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RELATIVISTIC TWO BODY PROBLEM AND FINITE-DIFFERENCE CALCULUS

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I. Introductory Remarks

In the previous work [1] we obtained the relativistic Schrodinger equation for a system of two scalar particles with equal masses

$$(2E_q - H_0)\,\Psi_q(\vec{r}) = \int V(\vec{r};\vec{r}\,';E_q)\,\Psi_q(\vec{r}\,')\,d\vec{r}\,', \qquad (1.1)$$

where $2E_q = 2\sqrt{q^2 + m^2}$ is the total energy of the particles in the c.m. system, $\Psi_q(\vec{r})$ is the wave function of their relative motion, $V(\vec{r};\vec{r};E_q)$ is the "quasipotential", determined generally from field theory x) and the quantity

$$H_0 = 2m\,\mathrm{ch}\left(i\frac{\partial}{\partial rm}\right) + \frac{2i}{r}\,\mathrm{sh}\left(i\frac{\partial}{\partial rm}\right) - \frac{\Delta_{\theta,\phi}}{mr^2}\,e^{i\frac{\partial}{\partial mr}} \qquad (1.2)$$

is ($\Delta_{\theta,\phi}$ is the angular part of the Laplacian operator) the relativistic analog of the free Hamiltonian. It was shown that the eigenfunctions of the operator H_0 are the "plane waves" in a Lobachevsky momentum space

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x) We use here the terminology of Logunov and Tavkhelidze [2], because in the p-representation the equation (1.1) is analogous to the quasipotential equation suggested in [2]. For details of connection between the quasipotential approach of Logunov and Tavkhelidze and the formalism of equation (1.1) see [3-5].

$$H_0\,\xi(\vec q;\vec r) = 2E_q\,\xi(\vec q;\vec r),$$

$$\xi(\vec q;\vec r) = \left(\frac{E_q - \vec q\,\vec n}{m}\right)^{-1-irm}; \quad \vec r = r\vec n, \quad \vec n^2 = 1, \tag{1.3}$$

In analogy with the case of usual plane waves $e^{i\vec q\vec r}$, which are the kernels of a Fourier transformation mapping the non-relativistic momentum space onto the non-relativistic coordinate space, the functions $\xi(\vec q;\vec r)$ are the kernels of the Shapiro integral transformation, mapping the Lobachevsky momentum space (the upper sheet of the hyperboloid $q_0^2 - \vec q^{\,2} = m^2$ onto some three-dimensional relativistic $\vec r$ space). From the group-theoretical point of view, the Shapiro transformation is the expansion in matrix elements of the principal series of unitary representations of the Lorentz group. The radius-vector $\vec r$ in (1.2) and (1.3) is connected in a simple way with the value of the Casimir operator of the group in these representations.

In /1/ it was established that with equation (1.1) we can consider also the equation with the local quasipotential $V(\vec r;E_q)$ x)

$$\frac{H_0}{2m}(2E_q - H_0)\,\Phi_q(\vec r) = V(\vec r;E_q)\,\Phi_q(\vec r) \tag{1.4}$$

x) We have slightly changed the normalization of the wave function Φ and the quasipotential $V(\vec r;E_q)$ in contrast with /1/, to simplify the investigation of the non-relativistic limit.

if we assume the existence of the spectral representation of the scattering amplitude in terms of the momentum transfer /7/.

Expanding $\Phi_q(\vec r)$ in Legendre polynomials

$$\Phi_q(\vec r) = \sum_{\ell=0}^{\infty} i^\ell \frac{R_\ell(r,q)}{qr}(2\ell+1)P_\ell\left(\frac{\vec q\,\vec r}{qr}\right) \tag{1.5}$$

it is easy to obtain from (1.4) the one-dimensional equation for the radial wave function $R_\ell(r,q)$: x)

$$\frac{H_0^{rad}}{2}(2E_q - H_0^{rad})R_\ell(r,q) = V(r;E_q)R_\ell(r,q), \tag{1.6}$$

where

$$H_0^{rad} = 2\,\mathrm{ch}\,i\frac{d}{dr} + \frac{\ell(\ell+1)}{r(r+i)}e^{i\frac{d}{dr}}. \tag{1.7}$$

