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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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ON THE ANALYTIC CONTINUATION IN THE COUPLING CONSTANT IN THE NONRENORMALIZABLE FIELD THEORY MODEL

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1. Introduction

Recently an essential progress has been achieved in the interpretation of the structure of nonrenormalizable theories intensively investigated by the "model" and "axiomatic" approaches. The model approach consists in that the exact solutions of approximate (or exact) equations are studied in various simple models of nonrenormalizable field theories. This approach has resulted in the accumulation of a rather large amount of information which allows one to make a detailed comparision of renormalizable theories with nonrenormalizable ones going beyond the framework of perturbation theory.

The results of investigation of the models have underlied the axiomatic approach to nonrenormalizable theories. The main task of this approach consists in such a modification of usual axioms of the general quantum field theory which would allow to include into the axiomatics the properties of nonrenormalizable theories which are characteristic of investigated simple models. The necessity of modifying of the ordinary axiomatics is explained, first of all, by the fact that in all reasonable nonrenormalizable models the asymptotic behaviour of the amplitudes in the momentum variables off the mass shell turns out to be exponential (see, e.g. /1-3/, in ref. /3/ references to other papers can be found). This contradicts the usually used postulate of the polynomial boundedness which is, as is well-known, equivalent to the fact that the amplitudes in co-ordinate space are tempered distributions (see e.g. ref. /4/). The present situation in nonrenormalizable theories is such that it seems most advisable to use a certain combination of the two approaches described. In other words, we shall investigate the solutions for model theories which obey the general modified axioms of the field theory.

The exclusion of the polynomial boundedness and the inclusion of nonrenormalizable field theories into the axiomatics is explained not only by the interests of the theory. Experimental investigations of the large momentum transfer form factors and of the fixed angle elastic scattering amplitudes at high energies gave the exponential decrease behaviour for these quantities (see, e.g. ref. /6/ x)). As far as we know, one has succeeded in finding such a behaviour in no one of remormalizable theories. On the contrary, in nonrenormalizable models the exponential behaviour occurs in almost all models and there are various arguments in favour of the fact that it is just this property that distinguishes essentially nonrenormalizable theories from renormalizable ones /1-3/.

2. *Localizability and Asymptotics*

We consider briefly the modification of the axioms of quantum field theory, which is necessary for the description of nonrenormalizable theories xx). New formulations of the postulate replacing the polynomial bounded-

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x) Using the analyticity conditions we can show that this is the reason for their exponential increase in some other domain of the variables, perhaps in the unphysical one.

xx) The convinction is widely spread that for all nonrenormalizable theories the locality postilate is not fulfilled. As we shall see this convinction is not quite founded and, generally speaking, is wrong.

ness were suggested by Meiman /7/ and later on by Jaffe /8/. Meiman started from the necessity of retaining the causality condition, while Jaffe started from the condition of field localizability. We give here the Jaffe's formulation and its simplest consequences (very similar consequences follow from the Meiman formulation).

The field $A(x)$ is called localizable at the point x_0 if for sufficiently smooth functions $f(x)$ with the support localized in an arbitrarily small vicinity of the point x_0 there exists an operator (in Hilbert space of states)

$$A[f] = \int d^4x\, f(x) A(x) . \qquad (2.1)$$

For localizable fields the usual locality condition may be formulated as

$$[A[f], A[g]]_{\pm} = 0 , \quad \text{if} \quad \text{supp}[f] \approx \text{supp}[g] \qquad (2.2)$$

or in the limiting form.

$$[A(x), A(y)]_{\pm} = 0 , \quad \text{if} \quad x \approx y , \qquad (2.2)$$

where $\approx$ denotes "space-like". A field is called localizable, if it is localizable at any point of space-time. The localizability postulate replaces the postulate of polynomial boundedness, the remaining postulates being unaffected. From the requirement of localizability under some additional assumptions on the choice of the space-time functions $f(x)$ one can obtain restrictions on the increase of the Green functions, the vertex function and the scattering amplitude and, owing to analyticity, on their decrease, too /9/. For example, for the form factor of a scalar particle these restrictions are of the form /8/:

$$|F(q^2)| \leq C\, g(|q^2|); \qquad (2.3)$$

$$|F(q^2)| \underset{q^2\to -\infty}{>} C\, e^{-a\sqrt{-q^2}} ; \quad |F(q^2)| \underset{q^2\to +\infty}{>} \frac{M}{g(q^2)} ,$$

