ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
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Дубна

E2 - 3891

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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SPECTRAL REPRESENTATION OF THE TWO-POINT FUNCTION FOR INFINITE-COMPONENT FIELDS

1968

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1. Usually, when one is dealing with fields describing particles of definite spin, the spectral representation of the two-point function is given by an integral in the total mass only[1,2]. If one is dealing with fields containing more than one spin, then it is necessary to decompose the two-point function, also with respect to the spin variable (i.e. with respect to the second Casimir operator of the Poincaré group). This is always the case when infinite-component fields are taken into account. We find here the form of such a representation for a field, transforming under an arbitrary irreducible representation of SL(2,C) (the result applies also to more general fields using their decomposition into irreducible components). The clue to the solution of the problem is the use of the formalism of homogeneous functions of two complex variables (instead of the tensor formalism) in the description of the irreducible representations of the Lorentz group (see e.g. [3] or Appendix A to [4]).

For finite component fields, Poincaré invariance and spectral conditions imply the equivalence between TCP and weak local commutativity[5]. For infinite component fields this is not the case[6]. In Sec. 4 we construct explicit examples of non-local, generalized infinite-component fields with an increasing mass spectrum (with respect to spin).

2. A field $\varphi(x)$ transforming according to the irreducible representation $[l_0, l_1]$ of SL(2,C) will be written as a homogeneous function $\varphi(x,z)$ of the complex (Lorentz) spinor $z=(z_1,z_2)$ (cf.[7])

$$\varphi(x, \lambda z) = \lambda^{\nu_1}\bar{\lambda}^{\nu_2}\varphi(x,z), \tag{1}$$

where the degree of homogeneity (ν_1, ν_2) is related to the number of the representation $[l_0, l_1]$ by

$$\nu_1 = l_1 + l_0 - 1, \qquad \nu_2 = l_1 - l_0 - 1 \tag{2}$$

(we recall that the single-valuedness of φ implies that $\nu_1 - \nu_2 = 2l_0$ is an integer). For the special case of finite-dimensional representations (for ν_1, ν_2 - non-negative integers) $\varphi(x,z)$ is a polynomial of z and $\bar{z}$, its coefficients being the ordinary (spinor or tensor) field components . In particular, the Pauli two-component spinor φ and the vector field A correspond in our notation to the polynomials

$$\varphi(x,z) = \sum_{a=1}^{2} \varphi^a(x)\, z_a\,,$$

$$A(x,z) = \sum_{\mu=0}^{3} A^\mu(x)\, z\sigma_\mu \bar{z}$$

(further the sum sign in similar expressions with repeated upper and lower index will be omitted). Consider the two-point function

$$\langle 0|\varphi(x,z)\Psi(y,w)|0\rangle = F_{\varphi\Psi}(x-y; z,w), \tag{3}$$

where φ and ψ are transforming under the representations $[\ell_0, \ell_1]$ and $[\ell_0', \ell_1']$ respectively. The spectral conditions allow us to write F in the form

$$F_{\varphi\psi}(x; z, w) = \int \vartheta(p)\, K(\underset{\sim}{p}; z, w)\, e^{-ipx}\, d^4p \quad , \tag{4}$$

where $\vartheta(p) = \theta(p^0)\,\theta(p^2)$ is the characteristic function of the forward cone and

$$\underset{\sim}{p} = \sigma_\mu p^\mu = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} . \tag{5}$$

Lorentz invariance implies that for each $A \in SL(2, C)$

$$K(A\underset{\sim}{p}A^*; zA^{-1}, wA^{-1}) = K(\underset{\sim}{p}; z, w) . \tag{6}$$

Furthermore, because of (1), K is a homogeneous function of z and w of degree (ν_1, ν_2) and (ν_1', ν_2'), respectively.

To satisfy (6) we require that the kernel K is a function of the invariants, which can be formed by the 4-vector p and the Lorentz spinors z and w .

First we find a complete set of independent invariants.

