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INVALIDITY OF TCP-THEOREM FOR INFINITE-COMPONENT FIELDS

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I. Introduction

It has been shown recently /1-4/, that infinite-component local fields may violate the right connection between spin and statistics. The only example of TCP nonvariant infinte-component field known up to now, the Majorana field /5,4,3/, is incompatible with the usual spectrum conditions. Here we give examples of local infinite-component fields transforming under irreducible unitary representations of SL(2,CL) which satisfy usual spectrum conditions and violate TCP. These examples do not contradict Epstein's proof /6/ of TCP in a theory of local observables, because his assumption of no mass degeneracy (with respect to spin) is not fulfilled in our case.

First of all, in Sect. 2, we derive some general properties of the two-point function, valid also for tempered local infinite-component fields. The main tool in this investigation is the Bogolubov-Vladimirov theorem /7/. We give the necessary conditions on the two-point function for the validity of TCP. We construct in Sect. 3 examples of free (Fermi and Bose) fields of mass m which violate these conditions. All our examples use the Majorana representations [x] $[0, 1/2]$ and $[1/2, 0]$ (but not the Majorana equations).

[x] For a detailed description of the Majorana representations the reader is referred to /4/. We are using here the notations $[\ell_0, \ell_1]$ of Gel'fand and Naimark /8/ for the irreducible representations of SL (2,C).

2. Two-Point Functions in Local and TCP-Invariant Theories

We shall deal with the quantum field theory in which all Wightman axioms /9/ except Lorentz invariance hold. We impose in particular (weak) locality and translation invariance including normal spectrum condition. Locality implies

$$< 0 \,|\, \phi(x)\psi(y) - \epsilon\psi(y)\,\phi(x) \,|\, 0> = 0 \quad \text{when } (x-y)^2 < 0, \qquad (1)$$

$$\epsilon = \pm 1$$

We start with the formulation of Bogolubov-Vladimirov theorem /7/ (we refer here to a part of the result of /7/ only).

<u>BV-Theorem.</u> Let $F_1(x)$, $F_2(x) \in S'(R^4)$ be two distributions satisfying the following conditions:

a) supp. $\tilde{F}_1(p) \subset \bar{V}_+$, $\bar{V}_+$ is the forward light cone,
   supp. $\tilde{F}_2(p) \subset \bar{V}_-$, $\bar{V}_- = -\bar{V}_+$;

b) $F_1(x) = F_2(x)$ for $x^2 < 0$.

Then a function $F(z)$ exists holomorphic in the extended tube

$$\mathcal{E}' = \{ z \in C^4 : z^2 \in m \} ,$$

where m is the complex plane with a cut along the real positive semiaxis, such that

$$F_1(x) = F(x + i0) \qquad (\equiv \lim_{y \in V_+,\, y \to 0} F(x + iy)),$$

$$F_2(x) = F(x - i0) .$$

Define

$$\begin{aligned} W_1(x-y) &= <0 \,|\, \phi(x)\,\phi^*(y) \,|\, 0> , \\ W_2(x-y) &= <0 \,|\, \phi^*(x)\,\phi(y) \,|\, 0> \end{aligned} \qquad (2)$$

and let

$$\phi(x)\phi^*(y) - \epsilon\phi^*(y)\phi(x) = 0 \qquad \text{when } (x-y)^2 < 0. \tag{3}$$

By the BV-theorem we can deduce that there is a function $F(z)$ holomorphic in the extented tube $\mathcal{E}'$ such that

$$\begin{aligned} W_1(x) &= F(x + i0), \\ W_2(x) &= \epsilon F(-x - i0). \end{aligned} \tag{4}$$

The statistics associated with $\phi(x)$ is connected with the question, whether $F(z)$ defined by (4), (2) is even or odd.

Proposition 1. If $F(z)$ defined by (4), (2) is even (resp. odd) in $\mathcal{E}'$: $F(-z) = \pm F(z)$ then ϵ in (3) is 1 (resp. -1).

Proof. Suppose that $F(-z) = \lambda F(z)$, $\lambda = \pm 1$. Due to (4) we have

$$W_2(x) = \epsilon F(-x - i0) = \epsilon\lambda F(x + i0) = \epsilon\lambda W_1(x),$$

i.e. $W_2(x) - \epsilon\lambda W_1(x) = 0$. Now it is obvious that the equality $\epsilon\lambda = -1$ is impossible since the sum of two positive-definite distributions $W_2(x) + W_1(x)$ cannot be zero (otherwise $\|\phi(f)|0>\| = \|\phi^*(f)|0>\| = 0$). This proves the assertion.

