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Разностные гипергеометрические уравнения и релятивистская кулонова проблема.

Введены конечно-разностные аналоги гипергеометрических функций и найдены уравнения, которым они удовлетворяют. Полученные результаты использованы для нахождения точного решения кулоновой проблемы в рамках квазипотенциального подхода.

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The Difference Hypergeometric Equations and the Relativistic Coulomb Problem.

Finite-difference analogs of hypergeometric functions are introduced and the equations which they satisfy are found. The results obtained are used in finding the exact solution to the Coulomb problem, in the framework of the quasipotential approach.

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1. Introduction

In works[1-6] the three dimensional formulation of the problem of the interaction of two relativistic particles with equal mass m was developed[x)]. It was established, in particular, that the radial wave function of the given two particle system $R_\ell(r,q)$ satisfies the difference equation

$$\frac{H_0^{rad}}{2}(2E_q - H_0^{rad})R_\ell(r,q) = V(r,E_q)R_\ell(r,q), \tag{1.1}$$

where

$$H_0^{rad} = 2\,\mathrm{ch}\,i\frac{d}{dr} + \frac{\ell(\ell+1)}{r(r+i)}e^{i\frac{d}{dr}} . \tag{1.2}$$

$2E_q$ - total energy of the system, and $V(r;E_q)$ is the quasipotential. The name for the function $V(r;E_q)$ is taken from the work of Logunov and Tavkhelidze[7], in which the quasipotential approach to the relativistic two body problem was suggested. Making some

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x) We use for convenience the unit system in which $\hbar = c = m = 1$.

simple transformations in the equation for the wave function, obtained in /7/, it is possible to get the following analog of the difference equation (1.1) in the quasipotential approach

$$(E_q^2 - (\frac{H_0}{2})^2) R_\ell(r,q) = V(r; E_q) R_\ell(r,q). \tag{1.3}$$

It is clear that when investigating equations of the type (1.1) and (1.3) it is necessary to use the methods of the finite difference calculus. Special attention was paid in /5/ to further development of these methods.

In particular there was introduced the concept of the generalized degree

$$r^{(\lambda)} = i^{\lambda} \frac{\Gamma(-ir+\lambda)}{\Gamma(-ir)}, \tag{1.4}$$

corresponding to the finite-difference differentiation operation

$$\Delta = \frac{e^{-i\frac{d}{dr}} - 1}{-i}. \tag{1.5}$$

And with its help the analogs of a number of important functions of continuous analysis were built. In the present work we shall find the finite-difference analogs of the hypergeometric functions, write down the equations which they satisfy and then use the results obtained in the problem of definition of the energy spectrum of the two-particle system, interacting under a relativistic Coulomb law.

II. The Hypergeometric Functions in the Finite-Difference Calculus

Let us construct first the generalization of the confluent hypergeometric function $\Phi(\alpha;\gamma;ar)$. As is well-known /9/

$$\Phi(\alpha;\gamma;ar) = \sum_{n=0}^{\infty} \frac{\alpha_{(n)}}{\gamma_{(n)}} \frac{a^n r^n}{n!} \tag{2.1}$$

where

$$\alpha_{(n)} \equiv \alpha(\alpha+1)\dots(\alpha+n-1). \tag{2.2}$$

Substituting in (2.1) the generalized degree instead of the usual degree r^n we get

$$\sum_{n=0}^{\infty} \frac{\alpha_{(n)}}{\gamma_{(n)}} \frac{a^n r^{(n)}}{n!} \equiv \Phi[\alpha;\gamma;a;r] \tag{2.3}$$

It is obvious that we can consider the quantity $\Phi[\alpha;\gamma;a;r]$ as the generalization of the confluent hypergeometric function in the spirit of the finite-difference calculus. Because, due to (1.5) and (2.2)

$$r^{(n)} = i^n (-ir)_{(n)} \, , \tag{2.4}$$

it follows from (2.3) that

$$\Phi[\alpha;\gamma;a;r] = \sum_{n=0}^{\infty} \frac{\alpha_{(n)} (-ir)_{(n)}}{\gamma_{(n)}} \frac{(ia)^n}{n!} \, , \tag{2.5}$$

