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REPRESENTATIONS OF THE CANONICAL (ANTI) COMMUTATION RELATIONS IN INFINITE TENSOR-PRODUCT SPACE

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I. Introduction

A fundamental problem of quantum field theory is that of classifying all inequivalent representations of the canonical (anti) commutation relations and understanding which of these representations is preferable in a concrete physical situation.

In Bose case the heuristic formulation of the problem is roughly speaking as follows. One has to find all pairs of operator-valued distributions, namely the field $\phi(\vec{k})$ and the conjugate momentum $\pi(\vec{k}')$ satisfying the conditions

$$[\phi(\vec{k}), \phi(\vec{k}')] = [\pi(\vec{k}), \pi(\vec{k}')] = 0 , \qquad (1)$$

$$[\phi(\vec{k}), \pi(\vec{k}')] = i\,\delta(\vec{k} - \vec{k}')$$

and

$$\phi^{+}(\vec{k}) = \phi(\vec{k}), \quad \pi^{+}(\vec{k}) = \pi(\vec{k}). \qquad (2)$$

That means that a representation of the canonical commutation relations (CCR) is a pair of linear maps of the real prehilbert test-function space T' into a set of operators on a Hilbert space $\mathcal{H}$

$$f \longrightarrow \phi(f) \qquad (f \subset T'),$$
$$g \longrightarrow \pi(g) \tag{3}$$

satisfying

$$[\phi(f), \phi(f')] = [\pi(g), \pi(g')] = 0$$
$$[\phi(f), \pi(g)] = i(f, g)_{T'}, \tag{4}$$

where $\phi(f)$ and $\pi(g)$ are self-adjoint and $(f,g)_{T'}$ is a scalar product in the Hilbert space $\bar{T}'$ which is the completion of T'.

If $\{e_i\}$ is a complete orthonormal system in $\bar{T}'$ we claim that T' is the set of all finite linear combinations $\sum_i z_i e_i$ (z_i real) and define

$$\phi_i \equiv \phi(e_i), \quad \pi_j \equiv \pi(e_j). \tag{5}$$

Now the problem is to find all unitary inequivalent sets

$$\{\phi_i, \pi_j\}, \quad (i,j) = 1,2,\ldots$$

of self-adjoint operators on $\mathcal{H}$ satisfying

$$[\phi_i, \phi_j] = [\pi_i, \pi_j] = 0,$$
$$[\phi_i, \pi_j] = i\,\delta_{ij} \tag{6}$$

or equivalently

$$[a_i, a_j] = [a_i^+, a_j^+] = 0$$
$$[a_i, a_j^+] = \delta_{ij}, \tag{7}$$

where

$$a_i = \frac{\phi_i + i\pi_i}{\sqrt{2}}$$

$$a_i^+ = \frac{\phi_i - i\pi_i}{\sqrt{2}} \qquad (8)$$

are the corresponding annihilation and creation operators.

Analogously in Fermi case one has to construct all unitary inequivalent sets of pairs of operators $\{b_i, b_i^+\}$, $i = 1,2,..$ which satisfy

$$b_i b_j + b_j b_i = b_i^+ b_j^+ + b_j^+ b_i^+ = 0$$

$$b_i b_j^+ + b_j^+ b_i = \delta_{ij} \qquad (9)$$

and b_i^+ is adjoint to b_i.

No troubles arise in the case of the canonical anticommutation relations (CAR), since it follows from (9) that b_i and b_i^+ (and consequently $b(f)$ and $b^+(g)$) should be bounded. Moreover $b(f)$ and $b^+(g)$ are continuous with respect to their arguments so that there is no need to restrict oneself to T' and one can choose $\bar{T}'$ for the test-function space.

On the contrary in the case of Bose fields it is easy to show that at least one of the operators $a(f)$ and $a^+(f)$ is unbounded and one must take care of the domains of definition in order that (4) and (7) make sense. To avoid this difficulty one introduces the CCR in Weyl's form that is one passed from $\phi(f)$ and $\pi(g)$ to $e^{i\phi(f)}$ and $e^{i\pi(g)}$ and rewrites the restrictions (4) in terms of these exponents.

So we reformulate the problem as follows.

Let T be a complex prehilbert space $T = T' + iT'$ with the inner product $(\ ,\ )_T : T \times T \to C^1$ (which confirming to the usual mathematical usage is linear in the first argument).

Let $\mathcal{U}(\mathcal{H})$ be a group of all unitary operators in some complex Hilbert space $\mathcal{H}$.

Definition 1. We say that a representation of the CCR is given if and only if a Weyl system is given. The mapping $W: T \to \mathcal{U}(\mathcal{H})$ is called a Weyl system if and only if for arbitrary $t, t' \in T$

$$\alpha)\quad W(t)\,W(t') = e^{\frac{i}{2}\,\mathrm{Im}(t,t')_T}\,W(t+t'),$$

$\beta)$ the function $\lambda \to W(\lambda t)$, $\lambda \in C^1$ is weakly continuous at $t = 0$.

One can easily connect Weyl systems with the operators $\phi(f)$ and $\pi(g)$ in (4). Namely given a Weyl system we construct two abelain groups $\mathcal{U}_1 \subset \mathcal{U}$ and $\mathcal{U}_2 \subset \mathcal{U}$:

$$\mathcal{U}_1 = \{\, u_1(f) = W(f), \qquad f \in T \text{ and is real}$$

$$\mathcal{U}_2 = \{\, u_2(g) = W(ig), \qquad g \in T \text{ and is real.}$$

According to $\beta)$ these groups generate two sets of self-adjoint operators $\phi(f)$ and $\pi(g)$ which satisfy the equations

$$u_1(f) = e^{i\phi(f)}\ ,$$

$$u_2(g) = e^{i\pi(g)}$$

and due to $\alpha)$ can be identified with the corresponding operators in (4).