Due to (1.3) and the relations

$$\xi(\vec q;\vec r) = \sum i^\ell \frac{s_\ell(r;\chi_q)}{qr}(2\ell+1)P_\ell\left(\frac{\vec q\,\vec r}{qr}\right), \tag{1.8}$$

$$s_\ell(r,\chi_q) = \sqrt{\frac{\pi}{2}\,\mathrm{sh}\,\chi_q}\,(-1)^{\ell+1}(-r)^{(\ell+1)}\,P^{-\frac12-\ell}_{-\frac12+ir}(\mathrm{ch}\,\chi_q), \tag{1.9}$$

$$\left(E_q = \mathrm{ch}\,\chi_q;\ (-r)^{(\ell+1)} \equiv i^{\ell+1}\frac{\Gamma(ir+\ell+1)}{\Gamma(ir)}\right)$$

x) We use the unit system, in which $\hbar = c = m = 1$. The transition to the non-relativistic case means that $r \gg 1$, $q \ll 1$.

one of the eigenfunctions of the operator (1.7) is the function $s_\ell(r, \chi_q)$

$$H_0^{rad}\, s_\ell(r,\chi_q) = 2E_q\, s_\ell(r,\chi_q) \tag{1.10}$$

2. The Operation of Finite-Difference "Differentiation" and the Concept of Generalized Degree

It is obvious from (1.7) that the operator H_0^{rad} is a combination of finite shift operators $e^{i\frac{d}{dr}}$ and $e^{-i\frac{d}{dr}}$. It is clear then, that it will be reasonable to investigate the properties of equation (1.6) using the methods of the finite-difference calculus.

Let us define the finite-difference "differentiation" operation Δ putting (cf. /8/)

$$e^{-i\frac{d}{dr}} = 1 - i\Delta . \tag{2.1}$$

Acting with Δ on some function $f(r)$ because of (2.1), we get

$$\Delta f(r) = \frac{f(r-i) - f(r)}{-i} . \tag{2.2}$$

It is obvious, that in the non-relativistic limit x)

$$\Delta f(r) \to \frac{d}{dr} f(r) \tag{2.3}$$

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x) In usual units (2.2) has the form

$$\Delta f(r) = \frac{f\left(r - \frac{i\hbar}{mc}\right) - f(r)}{-\frac{i\hbar}{mc}}$$

It is clear also, that we can rewrite equation (1.6) in terms of Δ.

It is easy to verify that the Δ-derivative of the product of two functions $f(r)$ and $\phi(r)$ is given by the expression

$$\Delta[f(r)\phi(r)] = [\Delta f(r)]\phi(r) + f(r)[\Delta\phi(r)] + \frac{1}{i}[\Delta f(r)][\Delta\phi(r)]. \tag{2.4}$$

It would be convenient for effective exploitation of the Δ-operation, to have some function whose behaviour under Δ-differentiation is similar to that of the usual degree function. It turns out that such generalized degree is the function

$$r^{(\lambda)} \equiv i^{\lambda} \frac{\Gamma(-ir+\lambda)}{\Gamma(-ir)} \tag{2.5}$$

(cf. 1.9).

Indeed, as it is easy to check,

$$\Delta r^{(\lambda)} = \lambda r^{(\lambda-1)} . \tag{2.6}$$

Furthermore, it is obvious that

$$r^{(0)} = 1 ,$$

If $\lambda = n$, where n is a positive integer, then

$$r^{(n)} = r(r+i)\ldots(r+(n-1)i), \tag{2.7}$$

and

$$r^{(-n)} = \frac{1}{(r-i)(r-2i)\ldots(r-ni)} \tag{2.8}$$

Passing to the nonrelativistic theory,

$$r^{(\lambda)} \to r^{\lambda} .$$

Let us note, that our definition of the generalized degree is more universal, than the corresponding definition in /8/.

In contrast with usual analysis the product of two generalized degrees is not again a generalized degree, but it can be shown that for arbitrary λ and μ

$$r^{(\lambda)} r^{(\mu)} = \frac{r^{(\lambda+\mu)}}{F(-\lambda, -\mu^{*}, -ir, 1)} \tag{2.9}$$

where F is the hypergeometric function.

Let us suppose now, that we know the Taylor expansion of some function $f(r+a)$

$$f(r+a) = \sum_{n=0}^{\infty} \frac{a^n}{n!} \frac{d^n f(r)}{dr^n} . \tag{2.10}$$

Let us construct the analogous expansion in terms of the Δ - operation /8/. It is clear, first of all, that

$$f(r+a) = e^{a\frac{d}{dr}} f(r) = (e^{-i\frac{d}{dr}})^{ia} f(r) =$$

$$= (1-i\Delta)^{ia} f(r) . \tag{2.11}$$

Expanding the binomial $(1-i\Delta)$ in a series in Δ, we get

$$(1-i\Delta)^{ia} = 1 + ia(-i\Delta) + \frac{ia(ia-1)}{2!}(-i\Delta)^2 + \dots$$

From here and from (2.11), keeping in mind (2.7), we finally obtain

$$f(r+a) = \sum_{n=0}^{\infty} \frac{a^{(n)}}{n!} \Delta^n f(r) . \tag{2.12}$$

A comparison of expansions (2.10) and (2.12) shows that these series are analogous in their structure.