One can construct four Lorentz vectors in terms of z and w: two real light-like vectors

$$\xi_\mu = z\sigma_\mu \bar{z} \qquad \text{and} \qquad \eta_\mu = w\sigma_\mu \bar{w} \tag{7}$$

and two complex conjugate zero-length vectors

$$\chi_\mu = z\sigma_\mu \bar{w} \quad \text{and} \quad \bar{\chi}_\mu = w\sigma_\mu \bar{z} . \tag{8}$$

The equality $\xi^2 = \eta^2 = \chi^2 = 0$ are a consequence of the identity

$$g^{\mu\mu} (\sigma_\mu)^{\alpha\dot{\beta}} (\sigma_\mu)^{\gamma\dot{\delta}} = 2\varepsilon^{\alpha\gamma}\varepsilon^{\dot{\beta}\dot{\delta}}, \quad \varepsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{9}$$

All non-vanishing scalar products of these vectors are proportional to the product of the complex conjugate invariants $z\varepsilon w$ and $\overline{z\varepsilon w}$:

$$\xi\eta = -\chi\bar{\chi} = 2|z\varepsilon w|^2, \; \xi\chi = \eta\chi = \xi\bar{\chi} = \eta\bar{\chi} = 0 . \tag{10}$$

More general, the tensor identity holds

$$\chi_\mu \bar{\chi}_\nu + \bar{\chi}_\mu \chi_\nu = \xi_\mu \eta_\nu + \eta_\mu \xi_\nu - g_{\mu\nu}\xi\eta ; \tag{11}$$

it implies in particular that

$$(\xi p)(\eta p) - (\chi p)(\bar{\chi} p) = \frac{1}{2} p^2 \xi\eta = p^2 |z\varepsilon w|^2 \tag{12}$$

For the existence of a single-valued homogeneous invariant function K it is necessary and sufficient that the difference $\ell_0 - \ell_0'$ is an integer. A straightforward argument which makes use of the identities (10)-(12) shows that K can always be written in the form

$$K(p; z, w) = K_{\varphi\psi}(\underset{\sim}{p}; z, w) = (z \varepsilon w)^{\ell_0 + \ell_0'} (z \underset{\sim}{p} \bar{w})^{\ell_0 - \ell_0'} \times$$
$$\times (z \underset{\sim}{p} \bar{z})^{\ell_1 - \ell_0 - 1} (w \underset{\sim}{p} \bar{w})^{\ell_1' - \ell_0 - 1} h_{\varphi\psi}(\mu, p^2), \qquad (13)$$

where

$$\mu = \cos\theta = 1 - 2p^2 \frac{|z \varepsilon w|^2}{z \underset{\sim}{p} \bar{z}\, w \underset{\sim}{p} \bar{w}} = 1 - \frac{p^2 \xi \eta}{(\xi p)(\eta p)} \qquad (14)$$

θ is the angle between $\underline{\xi}$ and $\underline{\eta}$ in the rest frame of p. This form of K is convenient if $\ell_0 \geq |\ell_0'|$ (which always can be achieved by possible interchanging of the roles of z and w).

3. Next we decompose $K_{\varphi\psi}$ in terms of the eigenfunctions of the spin square operator

$$\underline{S}^2 = \frac{1}{2} M_{\sigma\rho} M^{\sigma\rho} - M_{\sigma\mu} M^{\sigma\nu} \frac{p^\mu p_\nu}{p^2} = \ell_0^2 + \ell_1^2 - 1 - \frac{1}{p^2} M_{\sigma\mu} p^\mu M^{\sigma\nu} p_\nu$$

To find the eigenfunctions $K^{(s)}$ of $\underline{S}^2$ (with respect to the variables z) of the form (13) it is convenient to work in the rest frame in which

$$\underline{S}^2(\underline{p}=0) = \underline{M}^2 = \frac{1}{4}\left(z_a \frac{\partial}{\partial z_a} - \bar{z}_a \frac{\partial}{\partial \bar{z}_a} \right)^2 - \tag{15}$$
$$- \frac{1}{2}\left[z_a \varepsilon_{ab} \frac{\partial}{\partial \bar{z}_b}, \bar{z}_c \varepsilon_{cd} \frac{\partial}{\partial z_d} \right]_+ .$$

The equation

$$\left[\underline{M}^2 - s(s+1) \right] K^{(s)}(p=(m,0);\, z, w) = 0 \tag{16}$$

is equivalent to the following equation for the function $h^{(s)}_{\ell_0 \ell_0'}(\mu)$ connected with $K^{(s)}$ by (13) :

$$\left\{ (1-\mu^2)\frac{d^2}{d\mu^2} - 2\left[\ell_0' + (\ell_0+1)\mu\right]\frac{d}{d\mu} + \left[s(s+1) - \ell_0(\ell_0+1)\right]\right\} h^{(s)}_{\ell_0\ell_0'}(\mu) = 0 . \tag{17}$$

The solution of this equation regular for $\mu = \pm 1$ is

$$h^{(s)}_{\ell_0\ell_0'}(\mu) = P^{(\ell_0+\ell_0',\ \ell_0-\ell_0')}_{s-\ell_0}(\mu) , \tag{18}$$

where $P^{(\alpha,\beta)}_n(\mu)$ are the Jacobi polynomials (we have assumed here that $\ell_0 \geqslant |\ell_0'|$ as mentioned above) .

