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TEST OF CROSS-SECTION SUM RULES FOR NUCLEON MAGNETIC MOMENTS

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1. *Introduction*

In connection with forward Compton scattering it is well known that sum rules for the magnetic and electric moments of the target particles can be derived from a general low energy theorem and dispersion relations /1-5/. In this way quadratic or bilinear forms of electromagnetic moments are expressed through derivatives of dispersion integrals taken at zero energy. Usually the Optical theorem is employed to replace the imaginary part of the forward scattering amplitude by linear combinations of total cross-sections belonging to definite polarisations of the incident particles. Certain assumptions about the asymptotic behaviour of these cross sections being equivalent to the postulation of unsubtracted dispersion relations in the case of magnetic moment sum rules /3,4/ are necessary to guarantee the existence of the integrals.

A direct experimental test of the mentioned sum rules seems to be difficult since it requires the knowledge of the polarized cross-sections in a large energy region. It is therefore interesting to look for other possibilities of evaluation which are based on known data. Drell and Hearn /6/ and Pagels /2/ have estimated directly the continuum contributions of the nucleon magnetic moment sum rules by insertion of low energy and resonance approximations. They obtain evidence for the validity of the sum rules if there is no large contribution for the energy region above 1 GeV and if the nonresonant background is small (see also the more indirect test by Gerasimov[4] in connection with the decrease of the magnetic moment when the nucleon is bound). Another possibility of evaluation is studied in this paper. We consider nucleon and deuteron targets and investigate the isoscalar Drell-Hearn sum rule together with the deuteron magnetic moment sum rule[2,5] using the fact that all magnetic moment sum rules have the same cross-section structure, i.e. the imaginary parts of the forward scattering amplitudes are given by analogue cross-section expressions[3]. Since the deuteron binding energy plays no role in the asymptotic limit both sum rules are guaranteed by the same asymptotic condition. Moreover, assuming that for forward Compton scattering above the pion production threshold the deuteron may be replaced approximately by the free nucleons and neglecting electromagnetic interactions of order e^4 we combine the two sum rules to an equation containing only known terms and a low energy cross section integral between the deuteron photodisintegration threshold and the pion production threshold. This integral can be evaluated by standard methods with the help of low energy deuteron photodisintegration data[7]. The appearing polarized cross-sections are obtained conveniently from the helicity amplitudes. The result agrees with the predicted value (a 10%-deviation is due to the used approximations) showing the validity of the considered sum rules.

In sections 2 we study the sum rules and their combination to an evaluable equation. Section 3 deals with the calculation of the low energy integral. Finally in section 4 the result is discussed.

2. *The Magnetic Moment Sum Rules*

The kinematical structure of the γN forward scattering amplitude containing two independent invariants is[8]

$$F(\omega) = (\vec{e}_1^{\,*}\,\vec{e}_2)\,F_1(\omega) + i(\vec{\sigma}\,\vec{e}_1^{\,*} \times \vec{e}_2)\,F_2(\omega), \tag{1}$$

where $\vec{e}_1$ and $\vec{e}_2$ are the polarisation vectors of the two photons and ω is the photon energy in the laboratory system. From eq. (1) and the Optical theorem result the relations

$$\mathrm{Im}\,F_1(\omega) = \frac{\omega}{4\pi}\,\frac{\sigma^+(\omega) + \sigma^-(\omega)}{2}, \tag{2}$$

$$\mathrm{Im}\,F_2(\omega) = \frac{\omega}{4\pi}\,\frac{\sigma^+(\omega) - \sigma^-(\omega)}{2}. \tag{3}$$

The cross-sections σ^+ and σ^- belong to parallel and antiparallel polarisation of the initial particles, respectively. As well known, the assumption of an unsubtracted dispersion relation for the amplitude F_2 and the low energy theorem /9,10/

$$\left[\frac{\partial F_2}{\partial \omega}\right]_{\omega=0} = \frac{a}{2m^2}\,\kappa^2 \tag{4}$$

lead to the sum rules

$$\frac{2\pi^2 a}{m^2}\,\kappa_p^2 = \int_{m_\pi}^{\infty} \frac{\sigma_p^+ - \sigma_p^-}{\omega}\,d\omega, \tag{5}$$

$$\frac{2\pi^2 a}{m^2}\,\kappa_n^2 = \int_{m_\pi}^{\infty} \frac{\sigma_n^+ - \sigma_n^-}{\omega}\,d\omega, \tag{6}$$