3. The Analogs of Some Functions

Using the generalization of the degree function, we can introduce with the help of the corresponding series, the analogs of many functions, which are used in continuous analysis.

Let us define the generalized exponential function by the following expression

$$\exp[a;r] = \sum_{n=0}^{\infty} \frac{a^n r^{(n)}}{n!} . \tag{3.1}$$

It is obvious, because of (2.6) that

$$\Delta \exp[a;r] = a \exp[a;r] , \tag{3.2}$$

Comparing (3.1) with the hypergeometric function expansion, we conclude that

$$\exp[a;r] = F(-ir, 1; 1; ia) \tag{3.3}$$

which is equivalent to the equality

$$\exp[a;r] = (1-ia)^{ir} , \tag{3.4}$$

Keeping in mind (3.4) and (2.11), we can write the function $f(r+a)$ as follows

$$f(r+a) = \exp[\Delta; a] f(r) \tag{3.5}$$

Consequently, the operator Δ up to a factor i is the "generator" of translations along the r-axis in the representation defined by the exponentials $\exp[a;r]$.

Let us note that the plane waves $\xi(\vec{q};\vec{r})$, considered in §1, are also expressible in terms of generalized exponentials

$$\xi(\vec{q};\vec{r}) = (q_0 - \vec{q}\vec{n})^{-1-ir} =$$

$$= (q_0 - \vec{q}\vec{n})^{-1} \exp[i(q_0 - 1 - \vec{q}\vec{n}); -r], \qquad (3.6)$$

$$(\vec{r} = r\vec{n}).$$

Let us now define the generalized hyperbolic functions $sh[a,r]$ and $ch[a,r]$:

$$sh[a;r] = \frac{1}{2}\{\exp[a;r] - \exp[-a;r]\} = \sum_{n=0}^{\infty} \frac{a^{2n+1} r^{(2n+1)}}{(2n+1)!}$$

$$ch[a;r] = \frac{1}{2}\{\exp[a;r] + \exp[-a;r]\} = \sum_{n=0}^{\infty} \frac{a^{2n} r^{(2n)}}{(2n)!} \qquad (3.7)$$

and the generalized trigonometric functions $\sin[a;r]$ and $\cos[a;r]$

$$\sin[a;r] = \frac{1}{2i}\{\exp[ia;r] - \exp[-ia,r]\} =$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{a^{2n+1} r^{(2n+1)}}{(2n+1)!}; \qquad (3.8)$$

$$\cos[a;r] = \frac{1}{2}\{\exp[ia,r] + \exp[-ia;r]\}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{a^{2n} r^{(2n)}}{(2n)!}.$$

It is clear, that

$$\Delta\, sh[a;r] = a\, ch[a;r];$$

$$\Delta\, ch[a;r] = a\, sh[a;r]; \qquad (3.9)$$

$$\Delta \sin[a;r] = a\cos[a;r];$$

$$\Delta \cos[a;r] = -a\sin[a;r];$$

Let us consider now the so-called logarithmic derivative of the Γ function

$$\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}. \qquad (3.10)$$

After using the well-known functional relation

$$\Psi(z+1) - \Psi(z) = \frac{1}{z} \qquad (3.11)$$

and taking into account (2.2) and (2.8) we obtain

$$\Delta\Psi(ir+1) = \frac{1}{r-i} = r^{(-1)}. \qquad (3.13)$$

Consequently, the function $\Psi(ir+1)$ plays in the calculus under consideration the same role as $\ln r$ in continuous analysis. This conclusion is justified also by the similarity of the expansions of these functions in Taylor series (2.12) and (2.10) respectively

$$\Psi(i(r+a)+1) = \Psi(ir+1) + \frac{a}{r-i} - \frac{a(a+i)}{2(r-i)(r-2i)} +$$

$$+ \frac{1}{3}\,\frac{a(a+i)(a+2i)}{(r-i)(r-2i)(r-3i)} + \ldots = \Psi(ir+1) + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{r^{(-n)}a^{(n)}}{n},$$

$$\ln(r+a) = \ln r + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{r^{-n}a^{n}}{n}.$$

Let us now find the generalization of the step function $\theta(r)$:

$$\theta(r) = \begin{cases} 1 & r > 0 \\ 0 & r < 0. \end{cases} \tag{3.13}$$

The main property of $\theta(r)$ is expressed, as is well-known by the equality

$$\frac{d\theta(r)}{dr} = \delta(r), \tag{3.14}$$

which is most simply proved, using the integral representation

$$\theta(r) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{i\kappa r}}{\kappa - i\epsilon}\,d\kappa \tag{3.15}$$

Let us define taking into account (3.14) the generalized θ function $\hat{\theta}(r)$ to be that function for which the relation

$$\Delta\hat{\theta}(r) = \delta(r) \tag{3.16}$$

is valid.

