Nucl. Phys., 1968, v A113, n1, p. 215-22
25.I.68

ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2 - 3625

M.Micu

RECURSION RELATIONS FOR THE 3-j SYMBOLS

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1967 г.

E2 - 3625

M.Micu [x)]

RECURSION RELATIONS FOR THE 3-j SYMBOLS

x) On leave of absence from Institute of Atomic Physics, Bucharest, Rumania.

1. Introduction

The realization /1/ of the SU(2) group by boson operators represents the state with quantum number j, m in the form

$$[(j+m)!(j-m)!]^{-\frac{1}{2}} \mid \xi^{j+m} \eta^{j-m} > , \qquad (1)$$

where ξ and η are creation boson operators. In this basis the angular momentum operators become

$$L_+ = \xi \frac{\partial}{\partial \eta} , \quad L_- = \eta \frac{\partial}{\partial \xi} , \quad L_0 = \frac{1}{2}\left(\xi \frac{\partial}{\partial \xi} - \eta \frac{\partial}{\partial \eta} \right). \qquad (2)$$

The annihilation boson operators are written as derivative with respect to the creation operators.

In the papers /2,3/ an auxiliary creation boson operator ζ is introduced such that the $3-j$ symbols take the simple form of a scalar product

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = N < \prod_i \xi_i^{j_i+m_i} \eta_i^{j_i-m_i} \zeta_i^{J-2j_i} \left| \begin{vmatrix} \zeta_1 & \zeta_2 & \zeta_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \xi_1 & \xi_2 & \xi_3 \end{vmatrix}^J \right. > , \qquad (3)$$

where the normalization factor N is

$$N = \frac{(-1)^J}{J!}\left[(J+1)! \prod_i \{(j_i+m_i)(j_i-m_i)(J-2j_i)\}\right]^{-1/2}. \qquad (4)$$

Throughout this paper J denotes the sum $J = j_1 + j_2 + j_3$ and the relation $m_1 + m_2 + m_3 = 0$ is satisfied.

2. Recursion Relations

An operator ζ_1 from the left part of the scalar product acts in the right as annihilation operator $\frac{\partial}{\partial \zeta_1}$ and it gives

$$J(\eta_2 \xi_3 - \xi_2 \eta_3) \begin{vmatrix} \zeta_1 & \zeta_2 & \zeta_3 \\ \eta_1 & \eta_2 & \eta_3 \\ \xi_1 & \xi_2 & \xi_3 \end{vmatrix}^{J-1} . \tag{5}$$

The operator $\eta_2 \xi_3 - \xi_2 \eta_3$ which multiplies the $(J-1)$-th power of the determinant becomes $\frac{\partial}{\partial \eta_2} \frac{\partial}{\partial \xi_3} - \frac{\partial}{\partial \xi_2} \frac{\partial}{\partial \eta_3}$ in the left part of the scalar product. Comparison with the definition (3) and (4) gives the following relation

$$[(J+1)(J-2j_1)]^{½} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = [(j_2+m_2)(j_3-m_3)]^{½} \begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3-\frac{1}{2} \\ m_1 & m_2-\frac{1}{2} & m_3+\frac{1}{2} \end{pmatrix} - \tag{6}$$
$$- [(j_2-m_2)(j_3+m_3)]^{½} \begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3-\frac{1}{2} \\ m_1 & m_2+\frac{1}{2} & m_3-\frac{1}{2} \end{pmatrix} .$$

In a similar manner, acting with η_1 in the right and after that with $\xi_2 \zeta_3 - \zeta_2 \xi_3$ in the left part of the scalar product one obtains

$$[(J+1)(j_1-m_1)]^{½} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \tag{7}$$
$$= [(j_3+m_3)(J-2j_2)]^{½} \begin{pmatrix} j_1-\frac{1}{2} & j_2 & j_3-\frac{1}{2} \\ m_1+\frac{1}{2} & m_2 & m_3-\frac{1}{2} \end{pmatrix} - [(j_2+m_2)(J-2j_3)]^{½} \begin{pmatrix} j_1-\frac{1}{2} & j_2-\frac{1}{2} & j_3 \\ m_1+\frac{1}{2} & m_2-\frac{1}{2} & m_3 \end{pmatrix} .$$

All other recursion relations which can be obtained in this way follow directly from (6) and (7) by using the symmetries of the $3-j$ symbols /4/ relative to permutation of the three angular momenta and change of the sign of the magnetic quantum numbers.