It is well known that there is a maze of inequivalent representations of the CCR and the CAR. There exist different approaches to the description and classification of these representations/2-7/. In order to make this paper self-contained we restate some results by Garding and Wightaman/2,3/. The starting point of the scheme is to diagonalise the commuting set of self-adjoint operators N_i, which are in fact the operators of number of particles in i-th state, i.e. $N_i = b_i^+ b_i$ in the Fermi-case and $N_i = a_i^+ a_i$ in the Bose-case, where a_i^+ and a_i are the operators (7) generated by a Weyl system. A representation space is a direct-integral Hilbert space $\mathcal{H} = \int \oplus \mathcal{H}_\nu(a)\, d\mu(a)$ over a set Γ of all sequences $a = \{a_1, a_2, \ldots\}$ with a_i being non-negative integers 0,1 in the CAR-case and 0,1,2... in the CCR-case. Here $\mathcal{H}_\nu(a)$ is a ν-dimensional unitary space corresponding to a point $a \in \Gamma$ and μ is a quasi-invariant measure on Γ. The operators N_j are simply the multiplication operators, i.e.

$$(N_i f)(a) = a_i f(a), \quad f \in \mathcal{H}.. \tag{10}$$

The field operators, for example the annihilation operators in the CCR-case are defined by the formula of the type

$$(a_k f)(a) = \sqrt{a_k + 1}\; C_k(a) \sqrt{\frac{d\mu(T_k^+ a)}{d\mu(a)}}\; f(T_k^+ a), \tag{11}$$

where T_k^+ is an operator of increasing a_k by one i.e. if

$a = \{a_1, \ldots, a_k\}$ then

$$T_k^+ a = \{a_1, \ldots, a_{k-1}, a_k + 1, a_{k+1}, \ldots\} \tag{12}$$

and $C_k(a)$ is the set of unitary operators on $\mathcal{H}_\nu(a)$ satisfying some simple restrictions/3/. In the CAR-case the analogous formula holds for the field operators/2/.

In both cases a representation is given explicitly by a set $\{\nu(a), \mu, \{C_k(a)\}_1^\infty\} = \{$ dimension of the $\mathcal{H}_\nu(a)$, a quasi-invariant measure on Γ, a set of unitary operators $C_k(a)$ on $\mathcal{H}_\nu(a)\}$. Two representations $\{\ \}$ and $\{\ \}'$ are equivalent if and only if $\nu = \nu'$, μ is equivalent to μ' and $C_k(a)$ and $C'_k(a)$ satisfy a kind of equivalence relation which, for example the CAR-case, has the form

$$C_k(a) = U(a)\, C'_k(a)\, U^{-1}(T_k^+ a) \tag{13}$$

the operators $U(a)$ being unitary. Thus the properties of a representation are intimately connected with the properties of a measure μ.

In the present paper we consider the special class of representations- the direct-product representations. Let us remind several basic notions and definitions of the theory of infinite tensor-product spaces developed by von Neumann/1/ on which the definition of the direct-product representations is based. The infinite tensor-product $\prod_k \otimes \mathcal{H}_k$ of the set of Hilbert spaces $\mathcal{H}_k$ is a non-separable Hilbert space (complete direct-product space in von Neumann's terminology). It proves to be a direct orthogonal sum of the so called incomplete direct-product spaces $\prod^{\kappa} \otimes \mathcal{H}_k$ (IDPS). Each of the IDPS is a separable Hilbert space generated by a product-vector $\chi = \prod \otimes \chi_k$, i.e. $\prod^{\chi} \otimes \mathcal{H}_k$ is the closed linear subspace of $\prod \otimes \mathcal{H}_k$ spanned by all product-vectors which differ from χ in at most finite number of components χ_k.

<u>Definition 2.</u> Two product-vectors $\chi = \prod \otimes \chi_k$ and $\chi' = \prod \otimes \chi'_k$ are said to be equivalent $\chi \approx \chi'$ if and only if

$$\sum_{k=1}^{\infty} |(\chi_k, \chi'_k) - 1| < \infty . \tag{14}$$

Note, that if two product-vectors belong to the same IDPS, they are equivalent. Conversely, if two product-vectors are equivalent, they belong to the same IDPS.

<u>Definition 3.</u> Two product-vectors $\chi = \prod \otimes \chi_k$ and $\chi' = \prod \chi'_k$ are said to be weakly equivalent $\chi \overset{w}{\approx} \chi'$ if and only if

$$\sum_k \big| |(\chi_k, \chi'_k)| - 1 \big| < \infty . \tag{15}$$

It turns out that if $\chi' \overset{w}{\approx} \chi''$ then there exists a sequence of real numbers ϕ_k such that

$$\Pi\otimes \chi_k' = \Pi\otimes e^{i\phi_k} \chi_k'' . \tag{16}$$

Note also that if $\chi \sim \chi'$ then the scalar product is defined to be

$$(\chi , \chi') = \prod_{k=1}^{\infty} (\chi_k , \chi_k') \tag{17}$$

and the right-hand side converges. If $\chi \not\sim \chi'$ then the right-hand side of (17) may not converge but the scalar product is defined to be zero.

Now we construct the direct-product representations of the CCR and the CAR.

a) CCR: For all k let us choose $\mathcal{H}_k = L_2(x)$. Let $W^{(S)}$ be a Schrödinger-Weyl system, i.e. the mapping $W^{(S)} : C^1 \to \mathfrak{U}(\mathfrak{L}_2(x))$ of the set of complex numbers into a group of all unitary operators on $\mathfrak{L}_2(x)$ given by

$$(W^{(S)}(\xi + i\nu\eta)\chi)(x) = e^{i\xi\frac{\eta}{2}} e^{i\xi x} \chi(x+\eta) \tag{18}$$

with $\chi(x) \in \mathfrak{L}_2(x)$ and ξ , η real.