We shall have, in complete analogy with (3.15)

$$\hat{\theta}(r) = \frac{1}{2\pi i}\int\frac{e^{i\kappa r}\,d\kappa}{e^{\kappa} - 1 - i\epsilon} \tag{3.17}$$

from which, after integration

$$\hat{\theta}(r) = \lim_{\epsilon\to 0}\frac{(1+i\epsilon)^{ir-1}}{1-e^{-2\pi r}} = \frac{1}{1-e^{-2\pi r}}, \tag{3.18}$$

So, $\hat{\theta}(r)$ for $r \neq 0$ is a function with period i and has to be considered as constant with respect to Δ differentiation. At the point $r = 0$ the expression (3.18) has a pole, which leads finally to equality (3.16).

It is obvious, that in the nonrelativistic limit $|r| \gg 1$ the function (3.18) goes into (3.13). Let us note, that the identity $\theta(r) + \theta(-r) = 1$ for the function (3.13) is valid also for $\hat{\theta}(r)$

$$\hat{\theta}(r) + \hat{\theta}(-r) = 1. \tag{3.19}$$

The δ-function under Δ-differentiation has to be considered as a function of a complex variable, because the Δ-operation is accomplished by transition into the complex r-plane. Let us introduce the following representation for $\delta(r)$

$$\delta(r) = \lim_{\mu\to 0}\frac{1}{2\pi i}\left(\frac{1}{r-i\mu} - \frac{1}{r+i\mu}\right), \tag{3.20}$$

with (2.2) and (3.20) we shall have

$$\Delta\delta(r) = \frac{1}{2\pi i}\lim_{\mu\to 0} i\left[\frac{1}{r-i-i\mu} - \frac{1}{r-i+i\mu} - \frac{1}{r-i\mu} + \frac{1}{r+i\mu}\right] = \tag{3.21}$$

$$= -\frac{1}{i}\delta(r) + \frac{i}{\pi}\lim_{\mu\to 0}\frac{\mu}{(r-i)^2+\mu^2}.$$

Upon integration along the real axis in r-space, the second term in (3.21) can be considered as a continuous function of μ and consequently (3.21) is in fact equivalent to the relation

$$\Delta\delta(r) = -i\,\delta(r), \tag{3.22}$$

The same equality arises, when the identity $r\,\delta(r) = 0$ is Δ-differentiated formally according to the rule (2.4).

Let us pay attention to the fact that the relation (3.20), because of (3.11) and (2.2) can be written in the form

$$\delta(r) = \frac{1}{2\pi i}\lim_{\mu\to 0}\Delta[\Psi(ir+\mu) - \Psi(ir-\mu)], \tag{3.23}$$

We obtain from here, taking into account (3.16), the following expression for "smear" $\hat{\theta}$ -function

$$\hat{\theta}_\mu(r) = \frac{1}{2\pi i}[\Psi(ir+\mu) - \Psi(ir-\mu)] + \phi(r) \tag{3.24}$$

where $\phi(r)$ is some i periodic function, which we have to determine. We find easily, that $\phi(r) = -\frac{\mathrm{cth}\,\pi r}{2}$ by comparing (3.24) at $\mu = 0$ with (3.17) and (3.18).

4. The Solutions of the Free Schroedinger Difference Equation

It follows from (1.6) that when the interaction is switched off, the radial wave function $R^0_\ell(r, q)$ satisfies the equation

$$\frac{H_0^{rad}}{2}(2E_q - H_0^{rad})R^0_\ell(r,q) = 0. \tag{4.1}$$

As can easily be seen, equation (4.1) is fourth order with respect to the Δ-operator. According to the general theory of difference equations /8/, it follows from here that it must have four independent solutions.

However, it is necessary to know only the solutions of the second order equation

$$(2E_q - H_0)R^0_\ell(r,q) = 0 \tag{4.2}$$

because the other solutions of (4.1) can be obtained from them by going to the limit $E_q \to 0$.