where $g(z)$ is the integer function obeying the condition:

$$\int_1^{\infty} dz \frac{\log^+|g(z)|}{z^{3/2}} < \infty \qquad (2.4)$$

in this case $\log^{+}|x| \equiv \log|x|$ provided that $|x| > 1$ and $\log^{+}|x| \equiv 0$ provided that $|x| \leq 1$. Similar restrictions can be obtained on the increase and decrease of the Green functions and the scattering amplitudes. Thus, localizable fields satisfying the locality condition (2.2) can lead to the exponential behaviour of the Green functions, the vertex functions and the scattering amplitudes. In simple models of nonrenormalizable field theories the general restrictions found are fulfilled and, in addition, the exponential asymptotics appears in fact. There exist however models in which these restrictions are not fulfilled and which are consequently nonlocalizable (see, e.g. paper /10/). Unfortunately, no criterion has been found, as yet, which would allow to say, basing on the form of the Lagrangian, whether the theory is localizable or not. It is extremely necessary to find such a criterion for understanding more deeply the structure of nonrenormalizable theories as well as for applying them to the elementary particle physics.

The experimental data on the proton electromagnetic form factor do not contradict the estimates (1.3). They can be represented by the formula /5/

$$F(q^2) \cong 0{,}7\, e^{-1{,}67\sqrt{-q^2}} \qquad \text{where} \qquad F = \frac{G_M}{\mu_p} . \qquad (2.5)$$

However, errors are still very large and the formula can change as it was the case with the Orear formula /1/ for the proton elastic scattering at large angles. According to the recent data /6/, instead of the Orear formula it is necessary to use the function $e^{-c\sqrt{tu}}$ but such an asymptotics can not be obtained in localizable theories. However this does not mean that the localizability condition really contradicts the experiment. In fact, at momentum transfers of the order of the mass of two nucleons the coefficient suddenly decreases. If the interpretation of this discontinuity as in ref. /12/ is true, then similar discontinuities will occur also for further energy increase so that the asymptotic decrease may turn out to be much more slowly than $e^{-c\sqrt{tu}}$. To prove this hypothesis it would be desirable to find an analogous discontinuity in the proton form factor behaviour and besides to find a mechanism which provides this rapid decrease at finite intervals.

Thus, for the time being we have no quite convincing evidence for the violation of the axioms of the general localizable field theory. Therefore we shall use consequences drawn from these axioms in order to overcome difficulties which arise in constructing the solutions of concrete nonrenormalizable models and which are due to the increase of these solutions. In the previous paper we have constructed an exponentially increasing solution by means of the analytic continuation in the momentum variables (this increase is in accord with (2.3)). In the present paper we shall consider the application of the analytic continuation in the coupling constant. In a broader sense we try to find a general method for working with exponentially increasing functions by using some concrete models. A similar problem was considered by other methods in refs. /13,14/ where there was a difficulty in going over from the co-ordinate space to the momentum one.

3. *The Dyson Equation in an Exactly Solvable Model*

Let us consider the well-known exactly solvable model in which the interaction of spinor particles ψ and scalar particles ϕ is described by the lagrangian:

$$L_{int} = g\bar{\psi}(x)\gamma^n\psi(x)\frac{\partial\phi}{\partial x^n} . \qquad (3.1)$$

Under the gauge transformation of the field ψ such a theory reduces to a free theory for the transformed operators $\Psi = e^{ig\phi}\psi$ and therefore does not lead to physically observable effects. However, the Green function G of the spinor field, the vertex function Γ and so on in this theory are different from the free ones and, using them as an example, we may investigate a number of general features of nonrenormalizable theories. The calculation of G by means of this transformation was made in many papers (see refs./15,16/). Here we use a quite different method.

To obtain the exact solution for the theory we make use of the connection between the vertex function $\Gamma(p,q)$ and the Green function $G(p)$

$$\Gamma(p,q) = i[G^{-1}(p) - G^{-1}(q)], \qquad (3.2)$$

where q and p are the momenta of incoming and outgoing fermions, respectively. This is easily obtained from the definition of the vertex function in the co-ordinate representation:

$$\Gamma(x,y\,|\,z) = \frac{\partial}{\partial z^n} < T\psi(x)\bar{\psi}(y)j^n(z)>, \qquad (3.3)$$

where $j^n(z) = \bar{\psi}(z)\gamma^n\psi(z)$ and the operators ψ and $\bar{\psi}$ satisfy the Heisenberg equations of motion. Eq. (3.3) is easily obtained by means of variational derivatives (see, e.g. ref./17/). The identity (3.3) allows to resolve the set of Dyson-Schwinger equations for the Green functions and to obtain a closed equation for G (see, e.g. ref./17/).