A Weyl system $W: T \to \mathfrak{U}(\Pi\otimes \mathcal{H}_k)$ is called a <u>direct-product</u> representation of the CCR if

$$W(f) = \prod_k \otimes W_k^{(S)}(z_k), \tag{19}$$

where for every product-vector

$$(\prod_k \otimes W_k^{(S)}(z_k))\chi = \Pi\otimes (W^{(S)}(z_k)\chi) \tag{20}$$

and $z_k = (f , e_k)$ are the projections of the test-function $f \in T$ on unit vectors e_k which from an orthonormal basis $\{e_k\}$ in $\bar{T}$.

It is worthwhile saying that if the number M of freedoms is finite, the corresponding tensor-product space $\prod_k^M \otimes \mathcal{H}_k$ reduces to $\mathcal{L}_2(x_1, \ldots, x_M)$. In this case the field operators generated by the direct-product Weyl system take the usual Schrödinger form

$$\pi_k = \frac{1}{i} \frac{\partial}{\partial x_k} , \quad \phi_k = x_k \tag{21}$$

b) CAR: For all k let us choose $\mathcal{H}_k = E(2)$ where $E(2)$ is two-dimensional unitary space. A direct-product representation of the CAR is the set of operators $\{ b_i , b_i^+ \}$ in $\Pi \otimes \mathcal{H}_k$ which are defined by

$$b_i \Pi \otimes \chi_k = \prod^{i-1} \otimes (1 - 2\beta^+ \beta) \chi_j \otimes \beta \chi_i \times \prod_{i+1}^{\infty} \otimes \chi_j , \tag{22}$$

where β is a linear operator on $E(2)$ which in some fixed basis is given by a standard matrix:

$$\beta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} . \tag{23}$$

In § II we develop the general approach to the description of the representations of the CAR and the CCR in terms of the spectral measure generated by the set of the operators N_j . This approach is a kind of generalization of the original treatment in/2,3/ and is in fact a further development of considerations by Wightman and Schweber/8/.

In § III we consider the direct-product representations and derive a necessary and sufficient condition for unitary equivalence of any two direct-product representations (this condition has been previously obtained in/9/ by a rather special technique). Our next step is to construct a measure μ and unitary operators $C_k(a)$ for an arbitrary direct-product representation. It turns out that the corresponding measure is a product-measure. As a simple consequence of our consideration we obtain a necessary and sufficient condition for equivalence of any two product-measures.

In § IV several classes of canonical transformations are discussed.

II. The Spectral Measure Associated with a Representation of the Canonical (Anti) Commutation Relations.

Now we construct an operator-valued measure, defined on the Borel sets of Γ, that is the Descartes product of the spectra S_j of N_j which are the operators of the number of particles. This measure generates the spectral correspondence in which every N_j is represented by a linear function. Thus the whole construction is in fact the solution of the problem of the simultaneous diagonalization of the set $\{N_j\}_{j=1}^{\infty}$.

There exists the well-known standard procedure for diagonalization of any operator C^*-algebra $\mathfrak{A}$, based on the Gelfand isomorphism of $\mathfrak{A}$ onto a set of continuous functions defined on the space of all maximal ideals of $\mathfrak{A}$. Being a very powerful tool in general considerations, this procedure is not very convenient for our purposes. So, here we prefer an equivalent, but in a sense a more direct approach, which almost immediately gives the desired/2,3/ numerical measure μ on Borel sets of Γ.

Let a representation of the $C-R$ ($C-R$ means either CCR or CAR) be given in some Hilbert space $\mathcal{H}$. Thus we have the family $\{N_j\}_{j=1}^{\infty}$ of self- adjoint mutually commuting operators N_j. Let S_j be the spectrum of N_j and $M_j : \Sigma_j \to \mathcal{P}(\mathcal{H})$ be a spectral orthogonal measure corresponding to N_j. We mean that M_j is defined on the general Borel ring Σ_j of subsets of S_j its values belonging to the set $\mathcal{P}(\mathcal{H})$ of all projection operators on $\mathcal{H}$ and M_j is enumerably-additive function satisfying the following conditions

a) $M_j(S_j) = I$, i.e. identity operator in $\mathcal{H}$,

b) $M_j(E_j^1)M_j(E_j^2) = M_j(E_j^1 \cap E_j^2)$; $E_j^1, E_j^2 \subset \Sigma_j$

Note that in the CAR-case $S_j = \{a_j\} = \{0, 1\}$ and in the CCR-case $S_j = \{a_j\} = \{0, 1, 2, ...\}$, that is $\Gamma = \{a\}$ where a is a sequence $a = \{a_1, a_2 ...\}$ of non-negative integers a_j (0 and 1 in Fermi-case and 0,1,2 ... in Bose-case). In both cases Σ_j coincides with the collection of all subsets of S_j. M_j is a discrete spectral orthogonal measure, i.e.

$$M_j(E_j) = \sum_{a_j \in E_j} P_j(a_j), \qquad \text{where} \quad P_j(a_j)$$

is a projector in $\mathcal{H}$ corresponding to the point $a_j \in E_j \subset S_j$ and $\sum_{a_j \in S_j} P_j(a_j) = I$. The spectral resolution of the operator N_j takes the usual form

$$N_j = \sum_{a_j \in S_j} a_j M_j(\{a_j\}) = \sum_{a_j \in S_j} a_j P_j(a_j). \tag{24}$$