Another recursion relation is obtained by using the fact that the operator $\xi_1 \frac{\partial}{\partial \eta_1} + \xi_2 \frac{\partial}{\partial \eta_2} + \xi_3 \frac{\partial}{\partial \eta_3}$ gives zero when it acts on the determinant from eq. (3). When this operator acts in the left part of the scalar product, in the form $\eta_1 \frac{\partial}{\partial \xi_1} + \eta_2 \frac{\partial}{\partial \xi_2} + \eta_3 \frac{\partial}{\partial \xi_3}$ it gives a sum of three terms. By using the definition (3) and (4) it follows

$$[(j_1+m_1+1)(j_1-m_1)]^{1/2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + [(j_2+m_2)(j_2-m_2+1)]^{1/2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1+1 & m_2-1 & m_3 \end{pmatrix} + [(j_3+m_3)(j_3-m_3+1)]^{1/2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1+1 & m_2 & m_3-1 \end{pmatrix} = 0. \quad (8)$$

By the same method one obtains

$$[(j_3+m_3+1)(J-2j_3)]^{1/2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + [(j_2+m_2)(J-2j_2+1)]^{1/2} \begin{pmatrix} j_1 & j_2-\frac{1}{2} & j_3+\frac{1}{2} \\ m_1 & m_2-\frac{1}{2} & m_3+\frac{1}{2} \end{pmatrix} + [(j_1+m_1)(J-2j_1+1)]^{1/2} \begin{pmatrix} j_1-\frac{1}{2} & j_2 & j_3+\frac{1}{2} \\ m_1-\frac{1}{2} & m_2 & m_3+\frac{1}{2} \end{pmatrix} = 0, \quad (9)$$

$$[(j_3+m_3)(J-2j_3+1)]^{1/2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + [(j_2+m_2+1)(J-2j_2)]^{1/2} \begin{pmatrix} j_1 & j_2+\frac{1}{2} & j_3-\frac{1}{2} \\ m_1 & m_2+\frac{1}{2} & m_3-\frac{1}{2} \end{pmatrix} + [(j_1+m_1+1)(J-2j_1)]^{1/2} \begin{pmatrix} j_1+\frac{1}{2} & j_2 & j_3-\frac{1}{2} \\ m_1+\frac{1}{2} & m_2 & m_3-\frac{1}{2} \end{pmatrix} = 0, \quad (10)$$

$$[(J-2j_1)(J-2j_2+1)]^{1/2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} + [(j_2-m_2)(j_1-m_1+1)]^{1/2} \begin{pmatrix} j_1+\frac{1}{2} & j_2-\frac{1}{2} & j_3 \\ m_1-\frac{1}{2} & m_2+\frac{1}{2} & m_3 \end{pmatrix} + [(j_2+m_2)(j_1+m_1+1)]^{1/2} \begin{pmatrix} j_1+\frac{1}{2} & j_2-\frac{1}{2} & j_3 \\ m_1+\frac{1}{2} & m_2-\frac{1}{2} & m_3 \end{pmatrix} = 0 \quad (11)$$

by using the operators

$$\xi_1 \frac{\partial}{\partial \zeta_1} + \xi_2 \frac{\partial}{\partial \zeta_2} + \xi_3 \frac{\partial}{\partial \zeta_3} \, , \quad \zeta_1 \frac{\partial}{\partial \xi_1} + \zeta_2 \frac{\partial}{\partial \xi_2} + \zeta_3 \frac{\partial}{\partial \xi_3}$$

and $\xi_2 \frac{\partial}{\partial \xi_1} + \eta_2 \frac{\partial}{\partial \eta_1} + \zeta_2 \frac{\partial}{\partial \zeta_1}$, respectively.

There are many bilinear operators which give zero acting on the determinant from eq. (3) but the recursion relations which can be obtained result also from eqs. (8) - (11) and above mentioned symmetries of 3 - j symbols.

By combining the relations (6) - (11) in an appropriate manner and by using the symmetries of 3 - j symbols it is possible to obtain other recursion relations for 3 - j symbols.