where κ_p and κ_n are the two anomalous nucleon magnetic moments. The integration begins at the pion production threshold m_π and the integrals converge if the asymptotic condition

$$\sigma_{n,p}^+(\infty) = \sigma_{n,p}^-(\infty) \tag{7}$$

is valid. Considering in addition to eqs. (5) and (6) the corresponding deuteron magnetic moment sum rule and following ref. /5/ we write the T-matrix element for γd forward scattering in the form

$$T(k, d; k, d) = \sum_{i=1}^{4} I_i T_i \quad , \tag{7}$$

where the four independent kinematical invariants I_i are

$$I_1 = (\vec{U}_1 \vec{U}_2^*)(\vec{e}_1 \vec{e}_2^*) , \tag{9}$$

$$I_2 = (\hat{k} \vec{U}_1)(\hat{k} \vec{U}_2^*)(\vec{e}_1 \vec{e}_2^*) , (\hat{k} = \vec{k}/\omega) , \tag{10}$$

$$I_3 = (\vec{U}_1 \vec{e}_2)(\vec{U}_2^* \vec{e}_2^*) + (\vec{U}_1 \vec{e}_2^*)(\vec{U}_2^* \vec{e}_1) , \tag{11}$$

$$I_4 = (\vec{U}_1 \vec{e}_1)(\vec{U}_2^* \vec{e}_2^*) - (\vec{U}_1 \vec{e}_2^*)(\vec{U}_2^* \vec{e}_1) . \tag{12}$$

The deuteron and photon polarisation vectors are denoted by $\vec{U}_i$ and $\vec{e}_i (i = 1, 2)$, respectively. Sum rules can be derived for the amplitudes T_2 and T_4. The desired magnetic moment sum rule reads

$$\frac{4\pi^2 \alpha}{M^2} \left(1 - \frac{M}{2m} \mu_D\right)^2 = \frac{1}{M} \int_B^\infty \frac{\mathrm{Im}\, T_4(\omega)\, d\omega}{\omega^2} = \int_B^\infty \frac{\sigma_d^+ - \sigma_d^-}{\omega} d\omega \quad ^{x)} . \tag{13}$$

Here μ_D and M denote the magnetic moment and the mass of the deuteron. The limit B of the integral is the threshold for deuteron photodisintegration. Assuming now that for forward Compton scattering above the pion production threshold the deuteron may be replaced approximately by the free nucleons we get the cross-section relation

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x) We note that the substitution $\sigma_d^{\pm} \rightarrow \sigma_d^{s\pm}$, where s indicates the isoscalar part, leaves the sum rule unchanged because of the isoscalar Born term.

$$\sigma_d^{s\pm}(\omega) \approx \sigma_p^{\pm}(\omega) + \sigma_n^{\pm}(\omega), \quad \omega \gtrsim m_\pi . \tag{14}$$

The exact validity of eq. (14) in the asymptotic limit shows that the sum rule (13) also is guaranteed by the asymptotic assumption (7). Writing eq. (13) in the from

$$\frac{4\pi^2 a}{M^2}\left(1 - \frac{M}{2m}\mu_D\right)^2 = \int_B^{m_\pi} \frac{\sigma_d^{s+} - \sigma_d^{s-}}{\omega} d\omega + \int_{m_\pi}^{\infty} \frac{\sigma_d^{s+} - \sigma_d^{s-}}{\omega} d\omega \tag{15}$$

and inserting eqs. (14), (5) and (6) into the high energy integral we obtain the relation

$$\frac{4\pi^2 a}{M^2}\left(1 - \frac{M}{2m}\mu_D\right)^2 = \int_B^{m_\pi} \frac{\sigma_d^{s+} - \sigma_d^{s-}}{\omega} d\omega + \frac{2\pi^2 a}{m^2}(\kappa_p^2 + \kappa_n^2) \tag{16}$$

containing the polarized photodisintegration cross-sections only below the pion production threshold. Concluding this section we note the relation

$$M^{-2}\left(1 - \frac{M}{2m}\mu_D\right)^2 = 7{,}6 \cdot 10^{-4}\, m^{-2}(\kappa_p^2 + \kappa_n^2) \tag{17}$$

showing that the left hand side of eq. (16) may be neglected in our approximation. Thus results the sum rule

$$\frac{2\pi^2 a}{m^2}(\kappa_p^2 + \kappa_n^2) = -\int_B^{m_\pi} \frac{\sigma_d^{s+} - \sigma_d^{s-}}{\omega} d\omega \tag{18}$$

which will be evaluated in the following section.