Now we introduce the semiring $\tilde{R}$ of all subsets $\mathcal{E} \subset \Gamma = \prod_j S_j$ which have the form

$$\mathcal{E} = \prod_j E_j = E_1 \otimes E_2 \otimes \ldots, \tag{25}$$

where $E_j \subset \Sigma_j$, $j = 1, 2, \ldots$ and $E_j = S_j$ for all but a finite number of j. Next we define the mapping $\tilde{P} : \tilde{R} \to \mathcal{P}(\mathcal{H})$ by

$$\tilde{R} \ni \mathcal{E} \to \tilde{P}(\mathcal{E}) = \prod_j M_j(E_j), \tag{26}$$

where E_j is a j-component of $\mathcal{E} = \prod E_j$ in (25). One can easily check that this mapping is a spectral orthogonal function, i.e. $\tilde{\tilde{P}}$ is projector-valued, enumerably additive and satisfies the conditions

$$\tilde{\tilde{P}}(\Gamma) = I,$$

$$\tilde{\tilde{P}}(\mathcal{E}_1)\tilde{\tilde{P}}(\mathcal{E}_2) \quad \tilde{\tilde{P}}(\mathcal{E}_1 \cap \mathcal{E}_2). \tag{27}$$

to establish this fact one has simply to recall that each $M_j(E_j)$ is a projector in $\mathcal{H}$, $M_j(E_j) \neq I$ only for the finite number of j in (26) and that all $M_j(E_j)$ are mutually commuting in view of the commutativity of N_j.

Note that $\tilde{\tilde{P}}$ satisfies all requirements imposed on spectral measures except that of being defined on Borel ring. So our next step is to extend the domain of definition of $\tilde{\tilde{P}}$ up to Borel ring Σ generated by $\tilde{\tilde{R}}$ and to preserve all properties of $\tilde{\tilde{P}}$.

<u>Theorem.</u> The orthogonal spectral function $\tilde{\tilde{P}} : \tilde{\tilde{R}} \to \mathcal{P}(\mathcal{H})$ given by (26) can be extended up to spectral orthogonal measure $M : \Sigma \to \mathcal{P}(\mathcal{H})$, which is defined on the Borel ring Σ generated by $\tilde{\tilde{R}}$, so that $M(\mathcal{E}) = \tilde{\tilde{P}}(\mathcal{E})$ for all $\mathcal{E} \in \tilde{\tilde{R}}$. This extention is unique.

To prove this statement one should follow the same lines as in the case of the extention of numerical elementary measures/11/.

Now we summarize the properties of the so constructed measure M.

a) $M(\Gamma) = I.$

b) $M\left(\bigcup_{k=1}^{\infty} E^{(k)}\right) = \sum_{k=1}^{\infty} M(E^{(k)})$ if all

$E^{(k)} \in \Sigma$ and for all $k \neq k'$ $E^{(k)} \cap E^{(k')} = \phi$

The right-hand side converges absolutely in the strong operator topology.

c) $M(E) = M^{+}(E)$

d) $(M(E)f, f) \geq 0$ for all $f \in \mathcal{H}$

e) $M(\prod_j E_j) = \prod_j M_j(E_j)$

if all $E_j \in \Sigma_j$. The product on the right-hand side converges in the strong operator topology. (If $\Pi M_j(E_j) = 0$ it is still considered convergent. This convention is useful since M generated by continuous representations of the C-R takes the zero values on every one-point subset of Γ) .

The line e) states that the operator-valued measure M is always a product-measure. In general the same is not valid for the corresponding numerical measure on Γ .

The more detailed characteristics of M depend on the properties of $\{N_j\}_{j=1}^{\infty}$ and consequently on the properties of the representation of the C-R. The related characteristics will be studied in another paper. Here we note only that some of the requirements imposed on M by the properties of the representation have been in fact discussed by Schweber and Wightman/8/, though they have not considered the measure M as a whole but only its values on one-point subsets of Γ .

Having the spectral measure M , related to the family $\{N_j\}_{j=1}^{\infty}$ one can map isometrically the Hilbert space $\mathcal{H}$ into direct integral $\int_{\Gamma} \oplus \mathcal{H}(a) d\mu(a)$. Under this mapping every N_j will be represented by a multiplication operator in the corresponding variable a_j . The structure of $\int_{\Gamma} \oplus \mathcal{H}(a) d\mu(a)$ is determined by properties of a representation of the C-R. The most simple case is that when the set $\{N_j\}_{j=1}^{\infty}$ is maximum-abelian. Since in § III we treat precisely this case let us consider it in more detail.

The procedure is almost the same as in the case of only one operator. If the family $\{N_j\}_{j=1}^{\infty}$ is maximum-abelian, then there exists a vector $h \in \mathcal{H}$ ($\|h\| = 1$) which is cyclic with respect to $\{N_j\}_{j=1}^{\infty}$ that is the set $\{M(E)h\}_{E \in \Sigma}$ is dense in $\mathcal{H}$. Let $\int_{\Gamma} \oplus \mathcal{H}(\alpha)\, d\mu(\alpha)$ be a Hibert space of μ --squareintegrable functions on Γ , i.e. all $\mathcal{H}(\alpha)$ are one-dimensional and the inner product of any two such functions f_1 and f_2 is $\int_{\Gamma} \bar{f}_1(\alpha) f_2(\alpha)\, d\mu(\alpha)$. The numerical measure μ is given by

$$\mu(E) = (M(E)h, h). \tag{28}$$

Let us put in correspondence to any vector

$f = \sum_i \lambda_i M(E_i) h \in \mathcal{H}$ a function $f(\alpha) \in \int_{\Gamma} \oplus \mathcal{H}(\alpha)\, d\mu(\alpha)$

defined by

$$f \to f(\alpha) = \sum_i \lambda_i \chi_{E_i}(\alpha), \tag{29}$$

where $\chi_{E_i}(\alpha)$ is a characteristic function of the set $E_i \in \Sigma$.

