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ON THE HYPOTHESIS OF COMMUTATIVE SUPERSELECTION RULES

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Introduction

The study of quantum theory superselection rules has a great contemporary interest and in recent years much knowledge has been accumulated in this field (one is referred for bibliographical notes to [1],[2]). In our treatment of the problem we follow the paper of one of us [3]. We base the forthcoming exposition upon coherent sets of elementary properties (Propositions 1,2 ; the authors believe that the essence of §1 is the Definition).

The consideration of §1 permit to give the proof of the generalization of Wigner's theorem on symmetries ([4], Theorem 1-1) without the hypothesis of commutative superselection rules (under a symmetry any coherent set of physically realizable vectors is mapped onto a coherent one). In §2 we point out that any Hilbert space in which some superselection rules act can be decomposed into the direct sum of mutually orthogonal coherent subspaces and one subspace which is orthogonal to all physically realizable vectors. The wide-spread exposition of superselection rules in terms of the commutant of the set of all observables is presented.

§ 1. Coherent Sets in Hilbert Space.

<u>Definition.</u> We call a set $\mathfrak{F} \subset H$ ($\mathfrak{F} \neq \phi$, $0 \notin \mathfrak{F}$) coherent if there is no partition of $\mathfrak{F}$ into two nonvoid subsets $\mathfrak{F}_1$ and $\mathfrak{F}_2$ so that $\mathfrak{F}_1 \perp \mathfrak{F}_2$ ($\mathfrak{F}_1 \perp \mathfrak{F}_2$ means that any $f_1 \in \mathfrak{F}_1$ is orthogonal to any $f_2 \in \mathfrak{F}_2$).

Below the criterion of a set $\mathfrak{F} \subset H$ to be coherent follows. By $F = L(\mathfrak{F})$ we shall denote henceforth the closed linear hull of a set $\mathfrak{F}$.

<u>Proposition 1.</u> A set $\mathfrak{F} \subset H$ ($\mathfrak{F} \neq \phi$, $0 \notin \mathfrak{F}$) is coherent if the (von Neumann) algebra in a subspace $F = L(\mathfrak{F})$ generated by projections $P(f)$ in F on vectors $f \in \mathfrak{F}$ coinsides with the algebra of all bounded operators in F , i.e. if the commutant $\mathcal{P}'$ of the family $\mathcal{P} = \{P(f), f \in \mathfrak{F}\}$ of operators in F consists only of multiples of the identity.

Let $\mathfrak{F} \subset H$ be coherent and $C \in \mathcal{P}'$. From $CP(f) = P(f)C$ if follows $C^*P(f) = P(f)C^*$; one can see that each $f \in \mathfrak{F}$ is an eigen-vector of C and C^* :

$$Cf = \lambda(f)\cdot f \quad ; \quad C^*f = \overline{\lambda(f)}\cdot f \quad ; \quad \lambda(f) = \langle f, Cf \rangle \cdot \| f \|^{-2}$$

If we suppose that $\lambda(f)$ depends on f we can divide $\mathfrak{F}$ into nonvoid non-overlapping subsets $\mathfrak{F}_1$ and $\mathfrak{F}_2$ so that $\lambda(f_1) - \lambda(f_2) \neq 0$ for any $f_1 \in \mathfrak{F}_1$, $f_2 \in \mathfrak{F}_2$ and for these f_1, f_2

$$\langle f_1, f_2 \rangle = [\lambda(f_1) - \lambda(f_2)]^{-1} \cdot [\langle \overline{\lambda(f_1)} f_1, f_2 \rangle - \langle f_1, \lambda(f_2) f_2 \rangle]$$

i.e. $\mathcal{F}_1 \perp \mathcal{F}_2$. It is impossible according to the coherentness of $\mathcal{F}$. We obtain : if $C \in \mathcal{P}'$ then $Cf = \lambda f$ for any $f \in \mathcal{F}$; hence $Cf = \lambda f$ for any $f \in F = L(\mathcal{F})$.

Let $\mathcal{F} \subset H$ be not coherent then there exist $\mathcal{F}_1$ and $\mathcal{F}_2$ such that $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2, \mathcal{F}_1 \perp \mathcal{F}_2$. It is evident that the projection Q in F on $F_1 = L(\mathcal{F}_1) \subset F$ belongs to $\mathcal{P}'$ and differs from a multiple of the identity. The assertion is proved.