Comparing (2.5) with the usual hypergeometric series

$${}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{\alpha_{(n)} \beta_{(n)}}{\gamma_{(n)}} \frac{z^n}{n!} \, , \tag{2.6}$$

we come to the conclusion that

$$\Phi[\alpha;\gamma;a;r] = {}_2F_1(\alpha, -ir; \gamma; ia). \tag{2.7}$$

Taking into account the recurrence relation of Gauss for ${}_2F_1$

$$[\gamma - 2\beta + (\beta - \alpha) z]\, {}_2F_1(\alpha, \beta; \gamma; z) + \tag{2.8}$$

$$+ \beta(1-z)\, {}_2F_1(\alpha, \beta+1; \gamma; z) - (\gamma - \beta)\, {}_2F_1(\alpha, \beta - 1; \gamma; z) = 0$$

and the definition of the Δ -operation, it is possible to obtain the following difference equation for the function (2.7)

$$[(\gamma + 1 + ir)\Delta^2 + (i\gamma - \alpha a - a - iar)\Delta - i\alpha a]\,\Phi[\alpha; \gamma; a; r] = 0. \tag{2.9}$$

The transition to continuous analysis, or equivalently, to the non-relativistic limit in our unit system, corresponds to the following approximation

$$r \gg 1, \qquad a \ll 1, \qquad ra = \text{const}. \tag{2.10}$$

It is easy to be convinced, that in this region (2.9) turns into the confluent hypergeometric equation for the function (2.1)

$$\left[r \frac{d^2}{dr^2} + (\gamma - ar) \frac{d}{dr} - \alpha a \right] \Phi(\alpha; \gamma; ar) = 0. \tag{2.11}$$

As is well-known, besides the function $\Phi(\alpha;\gamma;ar)$ another solution of equation (2.11) is the expression

$$(ar)^{1-\gamma}\, \Phi(\alpha - \gamma + 1; 2 - \gamma; ar) =$$
$$= \sum_{n=0}^{\infty} \frac{(\alpha - \gamma + 1)_{(n)}}{(2-\gamma)_{(n)}} (ar)^{1-\gamma+n}. \tag{2.12}$$

It turns out, that if we replace in(2.12) the usual degrees by generalized ones, as a result the function

$$\sum_{n=0}^{\infty} \frac{(\alpha - \gamma + 1)_{(n)}}{(2-\gamma)_{(n)}} a^{1-\gamma+n}\, r^{(1-\gamma+n)}, \tag{2.13}$$

arises, which represents the second independent solution of the difference equation (2.9). Using the identity

$$r^{(1-\gamma+n)} = r^{(1-\gamma)}[r + i(1-\gamma)]^{(n)} \tag{2.14}$$

and taking into account (2.5), we can put (2.13) in the following form:

$$\sum_{n=0}^{\infty} \frac{(\alpha-\gamma+1)_{(n)}}{(2-\gamma)_{(n)}} a^{1-\gamma+n} r^{(1-\gamma+n)} =$$
$$= a^{1-\gamma} r^{(1-\gamma)} \Phi[\alpha-\gamma+1 ; 2-\gamma; ar + i(1-\gamma)]. \tag{2.15}$$

The remarkable property of the solutions (2.5) and (2.11) of equation (2.9) is that these functions are at the same time the two independent solutions of the differential hypergeometric equation in the argument a.

Let us look for the difference analog of the usual hypergeometric function (2.6). Putting $z = ar$ in (2.6) and passing to generalized degrees $r^{(n)}$ we get

$$\sum_{n=0}^{\infty} \frac{\alpha_{(n)} \beta_{(n)}}{\gamma_{(n)}} \frac{a^n r^{(n)}}{n!} \equiv {}_2F_1[\alpha, \beta ; \gamma; a; r]. \tag{2.16}$$

Using again the identity (2.4) and remembering the definition of the generalized hypergemetric series /9/ we conclude, that

$${}_2F_1[\alpha, \beta; \gamma; a; r] = \sum_{n=0}^{\infty} \frac{\alpha_{(n)}\beta_{(n)}(-ir)_{(n)}}{\gamma_{(n)}} \frac{(ia)^n}{n!} =$$
$$= {}_3F_1(\alpha, \beta, -ir; \gamma; ia). \tag{2.17}$$

Generally speaking, the expression of ${}_3F_1$ as a hypergeometric series is purely formal, because this series diverges at all values of the argument. Therefore we get from the formula (2.17) only the equality

$$ {}_2F_1[\alpha,\beta;\gamma;a;r] = {}_3F_1(\alpha,\beta,-ir;\gamma;ia), \tag{2.18} $$

and define the function ${}_3F_1$ in terms of Meijer's G-function /9/

$$ {}_3F_1(\alpha,\beta,-ir;\gamma;ia) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(-ir)} G^{31}_{23}\left(\frac{i}{a}, \middle| \begin{matrix} 1 & \gamma \\ \alpha & \beta-ir \end{matrix}\right), \tag{2.19} $$

