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Introduction

In previous publications (cf. [1,2] and references to previous work in [2]) we have introduced and discussed the two-dimensional expansions of relativistic amplitudes for binary scattering processes with respect to both independent kinematical parameters s and t (more precisely with respect to convenient combinations of these parameters). The amplitudes were decomposed into sums and integrals over the basis functions of irreducible representations of the homogeneous Lorentz group $O(3,1)$. In this approach the group $O(3,1)$ does not in general figure as the invariance group of the amplitude, but as the group of motions of the space of kinematical parameters.

In [2] we considered the relation between these two-dimensional expansions and the relativistic partial wave analysis [3,4] in terms of various little groups of the Poincaré group.

In this paper we wish to consider several futher features of the two-dimensional expansions. In particular, we shall consider expansions in the so-called H-system, corresponding to the reduction $O(3,1) \supset O(2,1) \supset O(2)$. This expansion can be considered as an integral transformation of the function $f(s,t)$ in velocity space (Mandelstam coordinates) to the function $A(\sigma, \ell)$ defined in the space of parameters - quantum numbers, which corresponds to the reduction mentioned above (The magnetic quantum number m of the smallest subgroup 0 (2) vanishes in the zero spin case). The next problem is to correlate the analytical properties of transform $A(\sigma, \ell)$ in the space of two complex variables (σ, ℓ) and the asymptotics of amplitude for real s or (and) $t \to \infty$. . In this manner we try to find the relation of relativistic expansions with Regge-pole theory and to see how moving Regge-poles are generated by the specific asymptotic behaviour of transform. This relation will be demonstrated for the potential scattering in Coulomb field.

Further we shall consider the specific case of elastic forward-direction scattering, when the 0(3, 1) group figures also as an invariance group, i.e. as the little group, corresponding to zero-vector momentum transfer. In this case we demand that the two-dimensional expansions for $t=0$ (and masses satisfying $m_1 = m_3$, $m_2 = m_4$) coincide with the Toller/3/ expansion for $t = 0$. This leads to an integral relation which must be satisfied by the Lorentz amplitudes (for arbitrary t). This ensures the possible[x] existence of a series of daughter poles/3-6/ in the complex angular momentum plane for $t=0$, corresponding to each "Lorentz pole". Making certain natural assumptions on the analytical properties of the Lorentz amplitudes, we obtain the same poles for arbitrary t, but these are fixed poles and not Regge trajectories. Thus we conclude, that we find no evidence for the existence of daughter Regge trajectories from pure kinematics.

In this paper we limit ourselves to the scattering of zero spin particles.

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x) Possible in the sense that they may be compensated by zeros of the expansion coefficients (transform).

2. The Reduction $O(3,1) \supset O(2,1)$ and the Behaviour of Amplitudes for Large Momentum Transfer

In this paper we shall concentrate on the H-system expansions, introduced in /1/. The explicit expressions for the basis functions in this system, as well as the direct and inverse expansion formulae are given in Applendix 1. These expansions can be interpreted in the following manner.

Let us write the usual Regge formula /7/ for the scattering amplitude (all four particles have zero spin):

$$f(s,t) = -\frac{1}{2i}\int_{\gamma-i\infty}^{\gamma+i\infty}(2\ell+1)\frac{1}{\sin\pi\ell}\,d\ell\; a(\ell,t)\,P_{\ell}(\operatorname{ch}\beta), \tag{1}$$

the connection between $\operatorname{ch}\beta$ and s,t is given e.g. in /2/ (if all masses are equal we have $\operatorname{ch}\beta = -1 - \frac{2s}{t-4m^2}$). For $\gamma = -\frac{1}{2}$, formula (1) is an expansion over the unitary representations of the little group $O(2,1)$ corresponding to the fixed value $t < 0$. In general γ is such, that $a(\ell,t)$ is holomorphic for $\operatorname{Re} l > \gamma$. Further, let us expand the Regge partial amplitude $a(\ell,t)$ into an integral, such that on substitution into (1), we obtain a double expansion over irreducible representations of the Lorentz group $O(3,1)$:[x]

$$a(\ell,t) = -\frac{i}{8\pi\sqrt{2}}\cos\pi\ell\int_{\delta-i\infty}^{\delta+i\infty}(\sigma+1)\,d\sigma\,\frac{\Gamma(\sigma-\ell+1)\Gamma(\sigma+\ell+2)}{\Gamma(\sigma+1)}\frac{1}{\operatorname{ch}\alpha}$$

$$\cdot\{A^{+}(\sigma,\ell)[P_{\ell}^{-\sigma-1}(-\operatorname{th}\alpha)+P_{\ell}^{-\sigma-1}(\operatorname{th}\alpha)] + A^{-}(\sigma,\ell)[P_{\ell}^{-\sigma-1}(-\operatorname{th}\alpha)-P_{\ell}^{-\sigma-1}(\operatorname{th}\alpha)]\} =$$

$$= \int_{\delta-i\infty}^{\delta+i\infty}\chi(\sigma,\ell,t)\,d\sigma \tag{2}$$

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x) Note the appearance of two functions A^{+} and A^{-}. The H-coordinates cover (for $\alpha > 0$) only one half of the upper sheet of hyperboloid ($u_3 > 0$) and also have to use the one half of the lower sheet. This leads to the introduction of two amplitudes, odd and even in $\operatorname{th}\alpha$ correspondingly.

The general connection between α and s, t is also given in/2/ and for all equal masses we have $\mathrm{sh}^2\alpha = -\frac{t}{4m^2}$.

Substituting (2) into (1) for $\gamma = -\frac{1}{2}$ and $\delta = -1$ we obtain an expansion of $f(s,t)$ in terms of the principal series of unitary representations of the Lorentz group. In this case ($\ell = -\frac{1}{2} + iq$, $\sigma = -1 + ip$) we can write the following inverse formulae

$$a(\ell, t) = a(\ell, \alpha) = -\cos\pi\ell \int_0^\infty f(\alpha, \beta) P_\ell(\mathrm{ch}\beta)\,\mathrm{sh}\beta\, d\beta \tag{3}$$

and

$$A^{\pm}(\sigma, \ell) = -\sqrt{2}\,\frac{\Gamma(-\sigma-\ell-1)\Gamma(-\sigma+\ell)}{\Gamma(-\sigma)\cos\pi\ell} \int_{-\infty}^{\infty} \mathrm{ch}\,\alpha\, d\alpha\, a(\ell, \alpha) \cdot$$
$$\cdot \left[P_\ell^{\sigma+1}(-\mathrm{th}\,\alpha) \pm P_\ell^{\sigma+1}(\mathrm{th}\,\alpha) \right] \tag{4}$$

If we try to go over to the nonunitary representations we have to be more careful with the convergence of (4).