One can easily check that

$$\|f\|^2 = \int_{\Gamma} |f(\alpha)|^2 d\mu(\alpha). \tag{30}$$

Since the set $\{M(E)h\}_{E \in \Sigma}$ is dense in $\mathcal{H}$ and the set $\{\chi_E(\alpha)\}_{E \in \Sigma}$ is dense in $\int_{\Gamma} \oplus \mathcal{H}(\alpha)\, d\mu(\alpha)$ the mapping (30) defines an isometric operator

$$U : \mathcal{H} \to \int_\Gamma \oplus \mathcal{H}(a)\, d\mu(a).$$

It is not difficult to prove that under this mapping the domain of definition $\mathfrak{D}(N_j)$ of every N_j transforms into the set of functions $f(a)$ satisfying $\int a_j^2 \, |f(a)|^2 \, d\mu < \infty$ and to every $N_j f$, $f \in \mathfrak{D}(N_j) \subset \mathcal{H}$ there corresponds a function

$$(N_j f)(a) = a_j f(a) \in \int_\Gamma \oplus \mathcal{H}(a)\, d\mu(a). \tag{31}$$

Thus the operator $N_j' = U N_j U^{-1}$ is a multiplication operator and in $\int_\Gamma \oplus \mathcal{H}(a) d\mu(a)$ a representation takes canonical form (11) by Carding and Wightman.

III. Direct-Product Representations of the Canonical (Anti) Commutation Relations.

Now we return to infinite tensor-product space $\Pi \otimes \mathcal{H}_k$ and to the direct-product representations defined by (18), (19), (20) and (22). For the sake of brevity we shall speak only about the CCR-case since all considerations of the CAR-case are almost the same.

Let $\mathcal{B}$ be an algebra of bounded operators on $\Pi \otimes \mathcal{H}_k$ which is generated by all operators of the form

$$\tilde{B}_k = I_1 \otimes \dots \otimes I_{k-1} \otimes B \otimes I_{k+1} \otimes \dots \tag{32}$$

that is for any product-vector $\chi \in \Pi \otimes \mathcal{H}_k$

$$\tilde{B}_k \chi = \prod^{k-1} \otimes \chi_i \otimes B \chi_k \otimes \prod_{k+1} \otimes \chi_j \tag{33}$$

and B is an arbitrary bounded operator on $\mathcal{H}_k$.

Let $\bar{\mathcal{B}}$ be a minimal weakly closed algebra containing $\mathcal{B}$. In particular it is obvious that any Weyl operator $W(f)$ defined by (18) is an element of $\mathfrak{B}$:

$$W(f) \in \mathfrak{B} \subset \bar{\mathfrak{B}} . \tag{34}$$

It has been shown by von Neumann (Theorem IX in /1/) that a bounded operator A on $\prod_k \otimes \mathcal{H}_k$ belongs to $\bar{\mathfrak{B}}$ if and only if A commutes with all projection operators $P(\Pi^{\chi} \otimes \mathcal{H}_k)$ on IDPS $\Pi^{\chi} \otimes \mathcal{H}_k$ and with all operators $U(\{\phi\}) = \prod_k \otimes e^{i\phi_k}$ (ϕ_k -real numbers)

$$U(\{\phi\})\chi = \Pi \otimes e^{i\phi_k} \chi_k$$

which are responsible for the relation (16) between weakly equivalent product-vectors.

It follows from (34) that the direct-product representations are reduced by every IDPS. The restriction W^{χ} of a Weyl system (19) to the IDPS $\Pi^{\chi} \otimes \mathcal{H}_k$ is again a Weyl, system and can be shown to be irreducible, that is every bounded operator on $\Pi^{\chi} \otimes \mathcal{H}_k$, which commutes with all $W^{\chi}(f)$ is a multiple of the identity. In the following we shall call W^{χ} an irreducible direct-product representation. Now the question is when two irreducible direct-product representations $W^{\chi'}$ and $W^{\chi''}$ are equivalent.

First we note that since the set $\{ W_k^{(S)}(z_k) \mid z_k \in C^1 \}$ defined by (18) is irreducible in $\mathcal{H}_k = \mathfrak{L}_2(x)$ the commutant

$$\{ W_k^{(S)}(z_k) \mid z_k \in C^1 \}' = \{ \alpha \ I \}$$

and hence the bicommutant $\{ W_k^{(S)}(z_k) \mid z_k \in C^1 \}''$ coincides with the algebra of all bounded operators on $\mathcal{H}_k$. This means that the algebra $\mathfrak{C}$ generated by $\{ W(f) \mid f \in T \}$ is dense in $\mathfrak{B}$ and consequently in $\bar{\mathfrak{B}}$. Now let two product-vectors χ' and χ'', $\chi' \overset{w}{\neq} \chi''$ be given. Suppose that the corresponding irreducible direct-product representations $W^{\chi'}$ and $W^{\chi''}$ are equivalent that is for every $f \in T$

$$W^{\chi''}(f) = U W^{\chi'}(f) U^{-1} \tag{35}$$

where U is an isometric operator, mapping $\Pi^{\chi'} \otimes \mathcal{H}_k$ on $\Pi^{\chi''} \otimes \mathcal{H}_k$.

Next we introduce a bounded on $\Pi \otimes \mathcal{H}_k$ defined by $A\phi = 0$ if $\phi \in \Pi^{\chi} \otimes \mathcal{H}_k$ and $\chi \approx \chi'$ and by $A\phi = \phi$ if $\phi \in \Pi^{\chi} \times \mathcal{H}_k$ and $\chi \not\approx \chi'$. One can easily check that A fulfills all conditions of the theorem, cited above, and thus

$$A \in \bar{\mathfrak{B}} \ . \tag{36}$$

Moreover

$$A^{\chi'} = 0, \qquad A^{\chi''} = I, \tag{37}$$

where A^{χ} is the restriction of A to IDPS $\Pi^{\chi} \otimes \mathcal{H}_k$.