We give below the conversion of proposition 1 under the assumption of TCP-invariance (proposition 2). Let Θ be an anti-unitary operator such that

$$\begin{aligned} \Theta^{-1}\phi(x)\Theta &= \eta(\phi)\phi^*(-x), \quad |\eta(\phi)| = 1, \\ \Theta^{-1}\phi^*(x)\Theta &= \eta(\phi^*)\phi(-x); \end{aligned} \tag{5}$$

taking hermitian conjugation of (5) one obtains

$$\eta(\phi^*) = \overline{\eta(\phi)}.$$

It is readily verified that TCP-invariance imposes the following condition on the two-point functions

$$W_{\phi,\psi}(x) \equiv \langle 0 | \phi(\tfrac{x}{2}) \psi(-\tfrac{x}{2}) | 0 \rangle = \overline{\eta(\phi)}\,\overline{\eta(\psi)}\, W_{\psi,\phi}(x) . \tag{6}$$

Let us mention that

if $W_{\phi,\phi} \neq 0$ (resp. $W_{\phi,\psi} \neq 0$) then (6) implies

$$\eta^2(\phi) = 1$$

$$(\text{resp. } \eta(\phi) \cdot \eta(\psi) = \pm 1).$$

Let $G(z)$ be the analytic function in $\mathfrak{k}'$ defined by

$$\begin{aligned} G(x+i0) &= W_{\phi\psi}(x), \\ G(-x-i0) &= \epsilon W_{\psi,\phi}(x). \end{aligned} \tag{7}$$

according to the BV-theorem . TCP invariance leads to the identity

$$G(-z) = \epsilon \eta(\phi) \eta(\psi) G(z). \tag{8}$$

Indeed, the functions

$$G_1(z) = G(-z), \qquad G_2(z) = \epsilon \eta(\phi) \eta(\psi) G(z)$$

are analytic in $\mathfrak{k}'$ and due to (7), (6) their boundary values from the forward tube coincide:

$$G_1(x+i0) = G_2(x+i0) .$$

In particular taking $\psi = \phi^*$ in (8) we obtain

Proposition 2. If

a) $\langle 0 | \phi(x) \phi^*(y) - \epsilon \phi^*(y) \phi(x) | 0 \rangle = 0$ when $(x-y)^2 < 0$,

b) there exists TCP symmetry

then $F(z)$ defined by (4),(2) satisfies

$$F(-z) = \epsilon F(z). \tag{9}$$

3. *Examples of Infinite-Component Fields Violating TCP-Theorem*

Let us consider one of the Majorana representations $[0,1/2]$ or $[1/2,0]$ acting in the Hilbert space X. These are the only irreducible representations of SL(2,C) for which a 4-vector of (unbounded) hermitian operators L_μ can be defined such that

$$V(A) L_\mu V(A^{-1}) = L_\nu \Lambda(A)^\nu_\mu .$$

The operator L_0 is positive definite (its eigenvalues being $s+1/2$, $s = \ell_0, \ell_0+1, \ell_0+2, \ldots$, ℓ_0 is 0 or 1/2). We define the free field as a linear operator valued functional of the vector $f \in X$ satisfying the anticommutation relations:

$$[\psi(x,f), \psi(y,g)]_+ = [\psi(x,f)^*, \psi(y,g)^*]_+ = 0,$$

$$[\psi(x,f)^*, \psi(y,g)]_+ = (f|(\frac{1}{m} L_\mu \frac{\partial}{\partial x_\mu} + \frac{1}{i} c) g) \times$$

$$\times D_m(x-y),$$

where $0 < c < 1/2$.

The field ψ is characterized completely by the two-point functions

$$\langle 0|\psi(x,f)^* \psi(y,g)|0\rangle = (f|(\frac{1}{m} L_\mu \frac{\partial}{\partial x_\mu} + \frac{1}{i} c) g) D^-_m(x-y) \tag{10}$$

$$<0|\psi(x,g)\psi(y,\ f)^{*}|0> = (f|(\frac{1}{m}L_\mu \frac{\partial}{\partial x_\mu} - \frac{1}{i}c)g\ )D^{-}_{m}(x-y). \tag{11}$$