3. *Calculation of the Low Energy Integral*

The polarized photodisintegration cross-sections σ_d^+ and σ_d^- can be expressed by the corresponding helicity amplitudes belonging to longitudinal deuteron polarisation. Using the notation of Jacob and Wick /11/ and Le Bellac et al./7/ we write them in the form

$$F_{1\pm} = F_{1,1,\pm\frac{1}{2},\pm\frac{1}{2}} = \frac{1}{8\pi\sqrt{2}} \sum_j D^j_{0,0}\, a_1^{(+)}(j), \tag{19}$$

$$F_{3\pm} = F_{1,-1,\pm\frac{1}{2},\pm\frac{1}{2}} = \frac{1}{8\pi\sqrt{2}} \sum_j D^j_{2,0} \; a_3^{(\pm)}(j), \tag{20}$$

$$F_{4\pm} = F_{1,1,\pm\frac{1}{2},\mp\frac{1}{2}} = -\frac{1}{8\pi\sqrt{2}} \sum_j D^j_{0,\pm 1} \; a_4^{(\pm)}(j), \tag{21}$$

$$F_{6\pm} = F_{1,-1,\pm\frac{1}{2},\mp\frac{1}{2}} = \frac{1}{8\pi\sqrt{2}} \sum_j D^j_{2,\pm 1} \; a_6^{(\pm)}(j), \tag{22}$$

where the four indices describe the spins of the photon, the deuteron and the two nucleons respectively. The well known parity formula

$$F_{-\lambda_1,-\lambda_2,-\mu_1,-\mu_2}(\theta,\phi) = F_{\lambda_1,\lambda_2,\mu_1,\mu_2}(\theta,\pi-\phi) \tag{23}$$

gives the other eight necessary helicity amplitudes. Since we need only the longitudinal deuteron polarisations the angle ϕ plays no role. Below the pion production threshold the dipole and quadrupole transitions are dominant. Neglecting higher multipoles and using appendix B of ref./7/ we obtain the following expressions for the coefficients

$$a_1^{(\pm)}(0) = \sqrt{\frac{2}{3}}\,[\mp M_1({}^1S_0) - E_1({}^3P_0)], \tag{24}$$

$$a_1^{(\pm)}(1) = \sqrt{\frac{3}{5}}\,[\pm\sqrt{3}\,M_2({}^1P_1) + E_2({}^3S_1) - \sqrt{2}\,E_2({}^3D_1)] \pm \sqrt{3}\,E_1({}^1P_1) - M_1({}^3S_1) + \sqrt{2}M_1({}^3D_1), \tag{25}$$

$$a_1^{(\pm)}(2) = \mp\sqrt{5}\,E_2({}^1D_2) - \sqrt{2}M_2({}^3P_2) + \sqrt{3}M_2({}^3F_2) + \sqrt{\frac{1}{3}}\,[\mp\sqrt{5}M_1({}^1D_2) + \sqrt{2}E_1({}^3P_2) - \sqrt{3}E_1({}^3F_2)], \tag{26}$$

$$a_1^{(\pm)}(3) = \frac{13}{6}\sqrt{\frac{1}{10}}\,[\mp\sqrt{7}\,M_2({}^1F_3) + \sqrt{3}\,E_2({}^3D_3) - 2\,E_2({}^3G_3)], \tag{27}$$

$$a_3^{(\pm)}(0) = a_3^{(\pm)}(1) = 0, \tag{28}$$

$$\begin{aligned} a_3^{(\pm)}(2) = &-\sqrt{\frac{2}{3}}\,[\pm\sqrt{5}\,E_2({}^1D_2) - \sqrt{2}\,M_2({}^3P_2) + \sqrt{3}\,M_2({}^3F_2)] + \\ &+ \sqrt{2}\,[\mp\sqrt{5}\,M_1({}^1D_2) + \sqrt{2}\,E_1({}^3P_2) - \sqrt{3}\,E_1({}^3F_2)], \end{aligned} \tag{29}$$

$$a_3^{(\pm)}(3) = 2\sqrt{\frac{1}{3}}\,[\mp\sqrt{7}\,M_2({}^1F_3) + \sqrt{3}\,E_2({}^3D_3) - 2\,E_2({}^3G_3)], \tag{30}$$

