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FAITHFUL REPRESENTATIONS OF ALGEBRAS OF TEST FUNCTIONS

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1. Introduction and Definitions

This article is a continuation of a recent paper /1/ about the topological $*$-algebra of test functions /2,3/ in the Wightman axiomatical quantum theory. The results are obtained for a more general class of "test functions" algebras R. It is shown that the topological $*$-algebras R are reduced (semisimple, in the sense of Rickart) and consequently, they can be faithfully represented as an algebra of (unbounded) operators in a Hilbert space H. If the topological algebra R is separable so the representation space H can be chosen separable, too.

We repeat the definitions of /1/. The topological $*$-algebra is the topological direct sum $R = \bigoplus_{n=0}^{\infty} R_n$, where $R_0 = C$ is the field of complex numbers and R_n for $n = 1, 2, \ldots$ is a locally convex linear topological space (over the complex field) of complex-valued functions on $M^{(n)} = M \times \ldots \times M$ (n times). M is a point set (the Minkowski space, for example). We assume $\|a_n\|_0 = \sup_{x_1, \ldots, x_n \in M} |a_n(x_1, \ldots, x_n)|$ to be a continuous norm on the topological space R_n. The multiplication in R for two elements $a = \sum_{n \geq 0} a_n$ and $b = \sum_{n \geq 0} b_n$ is defined by $a \cdot b = \sum_{n \geq 0} (ab)_n$ with $(a \cdot b)_n = \sum_{k+l=n} a_k(x_1, \ldots, x_k)\, b_l(x_{k+1}, \ldots, x_n)$ and the $*$-operation is defined by $a^* = \sum_{n \geq 0} (a^*)_n$ with $(a^*)_n(x_1, \ldots, x_n) = \bar{a}_n(x_n, \ldots, x_1)$ (the bar labbels the complex conjugate function). The existence of $a \cdot b$ and a^* in R and the continuity of these operations are assumed. With these operations R is a topological $*$-algebra over the complex field. In the usual cases in /2,3/, where $R_n = S(M^n)$ resp. $D(M^n)$ (the well-known Schwartz' spaces) and M is the Minkowski space, all assumptions are satisfied.

For two sequences $\alpha_0, \alpha_1, \ldots$ and $\beta_0, \beta_1, \ldots$ of positive numbers we define in R the norm $\|a\| = \sum_{n \geq 0} \alpha_n \|a_{2n}\|_0 + \sum_{n \geq 0} \beta_n \|a_{2n+1}\|_0$

Beside the basic-topology, this means the direct-sum topology, we regard in R a second topology defined by the norm $\| \ \|$ which is called the norm-topology in R. With respect to this topology R becomes a normed linear space, but not a normed algebra. This norm-topology is weaker than the topology in R, defined by the topological direct sum.

Let K_0 be the convex cover of the set of elements a^*a for $a \in R$. It is easy to see that K_0 is a cone in R, i.e. for k, $k' \in K_0$ and two arbitrary positive numbers s, t it is $sk + tk' \in K_0$ and if $k \in K_0$ and $k \neq 0$ then $-k \notin K_0$.

In /1/, Theorem 1, it is proved the following

<u>Lemma 1</u>

The sequence $\alpha_0, \alpha_1, \ldots$ in the definition of the norm $\| \ \|$ can be chosen so that in R the topological closure $K_{\| \|}$ of K_0 with respect to the norm $\| \ \|$ is a cone. Consequently, the topological closure $\bar{K}_0$ of K_0 in R with respect to the direct-sum topology is a cone, too, because $\bar{K}_0 \subset K_{\| \|}$ holds.

The sequence $\beta_0, \beta_1, \ldots$ can be chosen arbitrary positive. In the following $\| \ \|$ is always the norm from Lemma 1.

2. Positive Functionals on R

Theorem 1

For each $b \in R$, $b \neq 0$, there exists a positive continuous linear functional $W_b(a)$ on R with $W_b(b) \neq 0$ and $W_b(k) \geq 0$ for $k \in K_{\|\ \|}$. For $b \in K_{\|\ \|}$ the functional $W_b(a)$ can be chosen so that $|W_b(a)| \leq \| a \|$ holds. Consequently, the topological $*$-algebra R is reduced(/4/, p.270).

Proof:

Let first $b \neq 0$ be an element of $K_{\|\ \|}$. It is $0 \notin b + K_{\|\ \|}$, because $K_{\|\ \|}$ is a cone. Further let $U = \{ u : \| u \| < \delta \}$ be such a neighbourhood of the origin, that $U \cap (b + K_{\|\ \|}) = \emptyset$ holds. Now we define $L = \{ i(k_1 - k_2) : k_1, k_2 \in K_{\|\ \|},\ i^2 = -1 \}$ and $K_1 = \{ k + sb + su : k \in K_{\|\ \|},\ s \geq 0,\ u \in U \}$. L is a real linear space in R and K_1 a cone with the interior point b and we find $L \cap K_1 = \{0\}$ (the origin). For if $a = i(k_1 - k_2) = k + sb + su \in L \cap K_1$, $k_1, k_2, k \in K_{\|\ \|}$, $u \in U$, $s \geq 0$, it follows $a^* = -a$, i.e. $k + sb + su^* = -k - sb - su$ and finally $k + sb + su_1 = 0$, $u_1 = \frac{1}{2}(u^* + u) \in U$. If $s > 0$, then it would be $\frac{1}{s} k + b \in U$ and this is a contradiction to the construction of U. Therefore we have $s = 0$ and consequently $k = 0$, too, i.e. $a = 0$.