Using the expressions for the Legendre Functions near the singular points we have

$$P_\ell^{\sigma+1}(\pm\,\mathrm{th}\,\alpha) \underset{\alpha\to\pm\infty}{\to} \frac{1}{\Gamma(-\sigma)} e^{\pm\alpha(\sigma+1)}$$

$$P_\ell^{\sigma+1}(\pm\,\mathrm{th}\,\alpha) \xrightarrow[\substack{\alpha\to\mp\infty \\ \mathrm{Re}(\sigma+1)>0}]{} -\frac{\sin\pi\ell}{\pi}\Gamma(\sigma+1)\, e^{\mp\alpha(\sigma+1)} \tag{5}$$

$$P_\ell^{\sigma+1}(\pm\,\mathrm{th}\,\alpha) \xrightarrow[\substack{\alpha\to\mp\infty \\ \mathrm{Re}(\sigma+1)<0}]{} \frac{\Gamma(-\sigma-1)}{\Gamma(-\sigma+\ell)\;\Gamma(-\sigma-\ell-1)} e^{\pm\alpha(\sigma+1)} \;. \tag{5}$$

Substituting (5) into (4) we obtain the convergence condition

$$|a(\ell, \alpha)|_{|\alpha|\to\infty} < e^{-(1+|\mathrm{Re}\,\sigma+1|)|\alpha|} \tag{6}$$

Naturally $|\alpha| \to \infty$ implies $t \to -\infty$ and for equal masses we obtain the following restriction on the behaviour at infinite momentum transfer

$$|a(\ell, \alpha)| \equiv |a(\ell, t)| \underset{t\to-\infty}{<} (-t)^{-\frac{1+|\mathrm{Re}\,\sigma+1|}{2}} \tag{7}$$

For the unitary case $\mathrm{Re}\,\sigma + 1 = 0$ we face the weakest condition. In general case we have to choose not too large $|\mathrm{Re}\,\sigma + 1|$, in order to satisfy (7). As an example we take the nonrelativistic Coulomb amplitude (cf. Sec. 4). In this case $a(\ell, \alpha)_{\alpha\to\infty} = e^{-2\alpha}$ and (6) is satisfied.

If the integral (4) diverges the inverse transformation must be modified. This case will be treated separately.

3. Lorentz Amplitudes and Regge Poles

The function $a(\ell, t)$ can have a singularity if the integral (2) diverges. This can be due either to the behaviour of the integrated function for $|\sigma| \to \infty$, or to singularities lying on the integration path for finite σ . Let us consider these two cases separately.

a) "Lorentz poles" and fixed poles in the complex l-plane

Let us first consider singularities in the l-plane generated by singularities of the function $\chi(\sigma, \ell, t)$ in (2) lying on the integration path for finite $|\sigma|$. The Legendre functions $P^{\mu}_{\nu}(z)$ are analytical functions of ν and μ for all finite values of these indices (and have an essential singularity for $|\nu| \to \infty$ or $|\mu| \to \infty$). Thus $\chi(\sigma, \ell, t)$ can become infinite only if either the "normalization" Γ-functions $\Gamma(\sigma - \ell + 1)$, $\Gamma(\sigma + \ell + 2)$ have poles on the integration path, or if the Lorentz amplitudes $A^{\pm}(\sigma, \ell)$ have such singularities[x]. Since the only t-dependent (a-dependent) parts of $\chi(\sigma, \ell, t)$ are the functions $P_{\ell}^{-\sigma-1}(\mp \text{th}\, a)$ it is evident that the positions of the singularities of $a(\ell, t)$ generated by the singularities of $\chi(\sigma, \ell, t)$ at finite $|\sigma|$ do not depend on t i.e, these are fixed singularities in the complex ℓ-plane.

Fixed poles in the ℓ-plane if they do exist will dominate the high energy behaviour of the scattering amplitude for such a t for which they lie on the right-hand-side of the moving poles. Their properties can also be investigated using the integral representation (2) and we shall give examples of this below.

b) Lorentz Asymptotics and Regge trajectories

Moving poles of $a(\ell, t)$ (Regge trajectories) can only arise in (2) as a result of the integration over the infinite parts of the integration path. In order to investigate moving Regge poles we have to consider the Lorentz asymptotics, i.e. $A^{\pm}(\sigma, \ell)$ for $|\sigma| \to \infty$ and to evaluate the contribution of these asymptotics to $a(\ell, t)$.

To do this, let us write the integral in (2) as a sum of three terms

$$a(\ell, t) = \int_{\delta - i\infty}^{\delta - iN} \chi(\sigma, \ell, t)\, d\sigma + \int_{\delta - iN}^{\delta + iN} \chi(\sigma, \ell, t)\, d\sigma + \int_{\delta + iN}^{\delta + i\infty} \chi(\sigma, \ell, t)\, d\sigma = J_1 + J_2 + J_3, \quad (8)$$

x) The existence of singularities of the integrated functions is necessary, but not sufficient for the existence of singularities of $a(\ell, t)$.

where $N \gg 1$ is so large, that we can replace the integrated functions in J_1 and J_3 by their asymptotic expressions. The integral J_2 can only generate fixed singularities, which have already been discussed.

In order to estimate J_1 and J_3 we need the expression for $P_{\ell}^{-\sigma-1}(\mp \text{th}\, a)$ for $\text{Im}\, \sigma \to \pm\infty$, $\text{Re}\, \sigma$ fixed. Standard textbooks only list the asymptotic expressions for $\text{Re}\, \sigma \to \infty$, the expression which we need is obtained in Appendix II, namely

$$P_{\ell}^{\sigma+1}(\epsilon\, \text{th}\, a) \underset{|\text{Im}(\sigma+1)| \to \infty}{\approx} \frac{e^{-ip}}{\sqrt{2\pi}} (-ip)^{\delta + \frac{1}{2} + ip}\, e^{\epsilon(\delta + 1 + ip)a},$$

$$P_{\ell}^{-\sigma-1}(\epsilon\, \text{th}\, a) \underset{|\text{Im}(\sigma+1)| \to \infty}{\approx} \frac{e^{ip}}{\sqrt{2\pi}} (ip)^{-\delta - \frac{3}{2} - ip}\, e^{-\epsilon(\delta + 1 + ip)a}, \quad (9)$$

where $\sigma = \delta + ip$. Formula (9) holds for arbitrary finite a, including complex values.

Let us consider the integral J_3 :

$$J_3(\ell, a) = \frac{i \cos \pi\ell}{8\pi\sqrt{2}} \int_N^{\infty} p\, dp\, \frac{1}{\text{ch}\, a} \{ [A^{+}(\delta + ip, \ell) + A^{-}(\delta + ip, \ell)]\, e^{(\delta + 1 + ip)a} + \quad (10)$$

$$+ [A^{+}(\delta + ip, \ell) - A^{-}(\delta + ip, \ell)]\, e^{-(\delta + 1 + ip)a} \}$$

To continue the calculation we need some knowledge concerning the asymptotic behaviour of the Lorentz amplitudes $A^{\pm}(\sigma, \ell)$ i.e. we have to make certain physical assumptions, which will of course depend on the actual dynamics of the process.