$$a_4^{(\pm)}(0) = 0, \tag{31}$$

$$\begin{aligned} a_4^{(\pm)}(1) = &\sqrt{\frac{3}{5}}\,[\pm\sqrt{3}\,M_2({}^3P_1) + \sqrt{2}\,E_2({}^3S_1) + E_2({}^3D_1)] \mp \\ &\mp \sqrt{3}\,E_1({}^3P_1) - \sqrt{2}\,M_1({}^3S_1) - M_1({}^3D_1), \end{aligned} \tag{32}$$

$$\begin{aligned} a_4^{(\pm)}(2) = &\mp\sqrt{5}\,E_2({}^3D_2) - \sqrt{3}\,M_2({}^3P_2) - \sqrt{2}\,M_2({}^3F_2) \pm \\ &\pm \sqrt{\frac{5}{3}}\,M_1({}^3D_2) + E_1({}^3P_2) + \sqrt{\frac{2}{3}}\,E_1({}^3F_2), \end{aligned} \tag{33}$$

$$a_4^{(\pm)}(3) = \frac{13}{6}\sqrt{\frac{1}{10}}\,[\pm\sqrt{7}\,M_2({}^3F_3) + 2\,E_2({}^3D_3) + \sqrt{3}\,E_2({}^3G_3)], \tag{34}$$

$$a_6^{(\pm)}(0) = a_6^{(\pm)}(1) = 0\,, \tag{35}$$

$$a_6^{(\pm)}(2) = \pm\sqrt{\frac{10}{3}}\,E_2(^3D_2) + \sqrt{2}\,M_2(^3P_2) + 2\sqrt{\frac{1}{3}}\,M_2(^3F_2) \pm \pm\sqrt{10}\,M_1(^3D_2) + \sqrt{6}\,E_1(^3P_2) + 2E_1(^3F_2)\,, \tag{36}$$

$$a_6^{(\pm)}(3) = 2\sqrt{\frac{1}{3}}\,[\pm\sqrt{7}\,M_2(^3F_3) + 2E_2(^3D_3) + \sqrt{3}\,E_2(^3G_3)]\,. \tag{37}$$

The two necessary total cross-sections as functions of the helicity amplitudes read

$$\sigma_d^+ = 4\pi\alpha\left(\frac{m}{8\pi E}\right)^2 \frac{p}{\omega}\,\frac{1}{2}\int d\Omega \sum_{\substack{\lambda_1=\lambda_2\\ \mu_1,\mu_2}} |F_{\lambda_1\lambda_2\mu_1\mu_2}|^2 = \alpha\left(\frac{m}{8\pi E}\right)^2\frac{p}{\omega}\,\frac{1}{8}\sum_{j=0}^{3}\left(\frac{1}{2j+1}\right)[|a_1^{(+)}(j)|^2 + |a_1^{(-)}(j)|^2 + |a_4^{(+)}(j)|^2 + |a_4^{(-)}(j)|^2]\,, \tag{38}$$

$$\sigma_d^- = 4\pi\alpha\left(\frac{m}{8\pi E}\right)^2 \frac{p}{\omega}\,\frac{1}{2}\int d\Omega \sum_{\substack{\lambda_1=-\lambda_2\\ \mu_1,\mu_2}} |F_{\lambda_1\lambda_2\mu_1\mu_2}|^2 = \alpha\left(\frac{m}{8\pi E}\right)^2\frac{p}{\omega}\,\frac{1}{8}\sum_{j=0}^{3}\left(\frac{1}{2j+1}\right)[|a_3^{(+)}(j)|^2 + |a_3^{(-)}(j)|^2 + |a_6^{(+)}(j)|^2 + |a_6^{(-)}(j)|^2] \tag{39}$$

where $p = (E^2 - m^2)^{1/2}$ is the centre of mass nucleon momentum and where the solid angle integration can easily be performed after insertion of the explicit expressions (19)-(22). Retaining only the isoscalars in eqs. (24)-(37) we use the tables of ref./7/ containing the needed multipole transitions. The phase shifts appearing in the interference terms are given in ref./12/. What remains to do

after interpolation of the cross-sections between the discrete energy values of the table is the numerical integration leading to the result

$$\int_B^{m_\pi} \frac{\sigma_d^{s+} - \sigma_d^{s-}}{\omega}\,d\omega = -17{,}2\,\alpha\cdot 10^{-5}\,[\mathrm{MeV}]^{-2} \tag{40}$$