In a more symmetric form [3] the definition (3) may be written

$$\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = N < \prod_{ij} \xi_{ij}^{R_{ij}} \mid \begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{vmatrix}^{J} >. \qquad (12)$$

where

$$J = \Sigma_{i=1}^{3} R_{ij} = \Sigma_{j=1}^{3} R_{ij}$$

$$N = \frac{(-1)^{J}}{J!} \left[(J+1)! \prod_{ij} (N_{ij}!) \right]^{-\frac{1}{2}} \qquad (13)$$

and ξ_{ij} acts as a creation boson operator.

The determinant from the right-hand side of definition (12) can be considered as singledimensional representation of two $U(3)$ groups with the following generators

$$X_{ik} = \Sigma_{j=1}^{3} \xi_{ij} \frac{\partial}{\partial \xi_{kj}} \qquad (14)$$

for the first and

$$Y_{ik} = \Sigma_{j=1}^{3} \xi_{ji} \frac{\partial}{\partial \xi_{jk}} \qquad (15)$$

for the second group.

Since the definition (12) is invariant under the transformation

$$\xi_{ij} \to \xi_{ji} , \qquad R_{ij} \to R_{ji} \qquad (16)$$

we restrict ourselves to consider further only the first $U(3)$ group. Transformations generated by the subgroup $SU(2)$ ($i,k=1,2$) of this group give the following relation

$$\begin{pmatrix} R_{11} & R_{12} & R_{12} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} = \Sigma \; \Delta^{R'_{11} R'_{21}}_{R_{11} R_{21}} (\theta) \, \Delta^{R'_{12} R'_{22}}_{R_{12} R_{22}} (\theta) \, \Delta^{R'_{13} R'_{23}}_{R_{13} R_{23}} (\theta) \begin{pmatrix} R'_{11} & R'_{12} & R'_{13} \\ R'_{21} & R'_{22} & R'_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix}, \qquad (17)$$

where Δ denotes the transformation matrix which in a current notation /4/ is

$$\Delta^{a',b'}_{a,b}(\theta) = D^{\frac{1}{2}(a+b)}_{\frac{1}{2}(a'-b'),\frac{1}{2}(a-b)}(\theta). \qquad (18)$$

If we take θ to be a rotation of 180° around "y axis" we have

$$\Delta^{a'b'}_{ab}(\pi) = (-1)^{b}\,\delta_{ab'}\,\delta_{a'b} \qquad (19)$$

which leads to the fact, that in the form (12) the $3-j$ symbol is invariant under the permutations of lines and columns, the phase factor introduced by an odd permutation being $(-1)^{J}$.

By keeping in eq. (17) only the first power of that θ which multiplies the generator (14) for i=1, j=2 we obtain the recursion relation

$$[(R_{11}+1)R_{21}]^{1/2}\begin{pmatrix} R_{11}+1 & \cdot & \cdot \\ R_{21}-1 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} + [(R_{21}+1)R_{22}]^{1/2}\begin{pmatrix} \cdot & R_{12}+1 & \cdot \\ \cdot & R_{22}-1 & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} + [(R_{13}+1)R_{23}]^{1/2}\begin{pmatrix} \cdot & \cdot & R_{13}+1 \\ \cdot & \cdot & R_{23}-1 \\ \cdot & \cdot & \cdot \end{pmatrix} = 0, \qquad (20)$$

where the dots denote those elements which are not changed from a given value.

By using the symmetries /2/ of $3-j$ symbols we can put eq.(18) in the forms (8)-(11), the connection between the two notations of $3-j$ symbols being /3/

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} J-2j_1 & J-2j_2 & J-2j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{pmatrix}. \qquad (21)$$

Another type of recursion formula is obtained by acting with an operator, say ξ_{11}, from the left part of scalar product (12) in the right and carrying back in the left the term $\xi_{22}\xi_{33} - \xi_{32}\xi_{23}$ obtained when the determinant is derived with respect to ξ_{11}. The formula obtained in such a way

$$[R_{11}(J+1)]^{1/2}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = -(R_{22}R_{33})^{1/2}\begin{pmatrix} R_{11}-1 & \cdot & \cdot \\ \cdot & R_{22}-1 & \cdot \\ \cdot & \cdot & R_{33}-1 \end{pmatrix} + (R_{23}R_{32})^{1/2}\begin{pmatrix} R_{11}-1 & \cdot & \cdot \\ \cdot & \cdot & R_{23}-1 \\ \cdot & R_{32}-1 & \cdot \end{pmatrix} \qquad (22)$$

is equivalent to relations (3) and (4).