As quite an obvious example,/9/ which illustrates the generation of Regge trajectories by Lorentz asymptotics and may be also the usefulness of the suggested approach, let us consider the following assumption:

$$A^{+}(\sigma,\ell)+A^{-}(\sigma,\ell)\xrightarrow[p\to+\infty]{}\frac{e^{ipf_1(\ell)}}{p},$$

$$A^{+}(\sigma,\ell)-A^{-}(\sigma,\ell)\xrightarrow[p\to+\infty]{}\frac{e^{ipf_2(\ell)}}{p},\qquad (11)$$

where $f_1(\ell)$ and $f_2(\ell)$ are arbitrary functions such that

$$\operatorname{Im} f_i(\ell)>0,\quad i=1,2. \qquad (12)$$

Substituting (11) into (10) we obtain

$$J_3(\ell,a)=-\frac{\cos\pi\ell}{8\pi\sqrt{2}}\,\frac{1}{\operatorname{ch}a}\left\{\frac{e^{(\delta+1)a}\,e^{iN[f_1(\ell)+a]}}{f_1(\ell)+a}+\frac{e^{-(\delta+1)a}\,e^{iN[f_2(\ell)-a]}}{f_2(\ell)-a}\right\}. \qquad (13)$$

Thus the assumption (11) implies the existence of simple Regge poles, the trajectories of which can be obtained by solving the equations

$$f_1(\ell)+a=0$$

$$f_2(\ell)-a=0 \qquad (14)$$

with respect to ℓ .

The residues in these Regge poles can also be obtained directly from (13). The integral $J_1(\ell,a)$ can be considered in a completely analogous manner.

It is of course possible to make more general assumptions concerning the asymptotic behaviour of $A^{\pm}(\sigma,\ell)$. For instance, we can directly obtain multiple Regge poles, by putting for example

$$\overset{+}{A}(\sigma,\ell)+\overset{-}{A}(\sigma,\ell)\xrightarrow[p\to+\infty]{} p^{n}\, e^{ip\, f_1(\ell)} . \qquad (15)$$

4. An Example. Regge Trajectories from Lorentz Asymptotics for the Coulomb Potential

As a mathematical example illustrating the method, suggested in the previous section, let us consider the scattering of a charged particle in a non-relativistic Coulomb field.

The partial wave amplitude, obtained by solving the Schrödinger equation, can be written as/10/.

$$a(\ell,\alpha)=-\frac{1}{2m\,\mathrm{ch}\,\alpha}\left[\frac{\Gamma\left(\ell+1-\frac{e^2}{2m\,\mathrm{ch}\,\alpha}\right)}{\Gamma\left(\ell+1+\frac{e^2}{2m\,\mathrm{ch}\,\alpha}\right)}-1\right], \qquad (16)$$

where we have put the energy E equal to

$$E=-m^2\,\mathrm{ch}^2\alpha=\frac{t}{4}-m^2 . \qquad (17)$$

In this section we shall apply the relativistic expansions (1) - (4) and the relativistic kinematical variables α and β for amplitudes, obtained from a non-relativistic equation. There is no formal contradiction in this procedure, specially if we consider it only as a soluable model, demonstrating the generation of Regge poles by Lorentz asymptotics. We make no use of the 0(3,1) symmetry/11/ of the Coulomb scattering problem, which deserves separate attention in this context.

The amplitude $a(\ell,\alpha)$ satisfies the convergence condition (6) with $\mathrm{Re}\,\sigma=-1$ so that we shall put $\sigma=-1+ip$ (since we shall consider arbitrary complex ℓ , this is not equivalent to decompositions in terms of unitary representations of the Lorentz group.) In Principle, the Lorentz amplitudes $A^{\pm}(\sigma,\ell)$ for the Coulomb scattering can be obtained by substituting (16) into (4). However, there is no need to perform the integration explicitly, since we only need the leading term for $p \to \pm\infty$.

Using (4) and (9) we obtain

$$A^{\pm}(\sigma,\ell) \underset{|p|\to\infty}{\approx} \frac{\sqrt{2}\, i}{\cos\pi\ell\, p}\int_{-\infty}^{\infty} \mathrm{ch}\,\alpha\, d\alpha\, a(\ell,\alpha)\left[\mp e^{(\delta+1+ip)\alpha} - e^{-(\delta+1+ip)\alpha}\right] \qquad (18)$$

To evaluate the leading term in the Lorentz asymptotic it is not even necessary to calculate this integral. Indeed, we can apply the method used in Regge pole theory, namely shifting the integration path in (18) by adding an imaginary part to α in such a manner that the value of the integral for $|p| \to \infty$ decreases as $|\mathrm{Im}\,\alpha|$ increases. We can then replace the integral by a sum over the residues of the integrated function in the poles, crossed while shifting the path.

For the Coulomb amplitude this gives

$$A^{\pm}(p,\ell) \underset{p\to\pm\infty}{\approx} -\frac{i}{\sqrt{2}\cos\pi\ell\, mp}\int_{-\infty}^{+\infty} d\alpha\left[\frac{\Gamma\left(\ell+1-\frac{e^2}{2m\,\mathrm{ch}\,\alpha}\right)}{\Gamma\left(\ell+1+\frac{e^2}{2m\,\mathrm{ch}\,\alpha}\right)} - 1\right]\times \qquad (19)$$

$$\times\left[\mp e^{ip\alpha} - e^{-ip\alpha}\right] \approx \frac{i}{\sqrt{2}\cos\pi\ell\, mp}\left\{\sum_n \int_{c_n^-} d\alpha\, \frac{(-1)^n}{n!(\ell+1+n)\Gamma(2\ell+2+n)}\times\right.$$

$$\left.\times \frac{\mathrm{ch}\,\alpha}{\mathrm{sh}\,\alpha_n^-(\alpha-\alpha_n^-)}e^{-ip\alpha} \pm \sum_n \int_{c_n^+} d\alpha\, \frac{(-1)^n}{n!(\ell+1+n)\Gamma(2\ell+2+n)}\, \frac{\mathrm{ch}\,\alpha}{\mathrm{sh}\,\alpha_n^+(\alpha-\alpha_n^+)}e^{ip\alpha}\right\}$$

Here $\mathrm{ch}\,\alpha_n = \dfrac{e^2}{2m(\ell+1+n)}$, $\alpha_n^- = -\alpha_n^+ = \alpha_n$, $\mathrm{Im}\,\alpha_n < 0$.

$$\frac{1}{\mathrm{ch}\,\alpha - \frac{e^2}{2m(\ell+1+n)}} \approx \frac{1}{\mathrm{sh}\,\alpha_n(\alpha-\alpha_n)}, \qquad (20)$$

$c_n^{\pm}$ – closed contours, containing one pole $\alpha_n^{\pm}$.