Now let $\omega_n \in \mathfrak{C}$ and $\omega_n \to A$ weakly. Since for all n

$$\omega_n^{\chi''} = U \omega_n^{\chi'} U^{-1} \tag{38}$$

we get

$$A^{\chi''} = U A^{\chi'} U^{-1}$$

in contradiction to (37).

On the other hand if $\chi' \approx \chi''$ then $W^{\chi'}$ is obviously equivalent to $W^{\chi''}$ since in this case U can be constructed explicitely and proves to be an operator $U(\{\phi\}) = \prod_k \otimes e^{i\phi_k}$ with ϕ_k form (16).

So we have obtained the following.

<u>Theorem 1.</u> Two irreducible direct-product representations $W^{\chi'}$ and $W^{\chi''}$ defined on $\Pi^{\chi'} \otimes \mathcal{H}_k$ and $\Pi^{\chi''} \otimes \mathcal{H}_k$, respectively, are equivalent if and only if the corresponding product-vectors are weakly equivalent.

Another proof of this theorem has been given in/9/.

Our next task is to investigate the structure of the set

$$\{ \nu(a), \mu \{ C_k(a) \}_{k=1}^{\infty} \}$$

corresponding to an irreducible direct-product representation W^{χ} in the standard approach/3/.

According to § II the first step should be to find the operators N_j. Due to (18) and (19) N_j are given by

$$N_j = I_1 \otimes \ldots \otimes I_{j-1} \otimes n \otimes I_{j+1} \otimes \ldots, \tag{39}$$

where n is an operator of a number of particles in $\mathcal{H}_k = \mathfrak{L}_2(x)$ that is

$$(n\phi)(x) = \frac{1}{2}\left[-\frac{d^2\phi}{dx^2}(x) + (x^2-1)\phi(x)\right], \quad \phi \in \mathfrak{D}(n) \subset \mathfrak{L}_2(x). \tag{40}$$

The spectral function $M_j(\{a_j\})$ in (24) takes the form

$$M_j(\{a_j\}) = P_j(a_j) = I_1 \otimes \ldots \otimes I_{j-1} \otimes \pi(a_j) \otimes I_{j+1} \otimes \ldots \tag{41}$$

and $\pi(a_j)$ is a projector in $\mathcal{H}_j = \mathfrak{L}_2(x)$ on the one-dimensional subspace generated by h^{a_j}, where $h^{a_j} = h^{a_j}(x)$ is the a_j - th Hermit function:

$$(n h^{a_j})(x) = a_j h^{a_j}(x). \tag{42}$$

The second step is to understand whether the family $\{N_j\}_{j=1}^{\infty}$ is maximum abelian. Let us prove two simple statements.

a) There exists a dense set of vectors χ_j in $\mathcal{H}_j = \mathfrak{L}_2(x)$ which are cyclic with respect to n, that is for every such $\chi_j \in \mathfrak{L}_2(x)$ the set $\{\pi(a_j)\chi_j\}$ is dense in $\mathfrak{L}_2(x)$.

To prove the statement one simply notes that every $\chi_j \subset \mathfrak{L}_2(x)$ with all $q_j^m = (\chi_j, h^m)_{\mathfrak{L}_2} \neq 0$ is obviously cyclic.

b) In IDPS $\prod_k^{\chi'} \otimes \mathcal{H}_k$ generated by a product-vector $\chi' = \prod_k \otimes \chi'_k$ there exists a product-vector $\chi = \prod_k \otimes \chi_k$ with all $\chi_k \subset \mathfrak{L}_2(x)$ cyclic with respect to π. This χ is cyclic with respect to $\{N_j\}_{j=1}^{\infty}$.

Proof: According to a) we can choose a sequence $\chi \subset \mathfrak{L}(x)$ satisfying $\|\chi_k - \chi'_k\| < \epsilon^k$, $(0 < \epsilon < 1)$ with all χ_k cyclic with respect to π. One can check that inequality (14) holds for $\chi = \prod_k \otimes \chi_k$ and $\chi' = \prod_k \otimes \chi'_k$ and thus $\Pi^{\chi} \otimes \mathcal{H}_k = \Pi^{\chi'} \otimes \mathcal{H}_k$. On the other hand the cyclicity of χ is also guaranteed because the set of product-vectors which differ from χ only at the most in a finite number of components is dense in $\Pi^{\chi} \otimes \mathcal{H}_k$ and every such product-vector can be arbitrarily well approximated by a linear combinations of

$$P_j \chi = (I_1 \otimes \dots \otimes I_{j-1} \otimes \pi \otimes I_{j+1} \otimes \dots) \chi$$

(due to π-cyclicity of every χ_k).

Thus without loosing generality we can consider that for any π^{χ} a representation-space $\prod_k^{\chi} \otimes \mathcal{H}_k$ is generated by a cyclic product-vector $\chi = \Pi \otimes \chi_k$. In accordance with § II this means that all $\mathcal{H}(\alpha)$ in $\int_{\Gamma} \oplus \mathcal{H}(\alpha) d\mu(\alpha)$ are one-dimensional, i.e. dimension $\nu(\alpha) = 1$. The numerical measure μ on Γ corresponding to π^{χ} is given by

$$\mu(E) = (M(E)\chi, \chi) \tag{28^a}$$

and on the sets of the form

$$\Sigma \supset \mathcal{E} = \prod_j E_j \quad ; E_j \subset \Sigma_j \tag{43}$$

it takes values

$$\mu\ (\mathcal{E}\ ) = \prod_j \mu_j\ (E_j\ ), \tag{44}$$

where

$$\mu_j\ (E_j\ ) = \sum_{a_j \in E_j} (\ \pi\ (a_j\ )\ \chi_j\ ,\ \chi_j\ ). \tag{45}$$

