which disappears at $r = 0$, is the function (cf.(4.4))

$$R_0^{I}(r,q) = A^{I}(r)\sin \alpha r , \tag{4.24}$$

where the quantity a is determined by the equality (cf. (4.5))

$$\operatorname{ch} a = 1/2 \operatorname{ch} \chi_q + 1/2 \sqrt{\operatorname{ch}^2 \chi_q + 2 V_0}. \tag{4.25}$$

One of the solutions of equation (4.23) is expressible in the form

$$R_0^{II}(r, q) = A^{II}(r) \sin(\chi_q r + \delta_0). \tag{4.26}$$

The parameter δ_0 plays here obviously the role of the phase shift (cf. (2.7)). The value of δ_0 can be found from the connection condition (1.20) for solutions $R_0^I(r, q)$ and $R_0^{II}(r, q)$ at the point $r = a$. Simple calculations give

$$\delta_0 = \operatorname{arc\,tg} \left[\frac{\operatorname{sh} \chi_q \operatorname{ch} a + \sin 2\chi_q \operatorname{sh}^2 a}{\operatorname{ch} a \operatorname{sh} a} \operatorname{tg} a a \right] - \chi_q a, \tag{4.27}$$

In so far as we deal with a slow particle, it follows from (4.27) that

$$\delta_0 \simeq \chi_q \left(\frac{\operatorname{ch} a_0 + 2 \operatorname{sh}^2 a_0}{\operatorname{ch} a_0 \operatorname{sh} a_0} \operatorname{tg} a_0 a - a \right), \tag{4.28}$$

where the parameter a_0 is given by formula (4.11). In the same approximation we obtain the expression for the scattering cross section (see (2.15))

$$\sigma_0 = 4\pi a^2 \left(\frac{\operatorname{ch} a_0 + 2 \operatorname{sh}^2 a_0}{a \operatorname{ch} a_0 \operatorname{sh} a_0} \operatorname{tg} a_0 a - 1 \right)^2. \tag{4.29}$$

As is seen from (4.29), the cross section σ_0 becomes infinite at

$$a_0 a = \frac{\pi}{2},$$

which corresponds due to (4.12) to the occurrence of the first discrete level $E = 2$ in the well.

The authors are sincerely grateful to D.I.Blokhintsev, N.N.Bogolubov, Yu.A. Golfand, A.V.Efremov, A.N.Tavkhelidze, I.T.Todorov, R.N.Faustov and M.Freeman for their interest in the work and helpful discussions.

References

1. V.G.Kadyshevsky, ITF Preprint No.7, Kiev, 1967.
2. V.G.Kadyshevsky, Nuclear Physics B6, 125 (1968).
3. V.G.Kadyshevsky, M.D.Mateev, Nuovo Cimento 55A, 275 (1968).
4. V.G.Kadyshevsky, R.M.Mir-Kasimov, N.B.Skachkov, Nuovo Cimento 55A, 233 (1968).
5. V.G.Kadyshevsky. R.M.Mir-Kasimov, N.B.Skachkov, Preprint E2-3949, Dubna (1968).
6. I.S.Shapiro, Dokl. Akad. Nauk SSSR 106, 677 (1956). English transl. Soviet Phys. Doklady, 1, 91 (1956).
7. E.Fermi, C.N.Yang, Phys. Rev., 76, 1739 (1949).
8. N.N.Bogolubov, B.V.Struminsky, A.N.Tavkhelidze, Preprint D-1968, Dubna, 1965.
9. A.N.Tavkhelidze, Lectures on the seminar of High Energy Physics, Trieste, 1965.
10. L.D.Landau and E.M. Lifshits, Quantum Mechanics, Physmatgiz, 1963.

Received by Publishing Department
on June 26, 1968.