We shall establish now that any set $\mathfrak{M} \subset H$ $(0 \notin \mathfrak{M})$ can be represented (uniquely) as a union of some family $A = \{\alpha\}$ of coherent pairwise orthogonal subsets $\mathfrak{M}_\alpha \subset H$.

Proposition 2. Let $\mathfrak{M} \subset H (0 \notin \mathfrak{M})$. Then $M = L(\mathfrak{M})$ can be decomposed into direct sum of identity orthogonal subspaces decomposed into direct sum of mutually orthogonal subspaces

$$\mathfrak{M} = +_{\alpha \in A} \mathfrak{M}_\alpha$$

so that $\mathfrak{M} = {}_{\alpha \in A} \mathfrak{M}_\alpha$.

where $\mathfrak{M}_\alpha = M_\alpha \quad \mathfrak{M} (\alpha \in A)$ are coherent pairwise orthogonal subsets in H .

Prof. One introduces in $\mathfrak{M}$ the binary relation: $f \approx g$ if there is a coherent subset $\pi \subset \mathfrak{M}$ such that $f, g \in \pi$ (one is able even to demand the finiteness of a subsets π and this leads to the same). The relation is symmetric $(f \approx f)$, reflexive $(f \approx g \rightarrow g \approx f)$. We are going to prove the transitivity. Let $f \approx g$, $g \approx h$. Then there exist coherent subsets π_1 and π_2 in $\mathfrak{M}$ such that $f, g \in \pi_1$, $g, h \in \pi_2$. The subset $\pi = \pi_1 \quad \pi_2$ is coherent. To see this one takes an arbitrary division of π into π' and π'' : $\pi = \pi' \quad \pi''$, $\pi' \perp \pi''$. Suppose that $g \in \pi'$. As π_1 is the union of two orthogonal subsets $\pi_1 \quad \pi'$ and $\pi_1 \quad \pi''$ and the first of them is nonvoid ($g \in \pi_1 \quad \pi'$) the coherentness of π_1 entails the identity $\pi_1 \quad \pi'' = \phi$. For the same reason $\pi_2 \quad \pi'' = \phi$, hence $\pi'' = \pi \quad \pi'' = (\pi_1 \quad \pi'') \quad (\pi_2 \quad \pi'') = \phi$ and one should conclude that π doesn't admit partitions into nonvoid orthogonal subsets, i.e. π is coherent; $f, h \in \pi$ and $f \approx h$. The relation introduced is the equivalence relation and we can divide the set $\mathfrak{M}$ into classes $\mathfrak{M}_a (a \in A)$ of equivalent elements. These classes possess the properties:

1) any $f \in \mathfrak{M}$ belongs to some (unique) $\mathfrak{M}_a$;

2) $\mathfrak{M}_a \perp \mathfrak{M}_\beta$ when $a \neq \beta$ (if, $f, g \in \mathfrak{M}$ and $\langle f, g \rangle \neq 0$ then f and g are equivalent and belong to the same class);

3) $\mathfrak{M}_a (a \in A)$ is a coherent set in H (indeed, supposing the existence of a division of $\mathfrak{M}_a$ into nonvoid $\mathfrak{M}'_a$ and $\mathfrak{M}''_a$, $\mathfrak{M}'_a \perp \mathfrak{M}''_a$ one receives that vectors of $\mathfrak{M}'_a$ are not equivalent to vectors of $\mathfrak{M}''_a$).

We introduce now subspaces $M_a = L(\mathfrak{M}_a)$, $a \in A$. They are mutually orthogonal and $\mathfrak{M}_a = M_a \quad \mathfrak{M}$. We omit the proof of the next simple statement: if $\mathfrak{M} \subset H$ is a union of some family of pairwise orthogonal subsets $\mathfrak{M}_a$ $(a \in A)$ then $L(\mathfrak{M}) = \bigoplus_{a \in A} L(\mathfrak{M}_a)$. Thus one is drawn to the conclusion: $M = \bigoplus_{a \in A} M_a$; this completes the proof.

(For the sake of clearness we shall remind what is usually meant by the infinite direct sum of mutually orthogonal subspaces in Hilbert space. Let f_a $(a \in A)$ be a family of vectors in H; a series $\sum_{a \in A} f_a$ is summable with the sum f if for any $\epsilon > 0$ there is a finite subfamily $I \subset A$ the inequality for any finite subfamily $J \quad I$ taking place: $\| \sum_{a \in J} f_a - f \| < \epsilon$. The necessary and sufficient condition for the series of pairwise orthogonal vector $\sum_a f_a$ to be summable is the summability of the series $\sum_{a} \| f_{a} \|^{2}$; if the condition is fulfilled there can be at most countably many nonzero vectors f_{a} in the series. The direct sum of mutually orthogonal subspaces M_{a} $(a \in A)$ is the set of all sums of summable series $\sum_{a} f_{a}$, $f_{a} \in M_{a}$. Note that in a separable Hilbert space one is able to single out at the most a countable family of mutually orthogonal subspaces).