Using known relations for G-function

$$ z^{\sigma} G^{mn}_{pq}\left(z \middle| \begin{matrix} a_r \\ b_s \end{matrix}\right) = G^{mn}_{pq}\left(z \middle| \begin{matrix} a_r+\sigma \\ b_s+\sigma \end{matrix}\right); $$

$$ (1-a_1+b_1)\, G^{mn}_{pq}\left(z \middle| \begin{matrix} a_1,\dots a_p \\ b_1,\dots b_q \end{matrix}\right) = G^{mn}_{pq}\left(z \middle| \begin{matrix} a_1-1, a_2,\dots a_p \\ b_1, \dots b_q \end{matrix}\right) + $$

$$ + G^{mn}_{pq}\left(z \middle| \begin{matrix} a_1 \dots a_p \\ b_1+1,\dots b_q \end{matrix}\right); \tag{2.20} $$

$$ (a_p - a_1)\, G^{mn}_{pq}\left(z \middle| \begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix}\right) = $$

$$ = G^{mn}_{pq}\left(z \middle| \begin{matrix} a_1-1, a_2, \dots a_p \\ b_1, \dots b_q \end{matrix}\right) + G^{mn}_{pq}\left(z \middle| \begin{matrix} a_1 \dots a_{p-1}, a_p-1 \\ b_1, \dots b_q \end{matrix}\right), $$

and taking into account the formulas (2.19) we can establish that the function ${}_2F_1[\alpha,\beta;\gamma;a;r]$ satisfies the following difference equation of the third order

$$ \{[r-i(\gamma+2)]\Delta^3 + i[r-2i(\gamma+1)+a(\alpha+2+ir)(\beta+2+ir)]\Delta^2 + \tag{2.21} $$

$$ +[i\gamma-2a\alpha\beta - a(ir+2)(\alpha+\beta+1)\Delta - ia\alpha\beta]\}\, {}_2F_1[\alpha,\beta;\gamma;a;r] = 0. $$

In the non-relativistic limit (see (2.10)) the term containing Δ^3 vanishes and (2.21) goes into the hypergeometric differential equation for the function ${}_2F_1(\alpha,\beta;\gamma;\lambda;r)$. . In complete analogy with the case of the confluent function we find that another solution of equation (2.17) is the function (cf. (2.11))

$$\lambda^{1-\gamma} r^{(1-\gamma)} {}_2F_1\,[1+\alpha-\gamma, 1+\beta-\gamma; 2-\gamma; \lambda; r+i(1-\gamma)]\,. \tag{2.22}$$

III. The Relativistic Coulomb Problem

According to /4/ we can consider the function

$$V(r) = \pm\,\alpha\,\frac{\mathrm{cth}\,\pi r}{r}$$

as the relativistic generalization of the Coulomb potential, where α is the fine structure constant and the $\pm$ signs correspond respectively to repulsive and attractive fields.

Let us consider the problem of defining the discrete energy spectrum and the wave function of a particle in the relativistic Coulomb field of attraction

$$V(r) = -\frac{\alpha\,\mathrm{cth}\,\pi r}{r} \tag{3.1}$$

The solution of this problem would be simpler, if we work with equation (1.3). Substituting in this equation the integral (3.1) we get

$$\left(E_q^2 - \left(\frac{H_0^{rad}}{2}\right)^2\right) R_\ell(r,q) = -\frac{\alpha\,\mathrm{cth}\,\pi r}{r} R_\ell(r,q)\,. \tag{3.2}$$

Let us restrict ourselves to the case $\ell = 0$ to avoid over complicated calculations. It is convenient to parametrize the discrete energy spectrum of interest in the following way

$$2E_q = 2\cos\chi_q = 2 - |W|, \tag{3.3}$$

where $-|W|$ is the binding energy.

Putting in (3.2) $\ell = 0$ and taking into account (3.3), this equation becomes

$$\left(\cos^2\chi_q - \mathrm{ch}^2\left(i\frac{d}{dr}\right)\right)R_0(r,q) = -\frac{a\,\mathrm{cth}\,\pi r}{r}R_0(r,q). \tag{3.4}$$