Now we use

Lemma 2 (Mazur S.)

Let K be a convex set with an interior point b in a real locally convex space R and L a linear subspace of R which does not contain an interior point of K. Then there exists a linear continuous functional $f(a)$ on R with $f(k) \geq 0$ for $k \in K$, $f(b) > 0$ and $f(a) = 0$ for $a \in L$ /5/.

If we regard R as a normed linear space over the real field, it follows from this Lemma the existence of a real linear continuous functional $f(a)$ on R with $f(a) = 0$ for $a \in L$, $f(k) \geq 0$ for $k \in K_1$ and $f(b) > 0$. Then $W_b(a) = f(a) - i\, f(ia)$ is a linear functional on the complex linear space R, continuous with respect to the norm-topology in R and it holds $W_b(b) \neq 0$ and $W_b(k) = f(k) - i\, f(ik) = f(k) \geq 0$ for $k \in K_{\| \|}$, because $ik \in L$. This implies $W_b(a)$ is a positive functional on the algebra R. Evidently, we can choose $W_b(a)$ so that $|W_b(a)| \leq \|a\|$ holds. Of course, these functionals are continuous with respect to the basic-topology in R, too.

Because for every $b = a^*a \in K_{\| \|}$ a positive functional $W_b(a)$ with $W_b(b) \neq 0$ exists if $b \neq 0$, there exists such a functional for an arbitray $b \neq 0$ of R (/4/ p. 271) which is continuous with respect to the basic-topology. In general, it is not continuous with respect to the norm $\| \|$.

Corollary:

The set $\{ W_b : b \in K_{\| \|} \}$ of these positive functionals is a relatively bicompact set in the weak topology in R' (the dual space of R).

The Corollary follows from well-known facts /6/, because for these functionals $|W_b(a)| \leq \|a\|$ holds and, consequently, they are equicontinuous.

3. Faithful Representations of R

Theorem 2

The topological $*$-algebra R can be faithfully represented as a $*$-algebra of (unbounded) operators in a Hilbert space H.

If the algebra R is separable, the Hilbert space H can be chosen separable, too.

Remark:

In the usual cases for Wightman fields, where R is the tensor algebra over S (or D) /2,3/, i.e. $R_n = S^{4n}$ (or D^{4n}), R is separable.

Let us first recall the definition of a faithful representation.

Definition: A representation of a topological *-algebra R as (unbounded) operators in a Hilbert space H is given, if for every $a \in R$ there is a linear operator $A(a)$ in the Hilbert space H so that

1. for all $a \in R$ the domain $D(A(a)) = D$ is the same dense subspace of H and D is invariant for all $A(a)$, $A(a)D \subset D$, and it holds $D(A(a)^*) \supset D$.

2. for $a, b \in R$ and $\phi \in D$ it holds $A(ab)\phi = A(a)A(b)\phi$, $A(\alpha a + \beta b)\phi = \alpha A(a)\phi + \beta A(b)\phi$ and $A(a^*)\phi = A(a)^*\phi$.

3. $(A(a)\phi, \psi)$ with $\phi, \psi \in D$ is a continuous function on R.

The representation $A(a)$ is said to be faithful if $a \to A(a)$ is an one-to-one mapping.

Proof of Theorem 2

Let $\mathcal{F}$ be a system of positive functionals on R such that for each $a \in R$, $a \neq 0$, in $\mathcal{F}$ one can find a positive functional $W \in \mathcal{F}$ with $W(a^*a) \neq 0$. By Theorem 1 such a system $\mathcal{F}$ exists for R. Then for each positive functional $W \in \mathcal{F}$ by the Neumark-Gelfand-Segal construction there exists a cyclic representation $A_W(a)$ of R in a Hilbert space H_W with an invariant domain D_W and a cyclic vector ϕ_W. For this representation it holds $W(a^*a) = \| A_W(a)\phi_W \|^2$. Let $A(a) = \bigoplus_{W \in \mathcal{F}} A_W(a)$ be the direct sum all these representations $A_W(a)$. This is a representation of R in the Hilbert space $H = \bigoplus_{W \in \mathcal{F}} H_W$ with the invariant domain $D = \sum_{W \in \mathcal{F}} D_W$. This representation $A(a)$ is faithful, because for each $a \neq 0$ there exists a $W \in \mathcal{F}$ with $\| A(a)\phi_W \|^2 = \| A_W(a)\phi_W \|^2 = W(a^*a) \neq 0$.

Now we must yet prove the second assertion of the Theorem. If the algebra R is separable so the Hilbert spaces H_w are separable, too. Consequently, the second assertion would be proved , if for a separable R the system $\mathfrak{F}$ could be chosen countable. To prove this we use the

<u>Lemma 2</u>

If X is a separable linear topological space and G a bicompact set in the weak topology in X' (the dual space of X), then the weak topology in G can be given by a metric /6/.

Let G be the weak closure of the set $\{ W_b : b \in K_{\| \|} \}$. G contains only positive functionals and by the Corollary to Theorem 1 G is bicompact in the weak topology in R' . In consequence of the last Lemma, G is a bicompact metric space and therefore separable. Let $\mathfrak{F}$ be a countable dense subset of G , then $\mathfrak{F}$ has the desired properties.

R e f e r e n c e s

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