According to §1, one of the solutions of (4.2) is the function $s_\ell(r,\chi)$ given by relation (1.9). Since this equation does not change with the substitution

$$\ell \to -\ell - 1, \tag{4.3}$$

another solution of it will be the function

$$c_\ell(r, \chi_q) = (-1)^\ell\, s_{-\ell-1}(r,\chi_q) \tag{4.4}$$

or

$$c_\ell(r,\chi_q) = \sqrt{\frac{\pi}{2}\,\mathrm{sh}\,\chi_q}\;(-r)^{(-\ell)}\;P^{\frac{1}{2}+\ell}_{-\frac{1}{2}+ir}(\mathrm{ch}\,\chi_q). \tag{4.5}$$

In the non-relativistic limit, as is easily verified

$$s_\ell(r,\chi_q) \to \sqrt{\frac{\pi r q}{2}}\,J_{\ell+\frac{1}{2}}(r,q);$$

$$c_\ell(r,\chi_q) \to -\sqrt{\frac{\pi r q}{2}}\,N_{\ell+\frac{1}{2}}(r,q); \tag{4.6}$$

Like the spherical Bessel functions, s_ℓ and c_ℓ can be expressed in terms of elementary functions. For example for s_ℓ the following recurrence formulas are valid (cf./1/)

$$s_\ell(r,\chi_q) = (-1)^\ell \frac{(\mathrm{sh}\,\chi_q)^{\ell+1}}{r^{(\ell+1)}} \cdot r \left(\frac{d}{d\,\mathrm{ch}\,\chi_q}\right)^\ell \frac{\sin r\chi_q}{\mathrm{sh}\,\chi_q}, \tag{4.7}$$

$$s_\ell(r,\chi_q) = -\frac{(-r)^{(\ell+1)}}{(\mathrm{sh}\,\chi_q)^\ell}\left(\frac{1}{ir}\,\mathrm{sh}\, i\frac{d}{dr}\right)^\ell \frac{\sin r\chi_q}{r}. \tag{4.8}$$

It is easy to find the asymptotic behaviour of the function $s_\ell(r,\chi_q)$ for large $r\chi_q$ with the help of (4.7)

$$s_\ell(r,\chi_q) \to \sin\left(r\chi_q - \frac{\ell\pi}{2}\right), \quad r\chi_q \gg 1. \tag{4.9}$$

We get from here, using the relation (4.4)

$$c_\ell(r,\chi_q) \to \cos\left(r\chi_q - \frac{\ell\pi}{2}\right), \quad r\chi_q \gg 1. \tag{4.10}$$

So, the solutions $s_\ell(r,\chi_q)$ and $c_\ell(r,\chi_q)$ of equation (4.2) correspond to standing waves.

Let us introduce now the functions (comp. /9/)

$$e_\ell^{(1)}(r,\chi_q) = c_\ell(r,\chi_q) + i\, s_\ell(r,\chi_q), \tag{4.11}$$
$$e_\ell^{(2)}(r,\chi_q) = c_\ell(r,\chi_q) - i\, s_\ell(r,\chi_q).$$

It follows from (4.9) and (4.10) that

$$e_\ell^{(1,2)}(r,\chi_q) \to e^{\pm i\left(r\chi_q - \frac{\ell\pi}{2}\right)}, \quad r\chi_q \gg 1. \tag{4.12}$$

It is easy to establish, taking into account (1.9) and (4.2) and some relations from the theory of spherical functions /9/, that at arbitrary $r\chi_q$

$$e_\ell^{(1,2)}(r,\chi_q) = \frac{1}{i}\sqrt{\frac{2\,\mathrm{sh}\,\chi_q}{\pi}}\,(-r)^{(-\ell)}\, Q^{\frac{1}{2}+\ell}_{-\frac{1}{2}\mp ir}(\mathrm{ch}\,\chi_q). \tag{4.13}$$

It is clear that the choice of the solutions of equation (4.2) in the form (4.13) corresponds to the consideration of outgoing and incoming spherical waves. In the non-relativistic limit from (4.11) and (4.6) or from (4.13) we get immediately

$$e_\ell^{(1,2)}(r,\chi_q) \to \pm i\sqrt{\frac{\pi r q}{2}}\, H^{(1,2)}_{\ell+\frac{1}{2}}(rq). \tag{4.14}$$

As is well known, the cylindrical functions $J_{\ell+\frac{1}{2}}$, $N_{\ell+\frac{1}{2}}$ and $H^{(1,2)}_{\ell+\frac{1}{2}}$ are entire analytic functions in the plane of angular momentum ℓ. It is interesting to know the analytic properties in ℓ of the relativistic solutions s_ℓ, c_ℓ and $e_\ell^{(1,2)}$. This problem will be very easy if the s_ℓ, c_ℓ and $e_\ell^{(1,2)}$ are expressed in terms of hypergeometric functions.