A few words on the uniqueness of the decomposition given by the Proposition 2. We remark that if $\mathfrak{F}$ and $\mathfrak{G}$ are two coherent sets in H then either $\mathfrak{F} \quad \mathfrak{G} = \phi$ or $\mathfrak{F} \quad \mathfrak{G} \neq \phi$ and $\mathfrak{F} \quad \mathfrak{G}$ is coherent (the latter can be established just in the same way as the coherentness of $\pi_1 \quad \pi_2$ above). Given two representations of the set $\mathfrak{M} \subset H$:

$$\mathfrak{M} = \quad_{a \in A} \mathfrak{M}_{a} \;, \qquad \mathfrak{M} = \quad_{\beta \in B} \mathfrak{N}_{\beta}$$

($\mathfrak{M}_{a}$, $\mathfrak{N}_{\beta}$ - coherent ; $\mathfrak{M}_{a} \perp \mathfrak{M}_{a'}$, $a \neq a'$; $\mathfrak{N}_{\beta} \perp \mathfrak{N}_{\beta'}$, $\beta \neq \beta'$).
Using the remark one can easily establish a one-to-one mapping between A and B such that $a \to \beta$ entails $\mathfrak{M}_{a} = \mathfrak{N}_{\beta}$.

§ 2. *On Superselection Rules.*

Wick, Wigner, Wightman /5/ pointed out that in general not every vector of the quantum theory Hilbert space corresponds to a state of a system (i.e. is a "physically realizable vector"). Let $\mathfrak{M}$ be a set of all physically realizable vectors in H .

<u>Definition.</u> Subspace $F \subset H$ will be called coherent if it is a closed linear hull of some coherent subset $\mathfrak{F} \subset \mathfrak{M}$, and maximal coherent

if in addition it is not a proper subspace of some coherent subspace.

Due to the Proposition 2 the closed linear hull $M = L(\mathfrak{M})$ of the set $\mathfrak{M}$ can be decomposed (uniquely) into direct sum of maximal coherent mutually orthogonal subspaces $M_a (a \in A)$ each vector $f \in \mathfrak{M}$ belonging to some M_a. We point to the connection between this decomposition and the structure of observables in H. As to observables one usually says such words (and we adopt them): a) every maximal coherent subspace is an invariant subspace of any observable /5/,/6/; b) any projection $P(f)$ on a physically realizable vector f is an observable /4/.

Introducing $G = H \ominus M$, $M = \bigoplus_a M_a$ one receives $H = (\bigoplus_a M_a) \oplus G$. The collection of all observables is N. Any operator $T \in N$ (due to the assumption "a") has the blockwise form: $T = (\bigoplus_a T_a) \oplus S$, where T_a is a hermitian operator in M_a, S is a hermitian operator in G; in each subspace M_a operators T_a generate an irreducible set (the latter is the consequence of the assumption "b" and the Proposition 1). Therefore any $C \in N'$, N' the commutant of N, is of the form: $C = (\bigoplus_a C_a) \oplus D$, where C_a is a multiple of the identity in M_a, D is an operator in G.

The hypothesis of commutative superselection rules (which asserts N' is an abelian algebra) is fulfilled automatically if one demands that G, the subspace which is orthogonal to all physically realizable vectors, consists only of zero. In this case the superselection rules (i.e. the separation of coherent subspaces M_a in Hilbert space) can be characterized completely by N'; namely in this case the general form of $C \in N'$ is $C = \sum_a \lambda_a P_a$, where P_a is a projection in H on a coherent subspace M_a, λ_a is an arbitrary number.

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Note added in proof. The system of postulates adopted in our exposition doesn't describe the general situations in quantum theory. Namely, the postulates: 1) every physical state of a system is described by a normalized vector (ray) in Hilbert space, 2) a projection on a physically realizable vector is an observable, are too restrictive and are absent in the general theory of algebras of observables (see e.g. /7/); the structure of superselection rules for the general case is to be clarified. The authors would like to thank participants of the 5-th Annual Winter School for Theoretical Physics in Karpacz, 1968 for this critical remark.

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