All other two-point functions as well as the higher order trancated vacuum expectation values are zero. It can be checked in a straightforward way that the theory so defined satisfies all Wightman axioms (except finite componentness of the field). The positive definiteness of the metric in the Hilbert space of states follows from the positive definiteness of L_0 which implies

$$L_\mu p^\mu \geq m\ (\ \ell_0 + 1/2) \qquad \text{for} \quad p^0 = \sqrt{m^2 + \vec{p}^{\,2}}$$

$$(\ell_0 = 0, 1/2).$$

We can give explicitly the action of the field ψ in Fock space by decomposing it in terms of creation and annihilation operators of particles with fixed momentum $\vec{p}$ and spin s :

$$\psi(x,f) = \frac{1}{\sqrt{2}\,(2\pi)^{3/2}} \sum_{s=\ell_0}^{\infty} \int_{p_0=\omega} \sum_{\zeta=-s}^{s} \{(s+\frac{1}{2}-c)^{1/2}\, a_{s\zeta}(\vec{p}\,) \times \tag{12}$$

$$e^{-ipx} + (s+\frac{1}{2}+c)^{1/2}\, b^{*}_{s,-\zeta}(\vec{p}\,)\, e^{ipx}\,\}(u_{s\zeta}(\vec{p}), f\,)\frac{d^3p}{p^0},$$

where $\omega = \omega(\vec{p}\,) = (\,m^2 + \vec{p}^{\,2})^{1/2}$ and $u_{s\zeta}(\vec{p}\,)$ is normalized eigenvector of the total spin and its third projection (with eigenvalues s and ζ) which can be written as

$$u_{s\zeta}(\vec{p}\,) = V(B_p)|s,\zeta) \ . \tag{13}$$

Here $V(B_p)$ is the representation of the "boost" transformation

$$B_p = (\omega + m + \sigma_k p^j)[2m(\omega + m)]^{-1/2}. \tag{14}$$

Due to (10) and (11) $a^{(*)}_{s\zeta}$ and $b^{(*)}_{s\zeta}$ have to satisfy the canonical anticommutation relations

$$[a_{s\zeta}(\vec{p}), a^*_{s'\zeta'}(\vec{q})]_+ = [b_{s\zeta}(\vec{p}), b^*_{s'\zeta'}(\vec{q})]_+ = \tag{15}$$

$$= \delta_{ss'} \cdot \delta_{\zeta\zeta'} \cdot \omega(\vec{p})\,\delta(\vec{p} - \vec{q})$$

(all other anticommutators vanishing identically). We assume here the standard Fock representation of the canonical anticommutation relation in a Hilbert space $\mathcal{H}$ with an unique vacuum state $|0>$ such that

$$a_{s\zeta}(\vec{p})\,|0> = b_{s\zeta}(\vec{p})\,|0> = 0. \tag{16}$$

Eqs. (10) and (11) are a consequence of (12) - (16) because of the identities

$$\sum_{s\zeta} (s+1/2)(f\,|\,V(B_p)^*\,|\,s\,\zeta)(s\,\zeta\,|\,V(B_p)\,|\,g) = \tag{17}$$

$$= (f\,|\,V(B_p)^*\;L_0 V(B_p)\;|\,g) = (f\,|\,L_\mu\,p^\mu\;|\,g).$$

On the other hand the field ψ with vacuum expectation values (10), (11) clearly does not allow a TCP symmetry operation of the type (5¯) because the necessary condition for TCP-invariance (6) is violated. The function $F(z)$ defined by BV-theorem in this case has the form

$$F(z) = (f\,|\,g)\,F_0(z^2) + z^\mu\,(f\,|\,L_\mu\,g)\,F_1(z^2)$$

and hence is neither odd nor even in contradiction with (9). (Obviously for $t = 0$ the even term F_0 vanishes and the field becomes TCP invariant). We observe that although the field (12) is not

TCP invariant it describes particles and antiparticles with equal masses.

We mention that each term in the sum (12) with fixed s represents a non-local field. For the compensation of the non-local terms in the total anticommutator the mass degeneracy seems to be essential. As suggested in /2/ it is interesting to note that a theory with an infinitely degenerate mass level (with respect to spin) violates the compactness condition proposed by Haag and Swieca /10/.

We see from the above example that TCP violation is not necessarily connected with the wrong connection between spin and statistics: ψ is a local Fermi field describing half-integer spin particles for the representation $[1/2, 0]$ and integer spin particles for the representation $[0, 1/2]$. We can construct just as well an example of a free Bose TCP non-invariant field ϕ. For this aim we define the <u>commutator of ϕ</u> and ϕ^* by x)

$$[\phi(x,f)^*, \phi(y,g)] = (f \,|\, i(\tfrac{1}{m} L^{\mu} \partial_{\mu})^2 - \tfrac{c}{m} L^{\mu} \partial_{\mu} \,|\, g) \times \tag{18}$$
$$\times D_m(x-y), \qquad 0 < c < 1/2$$

(the commutator between two ϕ vanishing identically). It is easy to check that the free field ϕ so defined satisfies all requirements of local quantum field theory (except finite componentness) and violates TCP.

x) *This example was constructed in a discussion the second named author had with Prof. J. Bialynicki-Birula.*

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