Let us remember first, that $\frac{F(\alpha,\beta;\gamma;z)}{\Gamma(\gamma)}$ is an entire analytic function of the parameters α, β and γ for $|z|<1$. Taking into account this fact, and the relation

$$P_\ell^\mu(\mathrm{ch}\,\chi) = \left(\frac{\mathrm{ch}\,\chi-1}{\mathrm{ch}\,\chi+1}\right)^{-\frac{\mu}{2}}\left(\frac{\mathrm{ch}\,\chi+1}{2}\right)^{-\nu}\frac{F\left(-\nu,-\nu-\mu;1-\mu;\frac{\mathrm{ch}\,\chi-1}{\mathrm{ch}\,\chi+1}\right)}{\Gamma(1-\mu)} \tag{4.15}$$

we conclude, that the spherical function $P^{-\frac{1}{2}-\ell}_{-\frac{1}{2}+ir}(\mathrm{ch}\,\chi_q)$ at $|\mathrm{ch}\,\chi|>1$ is an entire function of ℓ. Consequently $s_\ell(r,\chi)$ (see (1.9)) is a meromorphic function of ℓ, because so is the generalized degree $(-r)^{(\ell+1)} = i^{\ell+1}\frac{\Gamma(ir+\ell+1)}{\Gamma(ir)}$.

The position of the poles is given by the equality

$$\ell = -1 - n - ir \tag{4.16}$$

where $n = 0, 1, 2, \ldots$. Then, in so far as we consider only $r \geq 0$ the poles of (4.16) occur in the third quadrant of the ℓ-plane.

Reasoning as earlier, it is easy to establish that the function $c_\ell(r, \chi)$ is also meromorphic in the ℓ-plane, its poles located in the first quadrant

$$\ell = n + ir, \qquad n = 0, 1, 2, \ldots \tag{4.17}$$

The fact that the functions $e_\ell^{(1,2)}$ in the ℓ-plane are meromorphic is obvious from formulas (4.11).

To conclude this section we write down explicitly those solutions of the initial equation, which are simultaneously solutions of the equation

$$H_0^{rad} R_\ell^{(0)}(r) = 0. \tag{4.18}$$

As in the preceeding case, it is convenient to group these solutions in pairs

$$s_\ell(r) = \frac{\Gamma\left(\frac{ir+\ell+1}{2}\right)\Gamma\left(-\frac{ir+2}{2}\right)\operatorname{sh}\frac{\pi r}{2}}{\Gamma\left(\frac{ir+1}{2}\right)\Gamma\left(-\frac{ir+\ell+2}{2}\right)} \equiv a_\ell(r)\operatorname{sh}\frac{\pi r}{2}$$

$$c_\ell = a_\ell(r)\operatorname{ch}\frac{\pi r}{2} \tag{4.19}$$

$$e_\ell^{(1)} = a_\ell(r)\, e^{\frac{\pi r}{2}}$$

$$e_\ell^{(2)} = a_\ell(r)\, e^{-\frac{\pi r}{2}} \tag{4.20}$$

5. The Wronskians and the Green-Functions

According to the general theory of finite-difference equations/8/, the arbitrary solutions y_1 and y_2 of a difference equation are linearly independent if the determinant

$$\begin{vmatrix} y_1(r) & y_2(r) \\ \Delta y_1(r) & \Delta y_2(r) \end{vmatrix} \equiv W(y_1; y_2) \tag{5.1}$$

differs from zero.

In the nonrelativistic limit it is obvious that

$$W(y_1; y_2) \to \begin{vmatrix} y_1(r) & y_2(r) \\ \frac{d}{dr} y_1(r) & \frac{d}{dr} y_2(r) \end{vmatrix}.$$

Then (5.1) has to be considered as the analog of the wronskian in finite-difference analysis.