Further we have

$$A^{\pm}(p,\ell) \underset{p\to\infty}{\approx} \frac{1}{\sqrt{2}\cos\pi\ell\, mp}\sum_n \frac{(-1)^n\, 2\pi i}{n!(\ell+1+n)\Gamma(2\ell+2+n)}\times$$

$$\times\left[-\mathrm{cth}\,\alpha_n^-\, e^{-ip\alpha_n^-} \pm \mathrm{cth}\,\alpha_n^+\, e^{ip\alpha_n^+}\right].$$

Performing the analogous procedure for $p \to -\infty$, we obtain

$$A^{+}(p,\ell) \underset{p\to\pm\infty}{\approx} \frac{4\pi}{\sqrt{2}\cos\pi\ell\, mp}\sum_n \frac{(-1)^n\, \mathrm{cth}\,\alpha_n}{n!(\ell+1+n)\Gamma(2\ell+2+n)}\, e^{\mp ip\alpha_n}, \qquad (21)$$

$$A^{-}(p,\ell) \underset{|p|\to\infty}{\approx} 0,$$

i.e. nonzero terms in A^- vanish more rapidly than those in A^+ when $|p|$ goes to infinity.

Substituting (21) into (10) and (8) with $J_1 = J_3$, we get

$$a(\ell,\alpha) \approx \frac{i\cos\pi\ell}{8\pi\sqrt{2}\,\mathrm{ch}\,\alpha}\int_N^{\infty} p\,dp\, \frac{4\pi}{\sqrt{2}\cos\pi\ell\, mp}\sum_n \frac{(-1)^n\,\mathrm{cth}\,\alpha_n}{n!(\ell+1+n)\Gamma(2\ell+2+n)}\cdot$$

$$2\left[e^{ip(\alpha-\alpha_n)} + e^{-ip(\alpha+\alpha_n)}\right] \approx \qquad (22)$$

$$\approx \frac{(-1)}{2m\,\mathrm{ch}\,\alpha}\sum_n \frac{(-1)^n\,\mathrm{cth}\,\alpha_n}{n!(\ell+1+n)\Gamma(2\ell+2+n)}\left[\frac{1}{\alpha-\alpha_n} - \frac{1}{\alpha+\alpha_n}\right].$$

So the knowledge of the leading term in the Lorentz asymptotics, gives us exactly the Coulomb amplitude in the vicinity of its poles.

5. Elastic Scattering at $t=0$ and Daughter Poles

In the previous sections we have shown how fixed and moving poles of the Regge partial amplitude are generated by singularities of the Lorentz amplitudes. We shall now apply these considerations to the problem of daughter Regge poles which have been discussed e.g. in/ 3-6/ in connection with the $0(3,1)$ symmetry of elastic scattering amplitudes for forward scattering. To clarify the problem, let us briefly repeat the group theoretical arguments leading to daughter Regge poles. The presentation is similar to that of Sciarrino and Toller/ 3/ but is more transparent since we only consider particles with zero spin.

For elastic forward direction scattering in the process $1+2\to 3+4$ (the masses satisfy $m_1=m_3$, $m_2=m_4$), the four-dimensional momentum transfer is a null vector $p_1-p_3=p_4-p_2=(0,0,0,0)$. In this case crossed channel partial wave analysis/ 3,4/ implies, that we write the scattering amplitude for $t=0$ as a function on the little group $0(3,1)$, and develop it in terms of the reduced transformation matrix elements/ 3,12/ $D^{\nu\rho}_{jj'\lambda}(\beta)$. Explicit expressions for $D^{\nu\rho}_{jj'\lambda}(\beta)$ are given in/ 3,12 / and elsewhere, for the scattering of zero spin particles we have

$$D^{0\rho}_{000}(\beta)=\sqrt{\frac{\pi}{2}}\,\frac{1}{\sqrt{\operatorname{sh}\beta}}\,P^{-\frac{1}{2}}_{-\frac{1}{2}+i\frac{\rho}{2}}(\operatorname{ch}\beta)=\frac{2}{\rho}\,\frac{\sin\frac{\rho\beta}{2}}{\operatorname{sh}\beta}\,, \qquad (23)$$

where β is the same variable that figures in the H-system expansion.

Thus in the considered case we can write

$$f(s,t)|_{t=0}=f(s,0)=\int_0^\infty \rho^2 d\rho\, a(\rho)\, D^{0\rho}_{000}(\beta). \qquad (24)$$

On the other hand, Regge pole theory is connected with expansions in terms of $0(2,1)$ representations and we can decompose the $0(3,1)$

representations with respect to the $0(2,1)$ subgroup. For spinless particles this is equivalent to putting

$$D^{0\rho}_{000}(\beta)=\int_0^\infty q\,\operatorname{th}\pi q\,dq\,C(\rho,q)\,P_{-\frac{1}{2}+iq}(\operatorname{ch}\beta). \qquad (25$$

Using the orthogonality property

$$\int_0^\infty P_{-\frac{1}{2}+iq}(\operatorname{ch}\beta)P_{-\frac{1}{2}+iq'}(\operatorname{ch}\beta)\operatorname{sh}\beta\, d\beta=\frac{1}{q\,\operatorname{th}\pi q}\,\delta(q-q'), \qquad (26$$

we obtain

$$C(\rho,q)=\frac{2}{\rho}\int_0^\infty \sin\frac{\rho\beta}{2}\,P_{-\frac{1}{2}+iq}(\operatorname{ch}\beta)\,d\beta \qquad (27$$

The integral (27) is calculated in Appendix III and we finally obtai

$$f(s,0)=\frac{1}{2\pi^2}\int_0^\infty a(\rho)\operatorname{sh}\frac{\pi\rho}{2}\,\rho\, d\rho\int_0^\infty q\,\operatorname{th}\pi q\, dq\,\Gamma\Big(\frac{\frac{1}{2}+iq-\frac{i\rho}{2}}{2}\Big)$$

$$\times\Gamma\Big(\frac{\frac{1}{2}-iq-\frac{i\rho}{2}}{2}\Big)\Gamma\Big(\frac{\frac{1}{2}-iq+\frac{i\rho}{2}}{2}\Big)\Gamma\Big(\frac{\frac{1}{2}+iq+\frac{i\rho}{2}}{2}\Big)P_{-\frac{1}{2}+iq}(\operatorname{ch}\beta \qquad (28$$

This is an expansion of $f(s,0)$ in terms of unitary representations. Since we wish to consider Lorentz poles in the complex ρ-plane and Regge poles in the complex ℓ-plane we need a more general expansion.