4. Discussion

Comparing the result (40) with the predicted value given by the left hand side of eq. (18)

$$\frac{2\pi^2\alpha}{m^2}(\kappa_p^2 + \kappa_n^2) = 15{,}4\,\alpha\cdot 10^{-5}\,[\mathrm{MeV}]^{-2} \tag{41}$$

we note satisfactory agreement. The 10%-discrepancy should be due to used approximations and errors of the included parameters. These are:

1) replacement of the deuteron by the free nucleons in the region $\omega > m_\pi$,

2) neglect of octupoles and higher transitions in the region $\omega < m_\pi$ and neglect of e^4- corrections,

3) errors of the phase shifts, the n-p effective range r_t and of the d-state probability contained in the tabulated multipole transitions,

4) uncertainty of the cross-section interpolation between the discrete energy values.

Rough estimations indicate that these uncertainties can lead to an error of about 10% in eq. (40).

The result (18) was obtained by insertion of the isoscalar Drell-Hearn sum rule

$$\frac{\pi^2\alpha}{m^2}(\kappa_p^2 + \kappa_n^2) = \int_{m_\pi}^{\infty} \frac{\sigma_s^+ - \sigma_s^-}{\omega}\,d\omega\,, \quad \sigma_s^\pm = \frac{1}{2}(\sigma_p^\pm + \sigma_n^\pm) \tag{42}$$

into the deuteron magnetic moment sum rule (15) which shows the same behaviour of the integrand in the high energy limit. Thus our numerical result shows the validity of these two relations guaranteed by the asymptotic condition

$$\lim_{\omega\to\infty} [\, \sigma_s^{+}(\omega) - \sigma_s^{-}(\omega)\,] = 0 . \qquad (43)$$

The deuteron sum rule may be considered as a tool to test eq. (42). Concerning the validity of this equation we agree with the earlier results of Gerasimov/4/, Pagels/2/ and Drell and Hearn/6/. Since our result was obtained in a completely other way it supports these tests which are based on low energy and resonance approximations for pion photoproduction below 1 GeV and shows that the continuum in equation (42) contains no essential contribution from the high energy region above 1 GeV and from the nonresonant background.

The discrepancy between the isoscalar and measurable γd cross-sections is large near the threshold B where the dominant isovector transition $M_1({}^1S_0)$ contained in σ_d^{+} leads to a positive value of the difference $\sigma_d^{+} - \sigma_d^{-}$ while the corresponding isoscalar expression is negative in the whole energy region $B < \omega < m_\pi$.

Eq. (17) shows that the left hand side of the sum rule (15) is small of third order compared with the left hand side of the isoscalar sum rule (42). This means that the two integrals in eq. (15) almost compensate one another. So it seems difficult to calculate the deuteron magnetic moment from the integrals since now the insertion of the sum rule (42) would be too rough and in addition electromagnetic interactions of order e^4 should become important.

Concluding we remark that for the other scalar amplitude F_1 in eq. (1) only a subtracted dispersion relation may be postulated since the unpolarized γN cross-section appears in eq. (2). This fact also follows directly from the consideration of unpolarized γN forward scattering (see f.i. ref./2/) where an unsubtracted dispersion relation leads to the well known contradiction with Thomson's theorem.

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References

1. *Л.И.Лапидус, Чжоу Гуан-чжао. ЖЭТФ 39, 1286 (1960).*
2. *H. Pagels. Rockefeller University preprints (1967).*
3. *M. Hosoda and K. Yamamoto. Osaka University preprint (1966).*
4. *С.Б.Герасимов. ЯФ 2, 598 (1965).*
5. *K. Lewin and D. Robaschik. Phys. Lett., 25B 522 (1967).*
6. *S.D. Drell and A.C. Hearn. Phys.Rev.Lett., 16, 908 (1966).*
7. *M.Le Bellac, F.M. Renard and L.Tran Thanh Van. Nuovo Cimento, 33, 594 (1964); Nuovo Cimento, 34, 450 (1964).*
8. *M. Gell-Mann, M.L. Goldberger and W. Thirring. Phys.Rev., 95, 1612 (1954).*
9. *F. Low. Phys.Rev., 96, 1428 (1954).*
10. *M. Gell-Mann and M.L. Goldberger. Phys, Rev., 96, 1433 (1954).*
11. *M. Jacob and G.C. Wick. Ann.Phys., 7, 404 (1959).*
12. *R. Wilson. The Nucleon-Nucleon Interaction, Interscience Publishers, New York 1963.*

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