The Dyson- Schwinger equations for the fermion function and for the meson polarization operator are of the form:

$$G(p) = Z^{-1}S(p) + \frac{g^2}{(2\pi)^4}G(p)\int d^4q\,\Gamma(p,p-q)G(p-q)\hat{q}S(p)D(q) \qquad (3.4)$$

$$\Pi(k) = \frac{g^2}{(2\pi)^4}Sp\{\int d^4q\,G(q-k)\Gamma(q,q-k)G(q)\hat{k}\}, \qquad (3.5)$$

where $S(p)$ -is the free causal Green function of a fermion, $D(q)$ is the total Green function of a meson, $\Pi(k)$ is the polarization operator of a meson, Z is the renormalization constant of the fermion Green function.

First let us consider eq. (3.4). Using eq., (3.2) we get:

$$\Pi(k) = \frac{ig^2}{(2\pi)^4}\int d^4q\,Sp\{[G(q-k) - G(q)]\hat{k}\}.$$

By the obvious shift of the integration momentum q in the first term of the integrand it is easily seen that $\Pi(k) = 0$. This means that the meson Green function $D(k)$ coincides with the Green function $(m^2-k^2-io)^{-1}$ x). Now using again eq.(3.2) we get a linear integral equation for $G(p)$

$$G(p)=Z^{-1}S(p)-ig^2\int\frac{d^4q}{(2\pi)^4}G(q)\frac{(\hat{q}-\hat{p})}{m^2-(q-p)^2}S(p). \tag{3.6}$$

Equations of such a type were considered by us earlier[2,18,19] and we shall attempt to apply the method of ref.[2] to the study of this equation as well.

To simplify the calculation we consider the case when all masses are zero. Then $G(p)$ is of the form $G(p)=-\frac{\hat{p}}{p^2}f(p^2)$ hence it follows that f obeys the equation:

$$f(p^2)=Z^{-1}+g^2\int\frac{d^4q}{(2\pi)^4 i}\frac{q^2-(pq)}{q^2(p-q)^2}f(q^2). \tag{3.7}$$

We assume at first that in eq. (3.7) one can make the ordinary Wick rotation of the integration contour and thus pass to the Euclidean momenta p and q. Then eq. (3.7) takes on the following from:

$$f(x)=Z^{-1}-\frac{\lambda^2}{2}\left\{\frac{1}{x}\int_0^x dy\, f(y)\,y+\right.$$
$$\left.+2\int_x^\infty dy\, f(y)-x\int_x^\infty dy\,\frac{f(y)}{y}\right\}, \tag{3.8}$$

where $x=-p^2$, $y=-q^2$, $\lambda=\frac{g}{4\pi}$.

This equation reduces to the differential equation[3]

$$x^3f'''+3x^2f''+\lambda^2xf=0 \tag{3.9}$$

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x) The same result can be obtained from the Heisenberg equation of motion for ϕ: $(\Box-m^2)\phi=\partial_n j^n$. From the equations for ψ, $\bar{\psi}$ it follows that $\partial_n j^n=0$ i.e. ϕ is the free operator.

with the boundary conditions

$$x\,f(x) \underset{x\to\infty}{\to} 0\,, \qquad f(x) \underset{x\to 0}{\to} \mathrm{const}\;. \tag{3.10}$$

It is not difficult to see that this boundary problem has no solution. In fact one of the solutions decreases at infinity and two of them increase. Since the increasing solution should be rejected, then there remains only one solution by means of which it is impossible to satisfy the condition at the origin, it being $\approx \frac{1}{x}$ at $x \to 0$. Note that if in eq. (3.9) one changes sign of λ^2 then the problem (3.9)-(3.10) has a solution. In this case two solutions decrease and one increases. Therefore choosing a suitable linear combination of the two decreasing solutions we satisfy the boundary condition at the origin, too. In this case the initial integral equation has a solution only for unphysical (negative) values of λ^2 .
In the next section we shall try to make the analytic continuation to the physical values.