Putting $\sigma=-1+\frac{i\rho}{2}$, the integral (24) can be written in the form

$$f(s,0)=4i\sqrt{\frac{\pi}{2}}\int_{-1-i\infty}^{-1+i\infty}(\sigma+1)^2 d\sigma\, a(\sigma)\,\frac{1}{\sqrt{\operatorname{sh}\beta}}\,P^{-\frac{1}{2}}_{\frac{1}{2}+\sigma}(\operatorname{ch}\beta)= \qquad (29a$$

$$= \frac{8}{\sqrt{2\pi}} \int_{-1-i\infty}^{-1+i\infty} (\sigma+1)^2 \, d\sigma \, a(\sigma) \frac{1}{\sqrt{\mathrm{sh}\,\beta}} Q^{-\frac{1}{2}}_{\frac{1}{2}+\sigma} (\mathrm{ch}\,\beta). \qquad (29b)$$

Now let us generalize this expression to non-unitary representations (so that it will be possible to expand functions $f(s,0)$ that are not square integrable with respect to β). This procedure is somewhat arbitrary e.g. because (29a) and 29b) are equivalent only on the unitary integration path. However, let us generalize (29b):

$$f(s,0) = \frac{8}{\sqrt{2\pi}} \int_{\delta-i\infty}^{\delta+i\infty} (\sigma+1)^2 \, d\sigma \, a(\sigma) \frac{1}{\sqrt{\mathrm{sh}\,\beta}} Q^{-\frac{1}{2}}_{\frac{1}{2}+\sigma} (\mathrm{ch}\,\beta) \qquad (30)$$

and let us shift the integration path to the right, assuming that $a(\sigma)$ is a meromorphous function:

$$f(s,0) = \frac{8}{\sqrt{2\pi}} \int_{\delta'-i\infty}^{\delta'+i\infty} (\sigma+1)^2 \, d\sigma \, a(\sigma) \frac{1}{\sqrt{\mathrm{sh}\,\beta}} Q^{-\frac{1}{2}}_{\frac{1}{2}+\sigma} (\mathrm{ch}\,\beta) + \qquad (31)$$
$$\delta' \to +\infty$$
$$+ 2\pi i \frac{8}{\sqrt{2\pi}} \sum_k \rho(\sigma_k)(\sigma_k+1)^2 \frac{1}{\sqrt{\mathrm{sh}\,\beta}} Q^{-\frac{1}{2}}_{\frac{1}{2}+\sigma_k} (\mathrm{ch}\,\beta),$$

where $\rho(\sigma_k)$ are residues of $a(\sigma)$ in the "Toller poles". For large β (asymptotic energies) the pole terms will dominate the integral.

In Appendix III we express the "Lorentz" function $P^{-\frac{1}{2}}_{-\frac{1}{2}+\frac{i\rho}{2}}(\mathrm{ch}\beta)$ in terms of the "Regge" function $P_{-\frac{1}{2}+iq}(\mathrm{ch}\,\beta)$.. Generalizing this expression to non-unitary representations, i.e. complex ρ and q (the rigour of this procedure is questionable), writing both the P-functions in terms of Q-functions, approximating the integral by a series

of poles and comparing the asymptotic expressions for the individual terms (cf./ 3/), we obtain the asymptotic expansion

$$\frac{1}{\sqrt{\mathrm{sh}\,\beta}} Q^{-\frac{1}{2}}_{\frac{1}{2}+\sigma} (\mathrm{ch}\,\beta) \underset{\mathrm{ch}\,\beta \to \infty}{\approx} \frac{1}{\sqrt{2\pi}} \sum_n \left(\sigma + \frac{3}{2} + 2n\right) \frac{\Gamma(n+\frac{1}{2})\Gamma(-\sigma-n-1)}{\Gamma(-\sigma)\Gamma(\sigma+2)} \times \qquad (32)$$
$$\times \Gamma\left(\sigma + \frac{3}{2} + n\right) \frac{(-1)^n}{n!} Q_{\sigma+1+2n} (\mathrm{ch}\,\beta).$$

Substituting (32) into (31) we finally obtain the decomposition of "Toller pole" contributions to $f(s,0)$ into series of daughter Regge poles:

$$f(s,0) = \frac{8}{\sqrt{2\pi}} \int_{\delta-i\infty}^{\delta+i\infty} (\sigma+1)^2 \, d\sigma \, a(\sigma) \frac{1}{\sqrt{\mathrm{sh}\,\beta}} Q^{-\frac{1}{2}}_{\frac{1}{2}+\sigma} (\mathrm{ch}\,\beta) - \qquad (33)$$

$$- 8 \sum_k \rho(\sigma_k)(\sigma_k+1) \sum_n \left(\sigma_k + \frac{3}{2} + 2n\right)(-1)^n \frac{\Gamma(n+\frac{1}{2})\Gamma(-\sigma_k-n-1)\Gamma(\sigma_k+n+\frac{3}{2})}{n!\,\Gamma(-\sigma_k)\Gamma(\sigma_k+1)} Q_{\sigma_k+1+2n} (\mathrm{ch}\beta)$$

In the framework of one-dimensional little group expansions there is no way to determine, whether these daughter poles are fixed poles or the intersections of Regge trajectories with the axis $t=0$. We shall demonstrate in the following section, that the same families of daughter poles are generated by Lorentz poles in the normalization Γ-functions in the two-dimensional O(3,1) expansions, so that according to the arguments of § 3 they are fixed poles in the complex l-plane.

6. Two-Dimensional Expansions for Elastic Scattering at $t=0$ and the O(3,1) Little Group

An application of the O(3,1) little group expansion, reduced to the O(2,1) subgroup, at $t=0$ for equal masses leads to relation (28). On the other hand we can directly apply the H-system decomposition of/ 1/ to this case i.e. put $a=0$ (and $\phi=0$) in the formulae of Appendix I.

Thus we obtain

$$f(s,0)=\frac{\sqrt{2}}{8}\int_0^\infty \rho^2 d\rho \int_0^\infty q\,\mathrm{th}\,\pi q\,dq\,A^+(\rho,q)\frac{1}{2^{i\frac{\rho}{2}+2}\pi^{5/2}}\times$$

$$\times(\mathrm{ch}\,\pi q-i\,\mathrm{sh}\,\pi\frac{\rho}{2})\Gamma\left(\frac{\frac{1}{2}-iq-\frac{i\rho}{2}}{2}\right)\Gamma\left(\frac{\frac{1}{2}+iq-\frac{i\rho}{2}}{2}\right)\Gamma\left(\frac{1}{2}+iq+i\frac{\rho}{2}\right)\times \qquad (34)$$

$$\times\frac{\Gamma\left(\frac{1}{2}-iq+\frac{i\rho}{2}\right)}{\Gamma\left(1+\frac{i\rho}{2}\right)}P_{-\frac{1}{2}+iq}(\mathrm{ch}\,\beta)$$

Comparing (34) and (28) we find that the two expansions coincide, if the coefficients satisfy an integral relation

$$\int_0^\infty \rho^2 d\rho\, 2^{i\frac{\rho}{2}}\frac{\Gamma\left(\frac{\frac{1}{2}+iq+\frac{i\rho}{2}}{2}\right)\Gamma\left(\frac{\frac{1}{2}-iq+\frac{i\rho}{2}}{2}\right)}{\Gamma\left(1+\frac{i\rho}{2}\right)}\{A^+(\rho,q)- \qquad (35)$$

$$-a(\rho)\frac{i}{\sqrt{2\pi}}\mathrm{sh}\frac{\pi\rho}{2}\cdot 2^{-i\frac{\rho}{2}+3}\,\Gamma\left(\frac{i\rho}{2}\right)\Gamma\left(\frac{\frac{1}{2}-iq-\frac{i\rho}{2}}{2}\right)\Gamma\left(\frac{\frac{1}{2}+iq-\frac{i\rho}{2}}{2}\right)\}=0$$

Thus, the fact that for $t=0$ (and masses satisfying $m_1=m_3$, $m_2=m_4$) the group $O(3,1)$ is an invariance group of the amplitude, posess the restriction (35) on the "symmetrical Lorentz amplitude" $A^+(\rho,q)$ in the H-system.