In particular, for $\Psi = \varphi^*$ (and hence, $\ell_0 = -\ell_0' \geqslant 0$, $\ell_1' = \bar{\ell}_1$) we obtain

$$K_{\varphi\varphi^*}(p; z, w) = \sum_{s \geqslant \ell_0} \rho_s(p^2)\, K^{(s)}_{\ell_0\ell_1}(p; z, w) \equiv$$
$$\equiv (z\underset{\sim}{p}\bar{w})^{2\ell_0} (z\underset{\sim}{p}\bar{z})^{\ell_1-\ell_0-1} (w\underset{\sim}{p}\bar{w})^{\bar{\ell}_1-\ell_0-1} \sum \rho_s(p^2) P^{(0,\, 2\ell_0)}_{s-\ell_0}(\mu) . \tag{19}$$

It is important that the kernel $K^{(s)}_{\ell_0\ell_1}$ defined by (19) is positive-definite in this case because

$$K^{(s)}_{\ell_0\ell_1}(p;z,w) = A_s^{\ell_0\ell_1} \sum_{\zeta=-s}^{s} u_{s\zeta}(p,z)\,\overline{u_{s\zeta}(p,z)}\,, \tag{20}$$

where A is a positive constant and

$$u_{s\zeta}(p,z) = f_{s\zeta}(zB_p)\,, \qquad B_p = \frac{p^0 + m + \sigma_j p^j}{\sqrt{2m(p^0+m)}}$$

are the canonical basis vectors[4].

For finite dimensional representations of SL(2,c) $\ell_1 - \ell_0$ is a positive integer and $K^{(s)}_{\ell_0\ell_1}$ is a polynOmial with respect to p. In that case the (weak) locality condition for the two-point function is equivalent to TCP invariance[5] which implies

$$\langle 0|\varphi(x,z)\,\varphi(y,w)^*|0\rangle = \langle 0|\varphi(x,w)^*\,\varphi(y,z)|0\rangle\,. \tag{21}$$

For instance, for a local vector field A $\left([\ell_0,\ell_1]=[0,2]\right)$ (19) reduces to

$$K_{AA^*}(p,\xi,\eta) = \rho_0(p^2)(\xi p)(\eta p) + \rho_1(p^2)\left[(\xi p)(\eta p) - p^2\xi\eta\right] =$$

$$= \left\{\rho_0(p^2)\,p^\mu p^\nu + \rho_1(p^2)\left(p^\mu p^\nu - p^2 g^{\mu\nu}\right)\right\}\xi_\mu\eta_\nu =$$

$$= K_{\mathring{A}A}(p,\eta,\xi)$$

where ξ and η are defined by (7).

For infinite component fields the locality of the two-point function is rather an exception. As an example of a local infinite-component field we take the free field $\psi(x,z)$ of mass m, transforming under some unitary representation $[\ell_0, i\sigma]$ of SL(2,C) (of the principle series). In this case

$$K_{\psi\psi^*}(p;z,w) = \left(\frac{z\underset{\sim}{p}\bar{w}}{w\underset{\sim}{p}\bar{z}}\right)^{\ell_0}\left(\frac{z\underset{\sim}{p}\bar{z}}{w\underset{\sim}{p}\bar{w}}\right)^{i\sigma}\delta(z\epsilon w)\,\delta(p^2-m^2), \tag{22}$$

where $\delta(z\epsilon w)$ is a two-dimensional δ- function which is a homogeneous function of degree (-1,-1) of its complex argument (see[3]) . In this case the first two factors do not depend actually on p because of the δ- function. For real self-conjugate representations (ℓ_0 =0, ℓ_1 - real) the free local field φ of mass m corresponds to

$$K_{\varphi\varphi^*}(p;\xi,\eta) = (\xi\eta)^{\ell_1-1}\,\delta(p^2-m^2)\,. \tag{23}$$

Both examples (22) and (23) correspond to infinite mass degeneracy with respect to spin. This is no more chance. Grondsky and Streater showed recently[8] that an infinite-component field, with polynomially bounded two-point function in momentum space, can be local only for infinite mass degeneracy.

4. As stated above in the general case, when at least one of the mass values is nondegenerate with respect to spin and K is a tempered distribution with respect to p, the theory is non-local. Here we shall consider a class of examples of generalized free non-local fields with increasing mass spectrum in which the sum over s can be taken explicitly.