In the region of large r (3.4) reduces to the equation

$$\left(\cos^2\chi_q - \mathrm{ch}^2\, i\frac{d}{dr}\right)R_0^{as}(r,q) = 0, \tag{3.5}$$

among the solutions of which there is the exponentially decreasing

$$R_0^{as}(r,q) = e^{-r\chi_q}.$$

It is clear that the solution of equation (3.4) has to be sought in the form

$$R_0(r,q) = e^{-r\chi_q}\,y(r,q), \tag{3.6}$$

where the function $y(r,q)$ must not increase at large r faster than a polynomial of finite degree. Besides this we must require that

$$y(0,q) = 0$$

(cf. § 4 in /5/). Substituting (3.6) into (3.4) and going over to the new argument

$$\rho = \frac{r}{2} \tag{3.7}$$

we find the equation for the function $\phi(\rho,q) = y(2\rho,q)$

$$\left[(\beta^2+1)e^{-i\frac{d}{d\rho}} - 1 - \beta^2 e^{-2i\frac{d}{d\rho}}\right]\phi(\rho,q) = \tag{3.8}$$

$$= -\frac{2e^2\beta}{\rho - i}\,\mathrm{cth}\,2\pi\rho\, e^{-i\frac{d}{d\rho}}\,\phi(\rho,q),$$

where $\beta = e^{2i\chi_q}$, and the quantity e^2 is introduced temporarily instead of the fine structure constant α. In terms of the Δ-operation (3.8) is written as

$$\{ i(\rho - i)\Delta^2 + i[i(\rho - i)(1 - \frac{1}{\beta^2}) - \frac{2ie^2}{\beta} \operatorname{cth} 2\pi\rho]\Delta + \tag{3.9}$$

$$+ \frac{2ie^2}{\beta} \operatorname{cth} 2\pi\rho \} \phi(\rho, q) = 0.$$

Comparing (3.9) with (2.9) we come to the conclusion that our equation is a particular case of the confluent hypergeimetric equation in finite differences, the parameters α, γ and a being equal to

$$\alpha = -\frac{2ie^2\beta}{\beta^2 - 1} \operatorname{cth} 2\pi\rho = -\frac{e^2 \operatorname{cth} 2\pi\rho}{\sin 2\chi_q},$$

$$\gamma = 0, \tag{3.10}$$

$$a = \frac{\beta^2 - 1}{i\beta^2} = \frac{2\sin 2\chi_q}{e^{2i\chi_q}},$$

respectively.

Consequently, due to (2.7)

$$\phi(\rho, q) \sim {}_2F_1(\alpha, -i\rho; \gamma; ia), \tag{3.11}$$

where α, γ and a have been defined earlier.

Since the hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; z)$ tends to infinity at $\gamma \to 0$, we must introduce a compensating factor into the proportionality coefficient neglected in (3.11). Using the relation

$$\lim_{\gamma \to 0} \frac{{}_2F_1(\alpha, \beta; \gamma; z)}{\Gamma(\gamma)} = \alpha\beta z \; {}_2F_1(\alpha + 1, \beta + 1; 2; z)$$

We choose this factor to be $\frac{1}{\Gamma(\gamma)}$. As the result we get instead of (3.11)

$$\phi(\rho, q) = \rho\, {}_2F_1\left(1 - \frac{e^2 \operatorname{cth} 2\pi\rho}{\sin 2\chi_q}, -i\rho + 1; 2; \frac{2i \sin 2\chi_q}{e^{2i\chi_q}}\right). \tag{3.12}$$

At large ρ the function $\operatorname{cth} 2\pi\rho$ appearing in (3.12) tends to unity. Therefore, if the expression $1 - \frac{e^2}{\sin 2\chi_q}$ equals a negative integer

$$1 - \frac{e^2}{\sin 2\chi_q} = -n, \qquad n = 0, 1, 2, \ldots \tag{3.13}$$

then the function $\phi(\rho, q)$ shall behave as a polynomial of n-th degree. Formula (3.13) obviously defines the discrete energy spectrum. Taking into account (3.3) we can give relation (3.13) the following form

$$|W| = 4 \sin^2\left(\frac{1}{4} \arcsin \frac{e^2}{n}\right) = 2\left(1 - \sqrt{\frac{1}{2}\left(1 + \sqrt{1 - \frac{e^4}{n^2}}\right)}\right), \tag{3.14}$$

or

$$\frac{e^2}{2n} = \frac{1}{2} \sin\left(4 \arcsin \sqrt{\frac{|W|}{4}}\right) =$$

$$= \sqrt{|W|}\left(1 - \frac{|W|}{2}\right)\sqrt{1 - \frac{|W|}{4}}, \quad n = 1, 2, \ldots$$

From here, in the non-relativistic limit $|W| \ll 1$, follows the Bohr formula for the case under consideration (equal masses)

$$W = -|W| = -\frac{e^4}{2n^2} \cdot \frac{1}{2} \quad (n = 1, 2, 3, \ldots). \tag{3.15}$$

Let us write in conclusion the wave function $R_0(r,q)$, performing the transformations (3.6) and (3.7) in reverse order

$$R_0(r,q) = c(r)\, r e^{-\chi_q r}\, {}_2F_1\left(1-n \operatorname{cth} \pi r, 1-\frac{ir}{2}; 2; \frac{2i \sin 2\chi_q}{e^{2i\chi_q}}\right), \tag{3.16}$$

where $c(r)$ is a normalizing i-periodic coefficient.