On the grounds of (4.11) we have

$$W(s_\ell, c_\ell) = \frac{1}{2i} W(e_\ell^{(1)}; e_\ell^{(2)}), \tag{5.2}$$

$$W(s_\ell, c_\ell) = W(s_\ell, e_\ell^{(1)}) = W(s_\ell, e_\ell^{(2)})$$

As in the continuous case, the wronskian (5.2) can be calculated precisely. To do this, let us find first of all the equation for W. Considering for the sake of definiteness $W = W(e_\ell^{(1)}, e_\ell^{(2)})$ we shall get from (4.2)

$$\left[1+\frac{\ell(\ell+1)}{r(r+i)}\right]\{e_\ell^{(2)}(r-i,\chi)e_\ell^{(1)}(r+i,\chi)-e_\ell^{(2)}(r+i,\chi)e_\ell^{(1)}(r-i,\chi)\}=$$

$$=\frac{2E}{i}W(e_\ell^{(1)};e_\ell^{(2)}). \tag{5.3}$$

On the other hand, using again (4.2) and performing some simple transformations of the argument, we obtain

$$\frac{2E}{i}W(e_\ell^{(1)};e_\ell^{(2)})=e_\ell^{(1)}(r,\chi)e_\ell^{(2)}(r-2i,\chi)-e_\ell^{(1)}(r-2i,\chi)e_\ell^{(2)}(r,\chi)=$$

$$=e^{-i\frac{d}{dr}}\{e_\ell^{(2)}(r-i,\chi)e_\ell^{(1)}(r+i,\chi)-e_\ell^{(2)}(r+i,\chi)e_\ell^{(1)}(r-i,\chi)\}. \tag{5.4}$$

Now, from (5.3) and (5.4) we can conclude, that

$$e^{-i\frac{d}{dr}}W(e_\ell^{(1)},e_\ell^{(2)})=\left(1+\frac{\ell(\ell+1)}{r(r-i)}\right)W(e_\ell^{(1)},e_\ell^{(2)}),$$

or finally

$$\Delta W(e_\ell^{(1)},e_\ell^{(2)})=i\frac{\ell(\ell+1)}{r(r-i)}W(e_\ell^{(1)};e_\ell^{(2)}). \tag{5.5}$$

The solution of equation (5.5) as is easily verified has the form

$$W(e_\ell^{(1)},e_\ell^{(2)})=\frac{(-r)^{(\ell+1)}}{(r)^{(\ell+1)}}A_\ell(r,\chi), \tag{5.6}$$

where $A_\ell(r,\chi)$ is some i-periodic function, which has to be determined[x)].

Let us make the substitution in (5.6)

$$r=\rho-in \qquad (n\text{ -integer})$$

and let n tend to infinity. Then obviously we shall have

$$A_\ell(\rho,\chi)=\lim_{n\to\infty}\left\{\frac{(\rho-in)^{(\ell+1)}}{(-\rho+in)^{(\ell+1)}}W(e_\ell^{(1)},e_\ell^{(2)})\right\}\Big|_{r=\rho-in} \tag{5.7}$$

Using the relation [10]

$$e^{-i\mu\pi}Q_\nu^\mu(z)=\sqrt{\frac{\pi}{2}}(z^2-1)^{-1/4}\left[z-\sqrt{z^2-1}\right]^{\nu+\frac{1}{2}}\cdot$$

$$\cdot\frac{\Gamma(1+\nu+\mu)}{\Gamma(\nu+3/2)}F\left(\frac{1}{2}+\mu,\frac{1}{2}-\mu;\nu+\frac{3}{2};\frac{-z+\sqrt{z^2-1}}{2\sqrt{z^2-1}}\right),$$

it is easy to show, that

$$e_\ell^{(1,2)}(\rho-in,\chi)\underset{n\to\infty}{\to}e^{\pm i(\chi(\rho-in)-\frac{\ell\pi}{2})} \tag{5.8}$$

(comp. (4.12)). Substituting (5.8) in (5.7) we find

$$A_\ell(\rho,\chi)=2i(-1)^\ell\,\mathrm{sh}\,\chi.$$

x) This function plays the role of the arbitrary constant, arising upon integration of difference equation (5.5)

Equation (5.6) now takes the form

$$W(e_\ell^{(1)}, e_\ell^{(2)}) = \frac{2i(-r)^{(\ell+1)}}{r^{(\ell+1)}} (-1)^\ell \,\mathrm{sh}\,\chi . \qquad (5.9)$$

It is interesting to note the relation

$$\frac{W(e_\ell^{(1)}, e_\ell^{(2)})}{\begin{vmatrix} e_\ell^{(1)}(r,\chi) & e_\ell^{(2)}(r,\chi) \\ \frac{d}{d\chi} e_\ell^{(1)}(r,\chi) & \frac{d}{d\chi} e_\ell^{(2)}(r,\chi) \end{vmatrix}} = \frac{W(e_\ell^{(1)}; e_\ell^{(2)})}{W_\chi(e_\ell^{(1)}; e_\ell^{(2)})} = \frac{\mathrm{sh}\,\chi}{r} . \qquad (5.10)$$

If we take into account the fact that $\mathrm{sh}\,\chi$ is the modulus of the three dimensional momentum, then the similarity of (5.10) with the analogous relation between wronskians in the non-relativistic theory becames quite obvious.