Now let us proceed to the case of non-unitary representations. The H-system expansion in terms of unitary representations for $t=0$ can also be written as

$$f(s,0)=-\frac{i}{\pi^{5/2}\sqrt{2}}\int_{\delta-i\infty}^{\delta+i\infty}2^\sigma\frac{d\sigma}{\Gamma(\sigma)}\int_{\gamma-i\infty}^{\gamma+i\infty}(2\ell+1)d\ell\,\Gamma(\sigma-\ell+1)\Gamma\left(\frac{\sigma+\ell+2}{2}\right)\times \qquad (36)$$

$$\times A^+(\sigma,\ell)Q_{-\ell-1}(\mathrm{ch}\,\beta)$$

with $\delta=-1$ and $\gamma=-\frac{1}{2}$. Let us now generalize this expression to arbitrary γ and δ and show how this expansion leads to the same results, as the little group expansion of the preceding section. Indeed, consider large energies, i.e. $\mathrm{ch}\,\beta \to \infty$. Let us shift both the δ and ℓ integration paths simultaneously to the left (so that we do not cross any poles of the Γ-functions). Assuming that $A^{+}(\sigma, \ell)$ is meromorphic with respect to σ and holomorphous with respect to ℓ in some strip, the integral over the new path will be dominated for large β by the residues in the poles of $A^{+}(\sigma, \ell)$:

$$f(s,0) = (2\pi i)\frac{(-i)}{\pi^{5/2}\sqrt{2}} \sum_k \frac{2^{\sigma_k}}{\Gamma(\sigma_k)} \int_{\gamma'-i\infty}^{\gamma'+i\infty} (2\ell+1)\,d\ell \; R^{+}(\sigma_k, \ell) \times \tag{37}$$

$$\times \Gamma\left(\frac{\sigma_k - \ell + 1}{2}\right) \Gamma\left(\frac{\sigma_k + \ell + 2}{2}\right) Q_{-\ell-1}(\mathrm{ch}\,\beta),$$

where $R^{+}(\sigma_k, \ell)$ is the residue of $A^{+}(\sigma, \ell)$ in the k-th pole. Now let us shift the remaining integration path to the left, assuming that $R^{+}(\sigma_k, \ell)$ is holomorphous and only keep the terms, due to poles of the Γ-functions, lying to the left of the path $\mathrm{Re}\,\ell = \gamma'$

Finally we obtain

$$f(s,0) = \frac{(-8i)}{\sqrt{\pi}} \sum_k \frac{2^{\sigma_k}}{\Gamma(\sigma_k)} \sum_n \left(\sigma_k + 2n + \frac{3}{2}\right) R^{+}(\sigma_k, -\sigma_k - 2 - 2n) \times$$

$$\times \frac{(-1)^n}{n!} \Gamma\left(\sigma_k + \frac{3}{2} + n\right) Q_{\sigma_k + 1 + 2n}(\mathrm{ch}\,\beta)$$

Thus, in complete analogy with Toller et al./3/ we have obtained a series of Lorentz poles each of which gives rise to an infinite sequence of daughter poles. However since these poles have nothing to do with the Lorentz asymptotics discussed in section 3, we conclude that they are fixed poles in the ℓ-plane. Further we can relate the residue $R^{+}(\sigma_k, -\sigma_k - 2 - 2n)$ to the residue $\rho(\sigma_k)$ of the Toller amplitude.

Let us add a few remarks. In the derivation of (38) we have assumed certain analyticity properties of $A^{+}(\sigma, \ell)$ and $R^{+}(\sigma_k, \ell)$..

If these are not satisfied, e.g. if these functions are not holomorphous with respect to ℓ, we still reproduce the result (38) but obtain additional non-daughterlike terms.

So far the unitarity condition for the scattering amplitude has not been exploited in the framework of two-dimensional relativistic expansions. It is quite possible that unitarity will pose restrictions on the Lorentz amplitudes, excluding the generation of fixed poles (including fixed daughter poles) in the complex ℓ-plane (confront also /13/).

7. Conclusions

The main results of the present paper are the following:

1) Two-dimensional expansions of relativistic amplitudes should be a useful supplement to conventional Regge pole theory.

2) The coefficients in these expansions, which we call Lorentz amplitudes $A^{\pm}(\sigma,\ell)$ determine the behaviour of the Regge partial amplitude $a_\ell(t)$ in the vicinity of its singularities. Namely certain asymptotic expressions for $A^{\pm}(\sigma,\ell)$ generate moving Regge singularities, whereas "Lorentz poles" of $A^{\pm}(\sigma,\ell)$ for finite σ generate fixed poles in the l-plane.

3) The condition, that the two-dimensional expansions should coincide with the 0(3,1) little group expansion for $t=0$ in the equal mass case, implies that $A^{+}(\sigma,\ell)$ must satisfy a certain integral condition.

4) The daughter Regge poles introduced in /3,6/ for $t=0$ in the equal mass case, are generated by Lorentz poles for finite σ and it follows that they are fixed poles. Thus we find no evidence for daughter trajectories from pure kinematics, but we do not exclude possible dynamical mechanisms, forcing Regge trajectories to pass through the daughter poles.

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Appendix I

Expansions in the S and H systems.

Here we shall reproduce the expansions over the orthonormal sets of basis functions in the S and H systems and incidentally to remove some misprints made in /1/.

The spherical coordinate system corresponds to the reduction $O(3,1) \supset O(3) \supset O(2)$ (confront /2/). In the S -system the four-velocity is parametrized as

$$u=(u_0,u_1,u_2,u_3)=(\mathrm{ch}\, a\,,\ \mathrm{sh}\, a\ \sin\theta\cos\phi,\ \mathrm{sh}\, a\sin\theta\sin\phi,\ \mathrm{sh}\, a\cos\theta)\,. \tag{1.1}$$

The expansion of amplitude $f(u)$ in the general non-unitary case has the form

$$f(u)=\frac{i}{2}\sum_{\ell m}\int_{\delta-i\infty}^{\delta+i\infty}(\sigma+1)^2 d\sigma\, a_{\ell m}(\sigma)\frac{\Gamma(\sigma+1)}{\Gamma(\sigma+1-\ell)}\frac{1}{\sqrt{\mathrm{sh}\, a}}P^{-\ell-\frac{1}{2}}_{\sigma+\frac{1}{2}}(\mathrm{ch}\, a)Y_{\ell m}(\theta,\phi) \tag{1.2}$$