3. Other Recursion Relations

If the eq. (22) is multiplied by $(R_{22})^{1/2}$ and the last term from right side is replaced according to the relation (20) one obtains

$$[R_{11}R_{22}(J+1)]^{1/2}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}+(R_{22}+R_{23})(R_{33})^{1/2}\begin{pmatrix} R_{11}-1 & \cdot & \cdot \\ \cdot & R_{22}-1 & \cdot \\ \cdot & \cdot & R_{33}-1 \end{pmatrix}+[R_{23}R_{31}(R_{21}+1)]^{1/2}\begin{pmatrix} R_{11}-1 & \cdot & \cdot \\ R_{21}+1 & R_{22}-1 & R_{23}-1 \\ R_{31}-1 & \cdot & \cdot \end{pmatrix}=0 \tag{23}$$

which is the eq. (23) of ref. /3/.

The eq.(22) of ref. /3/ is obtained from (20) replacing R_{22} by $(R_{22}-1)$ and R_{12} by $(R_{12}+1)$ multiplying it by $R_{12}+1$ and modifying the last term according to eq.(22). One of the two new terms, thus introduced, is modified according to eqs.(20) and (22), finally giving

$$(R_{11}+R_{12}+1)(R_{22})^{1/2}\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}+[R_{13}(R_{23}+1)(R_{12}+1)]^{1/2}\begin{pmatrix} \cdot & R_{12}+1 & R_{13}-1 \\ \cdot & R_{22}-1 & R_{23}+1 \\ \cdot & \cdot & \cdot \end{pmatrix}+[R_{11}R_{33}(J+1)]^{1/2}\begin{pmatrix} R_{11}-1 & \cdot & \cdot \\ \cdot & R_{22}-1 & \cdot \\ \cdot & \cdot & R_{33}-1 \end{pmatrix}=0. \tag{24}$$

This result was obtained by Sharp /3/ as a generalized symmetry and by Louck /5/ as recursion relation.

By writting the eq. (22) three times, in the role of R_{11} being successively R_{11} , R_{22} and R_{33} , one obtains three equations from which it is possible to eliminate the two common $3-j$ symbols to find the relation

$$
(R_{22}-R_{33})[R_{11}R_{23}R_{32}]^{1/2}\begin{pmatrix} R_{11}-1 & \cdot & \cdot \\ \cdot & \cdot & R_{23}-1 \\ \cdot & R_{32}-1 & \cdot \end{pmatrix} + (R_{33}-R_{11})[R_{13}R_{22}R_{31}]^{1/2}\begin{pmatrix} \cdot & \cdot & R_{13}-1 \\ \cdot & R_{22}-1 & \cdot \\ R_{31}-1 & \cdot & \cdot \end{pmatrix} + \tag{25}
$$

$$
+(R_{11}-R_{22})[R_{12}R_{21}R_{33}]^{1/2}\begin{pmatrix} \cdot & R_{12}-1 & \cdot \\ R_{21}-1 & \cdot & \cdot \\ \cdot & \cdot & R_{33}-1 \end{pmatrix} = 0 .
$$

4. *Conclusions*

The realization of $U(2)$ and $U(3)$ groups by boson operators /1/ gives for $3-j$ symbols a simple definition which allows to obtain in a rapid manner the symmetries and recursion relations of this symbol. By taking into account all the symmetries of the $3-j$ symbol the number of independent recursion relations is only two, all other recursion relations follow from symmetry relations and algebraic combinations.

References

1. J.Schwinger. U.S.Atomic Energy Commission Rept., NYO-3071, 1952.
2. T.Regge. Nuovo Cimento, 10, 544 (1958).
3. R.T.Sharp. Nuovo Cimento, 47, 860 (1967).
4. A.R.Edmonds. Angular Momentum in Quantum Mechanics, Princeton University Press, 1957.
5. J.D.Louck. Phys. Rev., 110, 815 (1958).

Received by Publishing Department
on December 2, 1967.