The line (44) says that the measure μ corresponding to an irreducible direct-product representation W^{χ} is always a product-measure generated by a set of real numbers

$$\begin{aligned} P_j^m &= \mu_j(\ \{\ a_j\ = m\ \}\ ) = (\ \pi\ (m)\ \chi_j\ ,\ \chi_j\ ) \\ \sum_m\ P_j^m &= 1\ . \end{aligned} \tag{46}$$

If the expansion of $\chi_j(x)$ in Hermit functions is

$$\chi_j(x) = \sum_m q_j^m\ h^m(x) \tag{47}$$

then

$$p_j^m = |\, q_j^m\, |^2\ . \tag{48}$$

Now let us find the last element $\{\ C_k(a)\ \}_{k=1}^{\infty}$ of the canonical triplet $\{\ \nu(a),\ \mu,\ \{\ C_k(a)\ \}_{k=1}^{\infty}\ \}$. It turns out that unitary operators $C_k(a)$ defined in (11) are given by

$$C_k(a) = e^{\,i\,[\ \phi_k(a_k+1) - \phi_k(a_k)\ ]}\ , \tag{49}$$

where

$$\phi_k(a_k) = \arg q_k^{a_k} \tag{50}$$

Proof: Let again

$$\chi_k = \sum_{a_k \in S_k} q_k^{a_k} h^{a_k} \ ; \quad \chi = \prod_j \otimes \chi_j \ .$$

Now due to (29)

$$M_k(\{a_k = m\})\chi = \prod_j^{k-1} \otimes \chi_j \otimes q_k^m h^m \otimes \prod \otimes \chi_j \rightarrow$$

$$\rightarrow f(a) = \delta_{a_k, m} \in \int_\Gamma \oplus \mathcal{H}(a)\, d\mu(a).$$

On the other hand according to the definition of the direct-product representation

$$a_k M_k(\{a_k = m\})\chi = \sqrt{m}\, q_k^m \prod^{k-1} \otimes \chi_j \otimes h^{m-1} \otimes \prod_{k+1} \otimes \chi_j = \tag{51}$$

$$= \sqrt{m}\, q_k^m / q_k^{m-1} \quad M_k(\{a_k = m-1\})\chi \ .$$

Use again (29). We have

$$a_k M_k(\{a_k = m\})\chi \rightarrow (a_k f)(a) =$$

$$= \sqrt{m}\, q_k^m / q_k^{m-1} \delta_{a_k+1, m} \in \int_\Gamma + \mathcal{H}(a)\, d\mu(a) \ . \tag{52}$$

An immediate generalization of the above consideration is

$$(a_k f)(a) = \sqrt{a_k + 1}\; e^{i[\phi_k(a_k+1) - \phi_k(a_k)]} \sqrt{\frac{p^{a_k+1}}{p^{a_k}}}\, f(T_k^+ a), \quad (53)$$

where $\phi_k(a)$ are defined by (50). Comparing (53) with (11) we obtain (49). Conversely, for an arbitrary given triplet

$$\{\nu = 1, \mu, \{C_k(a) = e^{i[\phi_k(a_k+1) - \phi_k(a_k)]}\}\}$$

with a product-measure μ one can choose $\chi = \Pi \otimes \chi_k$ with

$$\chi_k = \sum_m e^{i\phi_k(m)} \sqrt{p_k^m}\; h^m$$

and thus find a corresponding irreducible direct-product representation.

So, we have proved the following

Theorem 2. A representation of the CCR is (equivalent to) an irreducible direct-product representation if and only if $\nu = 1$, μ is (equivalent to) a product-measure and $C_k(a)$ take the form (49).

Combining theorem 2, (48), (50) and theorem 1 we obtain the

Corollary 1. Two quasiinvariant measures μ and μ' on Γ generated by two sequences of real positive numbers p_j^m and $p_j'^m$, respectively, are equivalent if and only if

$$\sum_j \left(1 - \sum_m \sqrt{p_j^m\, p_j'^m}\right) < \infty \quad (54)$$

(Note that since $\sum_m p_j^m = \sum_m p_j'^m = 1$ the inner sum converges).

Proof: Take

$$\chi = \prod_j \otimes \chi_j \; ; \quad \chi_j = \sum_m \sqrt{p_j^m}\; h^m$$

and

$$\chi' = \prod_j \otimes \chi_j' \quad ; \quad \chi_j' = \sum_m \sqrt{p_j'^m}\, h^m .$$

According to Theorem 2 the related irreducible direct-product representations W^{χ} and $W^{\chi'}$ are equivalent if and only if μ and μ' are equivalent. Next, using Theorem 1 and (15) we obtain (54). Thus, the inequality (54) is a complete solution of the corresponding problem formulated in/3/. In fact this result was first obtained in the early paper by Kakutani/12/.

Theorem 1 and 2 are also valid in the case of CAR. The inequality (54) holds as well, but in this case the inner sum contains only two terms.

IV. Canonical Transformations.

The results of the preceding sections can be readily applied to pseudocanonical transformations. The basic question here is to understand whether a given pseudocanonical transformation is in fact canonical, i.e. is implemented by a unitary transformation.