In the unitary case $\sigma=-1+ip$ this goes into the expansion over the orthonormalized set of functions

$$f(u)=\sum_{\ell m}\int_0^{\infty}a_{\ell m}(p)\,\Phi_{p\ell m}(a,\theta,\phi)\,p^2\,dp\ ,$$

$$(\Phi_{p'\ell'm'},\Phi_{p\ell m})=\frac{\delta(p-p')}{p^2}\delta_{\ell\ell'}\,\delta_{mm'}\ , \tag{1.3}$$

where

$$\Phi_{p\ell m}(a,\theta,\phi)=\frac{\Gamma(ip)}{\Gamma(ip-\ell)}\frac{1}{\sqrt{\mathrm{sh}\, a}}P^{-\ell-\frac{1}{2}}_{-\frac{1}{2}+ip}(\mathrm{ch}\, a)\,Y_{\ell m}(\theta,\phi)\,. \tag{1.4}$$

The inverse formula follows immediately

$$a_{\ell m}(p)=\int_0^{\infty}\mathrm{sh}^2 a\, da\int_0^{\pi}\sin\theta\, d\theta\int_0^{2\pi}d\phi\ f(u)\,\Phi^*_{p\ell m}(a,\theta,\phi)\,. \tag{1.5}$$

The hyperbolic coordinate system corresponds to the reduction $O(3,1) \supset O(2,1) \supset O(2)$ /2/. In the H-system the four-velocity is parametrized as

$$u=(\mathrm{ch}\, a\ \mathrm{ch}\beta,\ \mathrm{ch}\, a\,\mathrm{sh}\beta\cos\phi,\ \mathrm{ch}\, a\,\mathrm{sh}\beta\sin\phi\,,\ \mathrm{sh}\, a)\,. \tag{1.6}$$

We have the following expansion formula

$$f(u)=\frac{1}{8\pi}\sum_{m=-\infty}^{\infty}\int_{\delta-i\infty}^{\delta+i\infty}(\sigma+1)d\sigma\int_{\gamma-i\infty}^{\gamma+i\infty}(\ell+\frac{1}{2})\mathrm{ctg}\,\pi\ell\ d\ell\ \frac{\Gamma(\sigma-\ell+1)\Gamma(\sigma+\ell+2)}{\Gamma(\sigma+1)\Gamma(\ell+1)}\cdot$$

$$\cdot\frac{\Gamma(1+\ell-m)}{\mathrm{ch}\, a}[a^+_m(\sigma,\ell)P^{-\sigma-1}_{\ell}(-\mathrm{th}\, a)+a^-_m(\sigma,\ell)P^{-\sigma-1}_{\ell}(\mathrm{th}\, a)]\ \cdot \tag{1.7}$$

$$\cdot\ P^m_{\ell}(\mathrm{ch}\,\beta)e^{im\phi}$$

In the unitary case $\sigma=-1+ip$, $\ell=-\frac{1}{2}+iq$ this goes into

$$f(u)=\sum_m\int_0^{\infty}p^2dp\int_0^{\infty}q\ \mathrm{th}\,\pi q\ dq\,[a^+_m(p,q)\,\Phi^+_{pqm}(a,\beta,\phi)\ +$$

$$+\ a^-_m(p,q)\,\Phi^-_{pqm}(a,\beta,\phi)] \tag{1.8}$$

and the inverse formula takes the form

$$a^{\pm}_m(p,q)=2\int_{-\infty}^{+\infty}\mathrm{ch}^2 a\, da\int_0^{\infty}\mathrm{sh}\,\beta\, d\beta\int_0^{2\pi}d\phi\, f(u)\,\Phi^{\pm *}_{pqm}(a,\beta,\phi)\ , \tag{1.9}$$

where the basis functions

$$\Phi^{\pm}_{pqm}(\alpha,\beta,\phi) = \frac{\Gamma(\frac{1}{2}+iq+ip)\,\Gamma(\frac{1}{2}-iq+ip)}{2\pi\,\Gamma(1+ip)} \cdot \frac{\Gamma(\frac{1}{2}-m+iq)}{\Gamma(\frac{1}{2}+iq)}$$

$$\cdot \frac{1}{\operatorname{ch}\alpha} P^{-ip}_{-\frac{1}{2}+iq}(\mp \operatorname{th}\alpha)\, P^{m}_{-\frac{1}{2}+iq}(\operatorname{ch}\beta)\, e^{im\phi} \tag{1.10}$$

satisfy the orthonormalization condition

$$(\Phi^{\pm}_{p'q'm'}, \Phi^{\pm}_{pqm}) = \frac{\delta(p-p')}{p^2} \cdot \frac{\delta(q-q')}{q\,\operatorname{th}\pi q}\,\delta_{mm'}\,, \tag{1.11}$$

$$(\Phi^{\pm}_{p'q'm'}, \Phi^{\mp}_{pqm}) = 0$$

It may be convenient to define amplitudes $A^{\pm}_{m}(p,q)$:

$$a^{+}_{m}(p,q)\,\Phi^{+}_{pqm} + a^{-}_{m}(p,q)\,\Phi^{-}_{pqm} =$$

$$= A^{+}_{m}(p,q)\,\frac{\Phi^{+}_{pqm}+\Phi^{-}_{pqm}}{\sqrt{2}} + A^{-}_{m}(p,q)\,\frac{\Phi^{+}_{pqm}-\Phi^{-}_{pqm}}{\sqrt{2}}\,, \tag{1.12}$$

where

$$A^{\pm}_{m}(p,q) = \frac{a^{+}_{m}(p,q) \pm a^{-}_{m}(p,q)}{\sqrt{2}}\,.$$

Appendix II

Asymptotic formula for Legendre Functions.

Here we shall derive the asymptotic expression for

$P^{\pm(\sigma+1)}_{\ell}(\epsilon\,\operatorname{th}\alpha)$ for $\operatorname{Im}\sigma \to \pm\infty$, $\operatorname{Re}\sigma$ fixed

$P^{\mu}_{\nu}(z)$ denotes the Legendre function on the cut/14/ tha takes arbitrary finite complex values, and $\epsilon = \pm 1$. We begin with the relation between $P^{\mu}_{\nu}(z)$ and Gauss hypergeometric function

$$P^{\pm(\sigma+1)}_{\ell}(\epsilon\,\operatorname{th}\alpha) = \frac{e^{\pm\epsilon(\sigma+1)\alpha}}{\Gamma[1\mp(\sigma+1)]}\,{}_2F_1\left[-\ell,\ell+1;1\mp(\sigma+1);\frac{1-\epsilon\operatorname{th}\alpha}{2}\right] \tag{II. 1}$$

The following formulae are obtained by a necessary modification of those in/15/, so that they become applicable to our special case.

For the Gauss function we have asymptotic series, which can be used also for $|\operatorname{th}\alpha|>1$:

$${}_2F_1\left[-\ell,\ell+1;1\mp(\sigma+1);\frac{1-\epsilon\operatorname{th}\alpha}{2}\right] \cong$$

$$= \frac{\Gamma[1\mp(\sigma+1)]}{[\mp(\sigma+1)]^{-\ell}\,\Gamma[1\mp(\sigma+1)+\ell]}\left[1+\sum_{\chi=1}^{\infty}\frac{d_{\chi}\,\Gamma(-\ell+\chi)}{\Gamma(-\ell)[\mp(\sigma+1)]^{\chi}}\right] \tag{II. 2}$$

for $|\arg[1\mp(\sigma+1)+\ell]| < \frac{\pi}{2}$.