4. *Analytical Continuation in the Coupling Constant*

We consider an explicit form of the solution of the problem (3.9), (3.10) for negative λ^2:

$$f_-(x, \tilde{\lambda}^2) = G^{20}_{03}(\tilde{\lambda}^2 x \mid 1, 0, -1)\,, \tag{4.1}$$

where $\tilde{\lambda}^2 = -\lambda^2$ and G is the Meyer function[20] the expansion of which for small $\tilde{\lambda}^2$ is of the form:

$$G^{20}_{03}(\tilde{\lambda}^2 x \mid 1, 0, -1) = 1 + \tilde{\lambda}^2 x\, \ell og(\tilde{\lambda}^2 x) \sum_{n=0}^{\infty} \frac{(\tilde{\lambda}^2 x)^n}{n!(n+1)!(n+2)!} -$$

$$- \tilde{\lambda}^2 x \sum_{n=0}^{\infty} \frac{(\lambda^2 x)^n}{n!(n+1)!(n+2)!} (\psi_n + \psi_{n+1} + \psi_{n+2}) + \ldots, \psi_n = \frac{\Gamma'(n+1)}{\Gamma(n+1)}\;. \tag{4.2}$$

The solution (4.1) is normalized by the condition

$$f_-(0, \tilde{\lambda}^2) = 1 \quad .$$

This solution has a branch point in $\tilde{\lambda}^2$ plane at $\tilde{\lambda}^2 = 0$. Therefore the transition from $\tilde{\lambda}^2 > 0$ to $\tilde{\lambda}^2 < 0$ i.e. to the physical values is, at first sight, ambiguous and depends on the method of the analytic continuation. In fact, the singularity is described by the function $\log \tilde{\lambda}^2$ which has a cut for $\tilde{\lambda}^2 < 0$. The result will depend on the side of the cut from which we come up to the physical values $\tilde{\lambda}^2 < 0$.

We will try to find an unambiguous receipt of the analytical continuation, which is singled out by the physical requirements. To this end we consider the complex $\tilde{\lambda}^2$ plane (see Fig.1). Making continuation by the method I we have:

$$f_I(x, \lambda^2)\Big|_{\lambda^2 = -\tilde{\lambda}^2} = 1 - \lambda^2 x \log(\lambda^2 x) \sum_{n=0}^{\infty} \frac{(-\lambda^2 x)^n}{n!(n+1)!(n+2)!} + \qquad (4.3)$$

$$+ \lambda^2 x \sum_{n=0}^{\infty} (-\lambda^2 x)^n \frac{(\psi_n + \psi_{n+1} + \psi_{n+2})}{n!(n+1)!(n+2)!} - i\pi\lambda^2 x \sum_{n=0}^{\infty} \frac{(-\lambda^2 x)^n}{n!(n+1)!(n+2)!} \, .$$

Making continuation by the method II we have:

$$f_{II}(x, \lambda^2) = f_I^*(x, \lambda^2) \, . \qquad (4.4)$$

None of these methods is satisfactory from the physical point of view since according to the causality and spectrality conditions the function $f(x, \lambda^2)$ must be real for $x > 0$. In order that $f(x, \lambda^2)$ be real we should use the following method of analytic continuation. As f we should take the half-sum of f_I and f_{II}

$$f(x,\lambda^2) = \frac{1}{2}\left[f_I(x,\lambda^2) + f_{II}(x,\lambda^2)\right] =$$

$$= \frac{1}{2}\; G^{20}_{03}\left(\lambda^2 x e^{i\pi}\,\middle|\, 1,0,-1\right) + \frac{1}{2}\; G^{20}_{03}\left(\lambda^2 x e^{-i\pi}\,\middle|\, 1,0,-1\right)$$

$$= 1 - \lambda^2 x \,\ell og\,(\lambda^2 x)\sum_{n=0}^{\infty} \frac{(-\lambda^2 x)^n}{n!(n+1)!(n+2)!} + \lambda^2 x \sum_{n=0}^{\infty} \frac{(-\lambda^2 x)^n(\psi_n + \psi_{n+1} + \psi_{n+2})}{n!(n+1)!(n+2)!} .$$