In this section we consider a special class of pseudocanonical transformations - the "finite-dimensional" transformations. Let a_k^+ and a_k $(k = 1, 2, \ldots)$ be creation and annihilation operators generated by a representation W of the CCR in some Hilbert space $\mathcal{H}$. Let another set $\{a_k'^+, a_k'\}_{k=1}^{\infty}$ of operators be given in $\mathcal{H}$ and

$$\begin{aligned} a_k' &= f_k(a_{k_1}, \ldots, a_{k_m}; a_{k_1}^+, \ldots, a_{k_m}^+) \\ a_k'^+ &= \bar{f}_k(a_{k_m}^+, \ldots, a_{k_1}^+; a_{k_m}, \ldots, a_{k_1}). \end{aligned} \qquad (55)$$

If f_k are chosen in such a way that $[a_k', a_q'^+] = \delta_{kq}$;

the transformation (55) is called pseudocanonical. If $m < \infty$ we shall call it "finite-dimensional". The pseudocanonical transformation (55) generates another representation W' of the CCR in $\mathcal{H}$ and this new representation is generally speaking inequivalent to the initial W. If W and W' are equivalent the representation is said to be canonical. It is important that the answer to the question, if the pseudocanonical transformation (55) is canonical, depends not only on the properties of functions f_k in (55), which reflect only the algebraic structure of the transformation, but also on the properties of the initial representation W in $\mathcal{H}$.

Let W in $\mathcal{H}$ be equivalent to some irreducible direct-product representation W^{χ} ($\chi = \Pi \otimes \chi_k$). Then the transformed representation W' is equivalent to another irreducible direct-product representation, namely to $W^{U\chi}$ where a unitary operator U takes the form (for simplicity we restrict ourselves to the case of one-dimensional transformation):

$$U = U_1 \otimes U_2 \otimes \ldots \otimes U_k \otimes \ldots . \qquad (56)$$

Here U is a unitary operator in $\mathcal{H}_k = \mathcal{L}_2(x)$ satisfying the conditions

$$a_k^1 = \tilde{U}_k^{-1} a \tilde{U}_k , \qquad (57)$$

$$a_k^{+1} = \tilde{U}_k^{-1} a_k^{+} \tilde{U}_k ,$$

where

$$\tilde{U}_k = I_1 \otimes \ldots \otimes I_{k-1} \otimes U_k \otimes I_{k+1} \otimes \ldots , \qquad (58)$$

Due to uniqueness (up to unitary equivalence) of the representation of CCR in the case of finite number of degrees of freedom, every U_k exists and is explicitly determined by a function f_k in (55). Thus, instead of comparing W and W' we can compare $W\chi$ and $W\chi'$.
Applying Theorem 1 we obtain the

Corollary 2. The pseudocanonical transformation (55) is canonical if and only if

$$\sum_k \left(1 - | (U_k \chi_k , \chi_k) | \right) < \infty , \tag{59}$$

where U_k are defined by (57), (58).

One can easily generatize this statement to include all finite-dimensional transformations. It is interesting that for every transformation (55) there exists an irreducible direct-product representation in which (55) is canonical.

Let us discuss simple examples. The most general one-dimensional linear pseudocanonical transformation is given by

$$a_k' = a_k \operatorname{ch} \nu_k e^{i(\phi_k + \psi_k)} + a_k \operatorname{sh} \nu_k e^{i(\phi_k - \psi_k)} + \lambda_k , \tag{60}$$

where ν_k , ϕ_k , ψ_k are real and λ_k is complex. In $\mathcal{H}_k = \mathcal{L}_2 (x)$ (that is in the case of only one degree of freedom) the transformation (60) is implemented by a unitary operator U_k :

$$a_k^1 = U_k^{-1} a_k U_k$$
$$U_k = U_k^{(1)} U_k^{(2)} U_k^{(3)} U_k^{(4)} . \tag{61}$$

Here

$$U_k^{(1)}(\lambda_k) = e^{\lambda_k a_k^+ - \bar{\lambda}_k a_k}$$

$$U_k^{(2)}(\phi_k) = e^{i\phi_k a_k^+ a_k}$$

$$U_k^{(3)}(\nu_k) = e^{\frac{\nu}{2}(a_k^+ a_k^+ - a_k a_k)} \qquad (62)$$

$$U_k^{(4)}(\psi_k) = e^{i\psi_k a_k^+ a_k} .$$

and

$$a_k^+ = \frac{1}{\sqrt{2}}\left(x - \frac{d}{dx}\right)$$

$$a_k = \frac{1}{\sqrt{2}}\left(x + \frac{d}{dx}\right) .$$

Now let all $\phi_k = \nu_k = \psi_k = 0$, that is

$$a_k' = a_k + \lambda_k \qquad (63)$$

and $\{a_k , a_k^+\}$ generate an irreducible direct-product representation W^χ . (The pseudocanonical transformations of this type has been considered by Shelupsky/10/). Let the representation space be

$$\mathcal{H} = \Pi^\chi \otimes \mathcal{H}_k$$

$$\chi = \prod_k \otimes h^{n_k} , \qquad (64)$$

where h^{n_k} is the n_k-th Hermit function. In this case the representation W^χ is called discrete and is completely determined by a sequence

$$\{n_k\} = (n_1, n_2, \ldots) . \tag{65}$$

Now

$$(U_k^{(1)}(\lambda_k)h^{n_k}, h^{n_k}) = e^{-\frac{|\lambda_k|^2}{2}} \sum_{m=0}^{n_k} \frac{(-|\lambda_k|^2)^m}{m!} \frac{n_k!}{(n_k-m)!\, m!} . \tag{66}$$

Using (66) one can show that the sum $\sum_k (1-|(U_k^{(1)}(\lambda_k)h^{n_k}, h^{n_k})|)$ converges if and only if

$$\sum_k^{\infty} (n_k+1)|\lambda_k|^2 < \infty . \tag{67}$$

Thus, the pseudocanonical transformation (63) in a representation space defined by (64) and (65) is canonical if and only (67) holds. For the Fock representation (all $n_k = 0$) the relation (67) reduces to

$$\sum_k |\lambda_k|^2 < \infty$$

The analogous calculations can be performed also in general case (60). The authors are indebted to V.S. Vladimirov, A.I. Oksak and B.M. Stepanov for valuable discussion.

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