(for the evaluation of d_{χ} see/15/) and for

$$|\arg[\mp(\sigma+1)-\ell]| < \frac{\pi}{2}$$

the corresponding formula is obtained by the interchange $\ell \to -\ell-1$. Taking only the leading term of this asymptotic expansion we obtained

$$P_{\ell}^{\pm(\sigma+1)}(\epsilon \,\mathrm{th}\, a) \underset{|p|\to\infty}{\approx} \frac{1}{[\mp(\sigma+1)]^{-\ell}\,\Gamma[1\mp(\sigma+1)+\ell]}\, e^{\pm\epsilon(\sigma+1)a}, \qquad (\mathrm{II}.3)$$

where $\sigma = \delta + ip$.

The asymptotics of the Γ-functions we calculate leaving only non-vanishing terms in the Stirling series for $\ln\Gamma(z)$. So it follows that

$$\Gamma(-\sigma+\ell) \underset{|p|\to\infty}{\approx} \sqrt{2\pi}\, e^{ip}(-ip)^{-\delta+\gamma-\frac{1}{2}-i(p-q)},$$

$$(\mathrm{II}.4)$$

$$\Gamma(\sigma+\ell+2) \underset{|p|\to\infty}{\approx} \sqrt{2\pi}\, e^{-ip}(ip)^{\delta+\gamma+\frac{3}{2}+i(p+q)},$$

where $\ell = \gamma + iq$.

And finally we have the leading term in the form

$$P_{\ell}^{\pm(\sigma+1)}(\epsilon \,\mathrm{th}\, a) \underset{|p|\to\infty}{\approx} \frac{e^{\mp ip}}{\sqrt{2\pi}}(\mp ip)^{\pm(\delta+1)-\frac{1}{2}\pm ip}\, e^{\pm\epsilon(\delta+1+ip)a}. \qquad (\mathrm{II}.5)$$

It is also convenient to write down the asymptotics of kernels in the direct and inverse integral representation used in the H-system

$$\frac{\Gamma(-\sigma-\ell-1)\,\Gamma(-\sigma+\ell)}{\Gamma(-\sigma)}\, P_{\ell}^{\sigma+1}(\epsilon \,\mathrm{th}\, a) \underset{|p|\to\infty}{\approx} \frac{e^{\epsilon(\delta+1+ip)a}}{(-ip)},$$

$$\frac{\Gamma(\sigma-\ell+1)\Gamma(\sigma+\ell+2)}{\Gamma(\sigma+2)} P_{\ell}^{-\sigma-1}(\epsilon\, \mathrm{th}\, a) \underset{|p|\to\infty}{\approx} \frac{e^{-\epsilon(\delta+1+ip)a}}{ip} . \qquad (II, 6)$$

We could write the next terms of the asymptotic series, but these are not necessary in the context of this paper.

Appendix III

Calculation of an integral .

We shall calculate the integral

$$J = \int_0^\infty \sin\frac{\rho\beta}{2}\; P_{-\frac{1}{2}+iq}(\mathrm{ch}\,\beta)\, d\beta . \qquad (III,1)$$

It is convenient to represent (III.1) as a combination of two terms

$$J = \frac{1}{2i}(I - I^*),$$

where

$$I = \int_0^\infty d\beta\; e^{i\frac{\rho\beta}{2}}\; P_{-\frac{1}{2}+iq}(\mathrm{ch}\,\beta), \qquad (III.2)$$

and to start calculation with this integral/ 16/

$$I = \frac{1}{2}\int_0^\infty dx\; e^{\frac{x}{2}(-\frac{1}{2}-iq+i\frac{\rho}{2})}\; F\left(\frac{1}{2}+iq, \frac{1}{2}; 1; 1-e^{-x}\right) =$$

$$= \frac{\Gamma\left(\dfrac{\frac{1}{2}+iq-\frac{i\rho}{2}}{2}\right)\Gamma\left(\dfrac{\frac{1}{2}-iq-\frac{i\rho}{2}}{2}\right)}{2\Gamma\left(\dfrac{\frac{3}{2}-iq-\frac{i\rho}{2}}{2}\right)\Gamma\left(\dfrac{\frac{3}{2}+iq-\frac{i\rho}{2}}{2}\right)} = \frac{1}{4\pi^2}\left(\mathrm{ch}\,\pi q + i\,\mathrm{sh}\frac{\pi\rho}{2}\right)\times \qquad (III.3)$$

$$\times\Gamma\left(\frac{\frac{1}{2}+iq-\frac{i\rho}{2}}{2}\right)\Gamma\left(\frac{\frac{1}{2}-iq-\frac{i\rho}{2}}{2}\right)\Gamma\left(\frac{\frac{1}{2}+iq+\frac{i\rho}{2}}{2}\right)\Gamma\left(\frac{\frac{1}{2}-iq+\frac{i\rho}{2}}{2}\right).$$

Substituting (III.3) into (III.2) we obtain

$$\int_0^\infty \sin\frac{\rho\beta}{2}\, P_{-\frac{1}{2}+iq}(\operatorname{ch}\beta)\,d\beta = \frac{\operatorname{sh}\frac{\pi\rho}{2}}{4\pi^2}\,\Gamma\left(\frac{\frac{1}{2}+iq-\frac{i\rho}{2}}{2}\right)\Gamma\left(\frac{\frac{1}{2}-iq+\frac{i\rho}{2}}{2}\right).$$

$$\cdot\,\Gamma\left(\frac{\frac{1}{2}+iq+\frac{i\rho}{2}}{2}\right)\Gamma\left(\frac{\frac{1}{2}-iq-\frac{i\rho}{2}}{2}\right). \qquad \text{(III.4)}$$

For the sake of completeness we shall consider also the integral

$$I' = \int_0^\infty d\beta\, e^{i\frac{\rho\beta}{2}}\, Q_{-\frac{1}{2}+iq}(\operatorname{ch}\beta) = \frac{\pi}{i}\int_0^\infty d\beta\, e^{\beta\left(-\frac{1}{2}+iq+i\frac{\rho}{2}\right)}\times$$

$$\times\, F\left(\frac{1}{2}-iq,\frac{1}{2};1;1-e^{-2\beta}\right) =$$

$$= \frac{\pi}{2i}\,\frac{\Gamma\left(\frac{\frac{1}{2}-iq-\frac{i\rho}{2}}{2}\right)\Gamma\left(\frac{\frac{1}{2}+iq-i\frac{\rho}{2}}{2}\right)}{\Gamma\left(\frac{\frac{3}{2}+iq-i\frac{\rho}{2}}{2}\right)\Gamma\left(\frac{\frac{3}{2}-iq-i\frac{\rho}{2}}{2}\right)} \qquad \text{(III.5)}$$