The question may appear why it is impossible to add to this solution an integer function which is also the solution of eq. (3.9) and which is real on the real axis. To answer this question we make a more detailed consideration of eq. (3.7). It is obvious that any solution of this equation is of the form: $f(x,\lambda) = F(g\ x)$, and in this case $F(t^2)$ satisfies the equation:

$$F(t^2) = Z^{-1} + \int \frac{d^4 s}{(2\pi)^4 i}\; \frac{s^2 - (s\,t)}{s^2\left[(s-t)^2 + i\epsilon g^2\right]}\; F(s^2)\, , \tag{4.6}$$

where $s = g q$, $t = g p$ In eq. (4.6) the change of g^2 in sign results only in deformation of the integration contour, i.e. the causal function is replaced by "anticausal" one. When $F(s^2)$ is decreasing function for which the usual spectral representation can be written, a rule can be easily formulated which allows one to find a solution with $\epsilon < 0$ using the solution with $\epsilon > 0$. To this end it is sufficient to insert in eq. (4.6) the spectral representation for F and take the integral over s . This reduces to the calculation of the self-energy diagram in which one of the propagators has ϵ with a changed sign. This calculation allows one to justify the rule (19) when eq. (20) has the solution.

If the solution does not exist it should be derived using the obtained receipt of analytical continuation which is meaningfull for the increasing asymptotics as well. The analytical continuation in g variable for the Green functions in the x - space was considered earlier in ref.[16]. A wrong result has been obtained there since one has not singled out the singularity in g at $g=0$ (see ref.[22]) and the analytic continuation has been made only by one way. In making calculations in the x-space we have to work, as was already mentioned, with nontempered distributions. So, when passing to the p-space there arise difficulties connected with an unambiguous determination of the Fourier transform for such such function (see refs.[16,13]). The advantage of the p-space is a more obvious correspondence with usual diagram methods and the possibility of applying customary and reliable methods of the theory of analytical functions.

Thus, we have found an unambiguous receipt of the analytical continuation of the function from the negative values of the squared charge to the positive ones. Now we consider the properties of the solution obtained in such a way. The solution has correct analytical properties, i.e. it is analytical in the whole complex plane x with the cut along the negative real axis. The discontinuity on this cut is

$$2\,i\,\mathrm{Im}\,f(x,\lambda^2) = -2i\lambda^2 x\,\pi \sum_{n=0}^{\infty} \frac{(-\lambda^2 x)^n}{n!\,(n+1)!\,(n+2)!} = \tag{4.7}$$

$$= -2\pi i\, G^{10}_{03}(\lambda^2 x \mid 1,0,-1), \qquad x<0 .$$

This function increases at $x \to -\infty$ as $\exp\{c\sqrt[3]{-\lambda^2 x}\}$ and therefore it is impossible to write for it a dispersion relation with a finite number of subtractions. The real part on the cut is

$$G^{20}_{03}(-\lambda^2 x \mid 1,0,-1), \qquad x<0 \tag{4.8}$$

and decreases at $x \to -\infty$. The function increases exponentially over the whole complex plane, in paticular, the asymptotic behaviour is of the form

$$\frac{a}{(\lambda^2 x)^{1/3}}\, e^{3 e^{i\frac{\pi}{3}} \sqrt[3]{\lambda^2 x}} + \frac{a^*}{(\lambda^2 x)^{1/3}}\, e^{3 e^{-i\frac{\pi}{3}} \sqrt[3]{\lambda^2 x}} \tag{4.9}$$

From here it is clear why we have not succeeded in obtaining this function as a solution of the boundary problem (3.9), (3.10).

In previous papers /2,3/ we have investigated in detail similar problems in which the solution was found however to be decreasing throughout the whole complex plane of the momentum variable. As an example we have also considered the problem in which the solution decreased only in a part of the p-plane /18/. In this case it was necessary to make an analytic continuation of the solution from the region of decrease to the region of increase. In the problem considered here the analytic continuation (under the condition that the general restrictions following from the axioms of the field theory should be obligatorily fulfilled) is the main tool for the construction of the solution. We may hope that this method will be also useful in investigating more realistic unrenormalizable theories.

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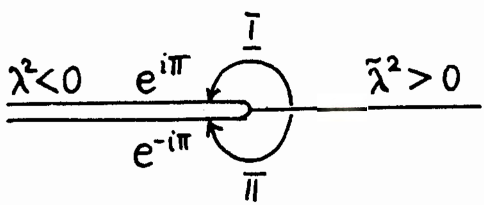

Fig.1.