In some calculations it is useful to keep in mind the equality

$$W(e_\ell^{(1)}, e_\ell^{(2)}) = \frac{ir - \ell - 1}{ir} \,\mathrm{sh}\,\chi \begin{vmatrix} e_\ell^{(1)}(r,\chi) & e_\ell^{(2)}(r,\chi) \\ e_{\ell+1}^{(1)}(r,\chi) & e_{\ell+1}^{(2)}(r,\chi) \end{vmatrix} , \qquad (5.11)$$

the analog of which in the non-relativistic case also exists.

Our next problem is to build the Greens functions of equation (4.2) from its free solutions. It is clear that $G_\ell(r,r';\chi)$ satisfies the equation (comp. (3.4)) from /1/

$$\left[2\,\mathrm{ch}\,\chi_q - 2\,\mathrm{ch}\, i\frac{d}{dr} - \frac{\ell(\ell+1)}{r^{(2)}} e^{i\frac{d}{dr}} \right] G_\ell(r,r';\chi_q) = \delta(r-r') . \qquad (5.12)$$

Then we can reason as in the non-relativistic formalism. Namely, due to (5.12) at $r < r'$ and $r > r'$ the Green-function satisfies the free equation and consequently must be a linear combination of functions s_ℓ, c_ℓ, $e_\ell^{(1)}$ and $e_\ell^{(2)}$ with i-periodic coefficients. At the point $r = r'$ these coefficients must have a singularity such that after "differentiation" with respect to r we get $\delta(r-r')$.

One can see, taking into account the correspondence principle and the definition of the θ function (see (3.16)-(3.18)) that it is in fact this function which must enter the expression for the coefficients.

It is easy to find the explicit expression for $G_\ell(r,r';\chi_q)$ from the integral representation (cf. (3.44) from /1/)

$$G_\ell(r,r';\chi_q) = -\frac{2}{\pi} \int_0^\infty \frac{d\chi \; s_\ell(r,\chi)\, s_\ell^*(r',\chi)}{2\,\mathrm{ch}\,\chi_q - 2\,\mathrm{ch}\,\chi + i\epsilon} . \qquad (5.13)$$

Let us note first of all, that

$$s_\ell^*(r,\chi) = (-1)^{\ell+1} \frac{r^{(\ell+1)}}{(-r)^{(\ell+1)}} s_\ell(r,\chi) . \qquad (5.14)$$

Then, instead of (5.13) we shall have

$$G_\ell(r,r';\chi_q) = \frac{2}{\pi}(-1)^{\ell+1} \frac{(r')^{(\ell+1)}}{(-r')^{(\ell+1)}} \int_0^\infty \frac{d\chi \; s_\ell(r,\chi)\, s_\ell(r',\chi)}{2\,\mathrm{ch}\,\chi_q - 2\,\mathrm{ch}\,\chi + i\epsilon} . \qquad (5.15)$$

The integral (5.15) can be obtained with the help of the theory of residues. We write down here the result

$$G_\ell(r,r';\chi_q) = -\frac{1}{W(e_\ell^{(1)}(r',\chi_q), e_\ell^{(2)}(r',\chi_q))} \times$$

$$\times\{\hat{\theta}(r-r')e_\ell^{(1)}(r,\chi_q)e_\ell^{(2)}(r',\chi_q)+\hat{\theta}(r'-r)e_\ell^{(1)}(r',\chi_q)e_\ell^{(2)}(r,\chi_q)-$$

$$-\hat{\theta}(r+r')e_\ell^{(1)}(r,\chi_q)e_\ell^{(1)}(r',\chi_q)-\hat{\theta}(-r-r')e_\ell^{(2)}(r,\chi_q)e_\ell^{(2)}(r',\chi_q)\}$$

The only difference between (5.16) and the corresponding expression of the usual theory /9/ is that in our Green-function there occurs an "acausal" term, proportional $\hat{\theta}(-r-r')$. In the non-relativistic limit this term disappears, obviously.

There is now, reason to consider the Green-function of the initial equation (4.1) separately, because it is expressible in terms of $G_\ell(r,r';\chi_q)$ and $G_\ell(r,r';0)$.

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