For this purpose we put

$$\rho_s(p^2) = \frac{c_s}{m_s^{\ell_1+\bar{\ell}_1}}\,\delta(p^2-m_s^2)\,, \quad m_s = m_0 + \varkappa s\,, \quad 0 < c \leq 1\,, \qquad (24)$$
$$m_0 > 0\,, \quad \varkappa > 0\,.$$

Substituting (19) with this ρ_s in (4) and putting in the $(s-\ell_0)$-th term of the sum $p = m_s n$ we obtain

$$F_{\varphi\varphi^*}(x; z, w) = \int (z\underset{\sim}{n}\bar{w})^{2\ell_0} (z\underset{\sim}{n}\bar{z})^{\ell_1-\ell_0-1} (w\underset{\sim}{n}\bar{w})^{\bar{\ell}_1-\ell_0-1} e^{-im_0 nx}$$
$$\times \sum_{s=\ell_0}^{\infty} \left(c\, e^{-i\varkappa nx}\right)^s P_{s-\ell_0}^{(0,\,2\ell_0)}(u)\,\frac{d^3n}{2n^0} =$$
$$= 2^{2\ell_0-1} \int (z\underset{\sim}{n}\bar{w})^{2\ell_0} (z\underset{\sim}{n}\bar{z})^{\ell_1-\ell_0-1} (w\underset{\sim}{n}\bar{w})^{\bar{\ell}_1-\ell_0-1} e^{-im_0 nx}$$
$$\times \frac{1}{R}\,(R+\tau+1)^{-2\ell_0}\,\frac{d^3n}{n^0} = F_{\varphi^*\varphi}(x; w, z)\,, \qquad (25)$$

where

$$R = (1 - 2\mu\tau + \tau^2)^{1/2}, \quad \tau = c\, e^{-i\varkappa n x}$$

and the range of integration is the upper unit hyperboloid: $n^0 = \sqrt{1 + \underline{n}^2}$. All other two-point functions (as well as all truncated Wightman functions) are zero by definition. The field $\varphi(x, z)$ so defined is TCP- invariant but not weakly local. For the mass degenerate limit $\varkappa = 0$ if we put $c = 1,\ m_0 = m,\ \ell_0 = 0,\ \ell_1 = \frac{1}{2}$ we find a local field of the type (23).

In the general case the two-point function (25) decreases exponentially for large space-like separations. Indeed, using the first equality (25) and the analyticity of the integrand in the strip $|\mathrm{Im}\, n_j| < 1$ for each $j (= 1, 2, 3)$, we find

$$|F_{\varphi\varphi^*}(0, \underline{x}; z, w)| \leq A_\varepsilon\, e^{-(m_0 - \varepsilon)|\underline{x}|}, \tag{26}$$

where ε is an arbitrarily small positive number and $A_\varepsilon > 0$.

It is interesting to speculate on the possibility of the existence of a local infinite-component field whose two-point function is p-space (19) is a Jaffe type generalized function[9] increasing faster than a polynomial.

The authors thank Prof. R.F.Streater for a prepublication copy of his paper.

References

1. S.Kamefuchi and H.Umezawa, Prog.Theor.Phys. 6 (1951) 543; G.Kallen, Helv.Phys.Acta 25 (1952) 417; H.Lehmann, Nuovo Cimento 11 (1954) 342; O.Steinmann, J.Math.Phys. 4 (1963) 583.
2. I.Raszillег and D.H.Schiller, Nuovo Cim. A48 (1967) 617, 635, 645.
3. И.М.Гельфанд, М.И.Граев, Н.Я.Виленкин. Интегральная геометрия и связанные с ней вопросы теории представлений (обобщенные функции, вып. 5), М., Физматгиз, 1962.
4. D.Tz.Stoyanov and I.T.Todorov, Self-Coupled Representations of the Lorentz Group and Infinite-Component Fermi Fields, ICTP Trieste, Preprint IC/67/58 (1967).
5. R.Jost, Helv.Phys.Acta 30 (1957) 409. See also: R.F.Streater and A.S.Wightman, PTC, spin and statistics and all that, W.A.Benjamin, Ic., New York-Amsterdam, 1964.
6. A.I.Oksak and I.T.Todorov, Invalidity of TCP-Theorem for Infinite Component Fields, Preprint JINR, Dubna, 1968.
7. Дао Вонг Дык и Нгуен Ван Хьеу, ЯФ 6(1967) 186.
8. I.T.Grodsky and R.F.Streater, Phys.Rev.Lett. 20 (1968) 695.
9. A.M.Jaffe, Phys.Rev. 158, (1967) 1454.

Received by Publishing Department
on May, 23, 1968.