IV. Some Remarks on the Geometric Properties of the Relativistic $\vec{r}$-Space

In earlier mentioned works /4-6/ there was introduced into consideration a new relativistic $\vec{r}$ -space. It was shown in particular that this $\vec{r}$ -space is connected with the three dimensional momentum space of Lobachevsky realized on the upper sheet of the hyperboloid $p_0^2 - \vec{p}^{\,2} = m^2 = 1$, by the Shapiro transformation /10/.

The question naturally arises, about what geometry this $\vec{r}$ -space obeys. It follows from the orthogonality and completeness conditions for the "plane waves" $\xi(\vec{p},\vec{r}) = (p_0 - \vec{p}\vec{r}\,)^{-1-ir}$ used in the Shapiro transformation

$$\frac{1}{(2\pi)^3} \int \xi(\vec{p},\vec{r})\, \xi^*(\vec{p},\vec{r}\,')\, d\Omega_p = \delta^{(3)}(\vec{r}-\vec{r}\,'), \tag{4.1}$$

$$\frac{1}{(2\pi)^3} \int \xi(\vec{p},\vec{r})\, \xi^*(\vec{p}\,',\vec{r})\, d\vec{r} = \sqrt{1+\vec{p}^{\,2}}\, \delta^{(3)}(\vec{p}-\vec{p}\,'),$$

that the metric of the relativistic $\vec{r}$ -space has to be euclidean. On the other hand the generators of its shifts can not be equal to $-i\dfrac{\partial}{\partial \vec{r}}$, because the eigenvalues of these operators form a euclidean $\vec{p}$ -space, not $\vec{p}$ -space of constant curvature.

It happens, that the following principal statement is true: the relativistic $\vec{r}$ -space is euclidean, but the generators of the shift transformations $\hat{\vec{p}} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$ are realized in it in the form of differential-difference operators, the common spectrum of which forms a p-space of Lobachevsky, and the common eigenfunctions of which are the "plane waves" $\xi(\vec{p}, \vec{r})$.

Explicitly, the quantities $\hat{p}_1, \hat{p}_2$ and $\hat{p}_3$ are the following

$$\hat{p}_1 = -\sin\theta\cos\phi\,(e^{i\frac{\partial}{\partial r}} - \frac{H_0}{2}) - i\,(\frac{\cos\phi\cos\theta}{r}\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial\phi})\,e^{i\frac{\partial}{\partial r}} ,$$

$$\hat{p}_2 = -\sin\theta\sin\phi\,(e^{i\frac{\partial}{\partial r}} - \frac{H_0}{2}) - i\,(\frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial\phi})\,e^{i\frac{\partial}{\partial r}} , \qquad (4.2)$$

$$\hat{p}_3 = -\cos\theta\,(e^{i\frac{\partial}{\partial r}} - \frac{H_0}{2}) + i\,\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\,e^{i\frac{\partial}{\partial r}} ,$$

where θ and ϕ are the angular coordinates of the vector $\vec{r}$ and H_0 is the relativistic free hamiltonian /4/.

$$H_0 = 2\,\mathrm{ch}\,i\frac{\partial}{\partial r} + \frac{2i}{r}\,\mathrm{sh}\,i\frac{\partial}{\partial r} - \frac{\Delta_{\theta,\phi}}{r^2}\,e^{i\frac{\partial}{\partial r}} . \qquad (4.3)$$

It is easy to see from (4.2) that

$$[\hat{p}_1\,\hat{p}_2] = [\hat{p}_2\,\hat{p}_3] = [\hat{p}_3\,\hat{p}_1] = 0. \qquad (4.4)$$

In the non-relativistic limit the operators (4.2) go into the usual generators of the translations $-i\frac{\partial}{\partial\vec{r}}$ in spherical coordinates. The other commutators of the euclidean motion group stay unchanged along with (4.4) in the relativistic $\vec{r}$ -space. We have in particular as in the case of the euclidean group

$$[L_i, \hat{p}_k] = i\epsilon_{ik\ell}\hat{p}_\ell \ . \tag{4.5}$$

where $\vec{L}$ is the angular momentum operator. It may be that the results obtained here will have an heuristic meaning for the development of the idea of space-time quantization, because we have shown in fact that the motion group of the (pseudo) euclidean space permits a representation containing organically a parameter with the dimension of length (in the given case it is $\hbar/mc$).

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