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POINCARE AND LORENTZ INVARIANT EXPANSIONS OF RELATIVISTIC AMPLITUDES

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Пуанкаре и Лоренц инвариантные разложения релятивистских амплитуд

Обсуждаются двойные разложения релятивистских амплитуд по неприводимым представлениям однородной группы Лоренца, которые были предложены в более ранней работе для произвольных значений кинематических переменных s и t. Изучается связь между этими разложениями и релятивистским фазовым анализом в терминах представлений различных малых групп группы Пуанкаре.

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Poincaré and Lorentz Invariant Expansions of Relativistic Amplitudes

This paper is devoted to a discussion of double expansions of relativistic amplitudes in terms of the irreducible representations of the homogeneous Lorentz group, suggested previously for arbitrary values of the kinematical variables s and t. The relation between these expansions and relativistic phase shift analysis in terms of various little groups of the Poincaré group is studied.

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POINCARE AND LORENTZ INVARIANT EXPANSIONS OF RELATIVISTIC AMPLITUDES

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1. Introduction

In the last few years much attention has been devoted to the problem of expansions of relativistic amplitudes for scattering and reaction processes[1-23]. Such expansions will clearly play an important role in the investigations of the general analytical properties of amplitudes and they can serve as natural means to further develop and generalize various approaches to the theory of strong interactions, such as the Regge pole method, asymptotic theorems etc. They help to give a natural group theoretical interpretation of such important concepts as complex angular momentum, signature, etc. and they have become specially interesting in view of recent developments in high-energy scattering theory, like the hypothesis of fixed poles, the possible existence of daughter trajectories[22,23] and cuts in the complex J -plane.

2. Two Approaches to Relativistic Expansions

In the paper of Vilenkin and Smorodinsky[3] (further refered to as SV) a general theory of Lorentz invariant expansions is developed. The ideology of this approach (approach I) can be stated in the following manner. Amplitudes for processes of the type $1 + 2 \to 3 + 4$ depend upon two independent kinematical variables. The usual Mandelstam parameters s and t are not convenient for relativistic expansions, since the region in which they are defined (a sector in the Mandelstam plane) does not carry any reasonable geometry and what is more, if one of these parameters is fixed, the other one does not give a convenient parametrization of any subgroup of the Lorentz or Poincaré groups. Instead of s and t we can choose the components of the 4-velocity $u_\mu = \frac{p_\mu}{m}$ of one of the particles and then consider the amplitude as a function defined on the upper sheet of the hyperboloid $u^2 = 1$ (in the zero-spin case the function does not depend on the azimuthal angle ϕ). Various possible parametrizations give rise to various expansions considered in /3,4/.

The meaning of relativistic invariance for these expansions is two-fold. The first point of view simply expresses the Lorentz invariance (and also the Poincaré invariance) of the amplitude, which means that we can place the kinematical graph /5/ describing the scattering, arbitrarily on the hyperboloid $u^2 = 1$, for instance by identifying the vertex of the hyperboloid with the velocity of one of the particles or of the centre-of-mass, brick-wall or any other system and by choosing a convenient direction of the space axes. Lorentz transformations from one system to another, will correspond to motions of the kinematical graph as a whole on the hyperboloid and invariance of the amplitude means that the transformations of the expansion coefficients under irreducible representations of the Lorentz group will be compensated by transformations of the relations connecting the chosen kinematic variables with s and t.

The second implication of the considered expansions, which we shall call "extended Lorentz invariance" is more interesting and expresses certain analytical properties of scattering amplitudes. Since the amplitude is now determined on a hyperboloid, the group of motions of which is isomorphous to the homogeneous Lorentz group, different values of the variables are connected by Lorentz transformations. The obtained expansions in terms of the basis functions of irreducible representations of the Lorentz group, make it possible in principle to express the values of the amplitude for arbitrary values of the parameters in terms of its value for one known set of parameters. It should be stressed, that contrary to usual Lorentz (or Poincaré) invariance, which deals with one and the same amplitude as seen by different observers, "extended Lorentz invariance" deals with different values (for different values of the kinematical parameters s and/or t) as seen by the same observer.

The assumed convergence of the expansions, in the direct sense (implying square-integrability for amplitudes expanded in terms of unitary representations of the principal series) or in some generalized one, is a new assumption and leads to new physical results, such as asymptotic theorems [7]. Thus the content of extended Lorentz invariance are certain convergence assumptions and these can be considered to be manifestations of the existence of certain (unknown) equations of motion or causality principles.

A different approach (approach II) was developed in the papers of Joos [10,11], Toller [12,13], Salam and others [14,15]. Their treatment was based directly on the Poincaré invariance of the scattering amplitude and they consider two different types of "coupling schemes" for the irreducible representations of the Poincaré group. Essentially, as stressed in [12,14], this is equivalent to the following procedure. The amplitude for a fixed value of the total momentum $p_1 + p_2$ (direct coupling) or momentum transfer $p_3 - p_1$ ("crossed channel coupling") is defined as a function over the little group of the Poincaré group, corresponding to this fixed momentum and then expanded in terms of the irreducible representations of this little group. Mathematically this corresponds to harmonic analysis of scattering amplitudes in terms of the $0(3)$ group representations for $p_1 + p_2$ fixed and in terms of $0(3), 0(2,1), E(2)$ or $0(3,1)$ representations for $p_3 - p_1$ fixed and respectively timelike, spacelike, lightlike and zero-vector.

The aim of this paper is to establish the relation between the two mentioned approaches and incidentally to stress that the first approach, based on the group of motions $0(3,1)$ of the space of independent kinematical variables is also explicitly invariant with respect to the Poincaré group.

To make the problem more clear, let us remind that Poincaré invariance implies that the amplitude depends only on two "essential" kinematical variables (s and t or suitable combinations of these) and that all other variables (e.g. all other components of the 4-momenta of the initial and final particles) can be fixed ad hoc. In SV the choice of the coordinate system is equivalent to the choice of specific values of the unessential parameters. The two essential parameters serve as the expansion variables. In the method of Toller, Salam et al. the restriction of the Poincaré group to one of its little groups corresponds to the attribution of a constant value to one of the essential parameters. The little group is parametrized by the second essential parameter (and the angle ϕ), which serves as the only expansion variable. This is equivalent to the more rigorous (but more complicated) procedure suggested by Joos[10,11], based directly on the decomposition of direct products of irreducible representations of the Poincaré group, using explicit expressions for the Clebsch-Gordan coefficients of this group[20], in order to obtain Poincaré invariant expansions of amplitudes.

3. Coordinate Systems and Little Groups

Let us now proceed to the explicit formulae for expansions in the individual coordinate systems and indicate their relation to the little group expansions.

To simplify the argumentation we shall only consider the binary scattering of zero-spin particles. A consideration of the general spin case is in progress[8].

In this paper we shall only consider three of the thirty four[24] possible expansions on a three-dimensional hyperboloid (seven of which are related to subgroups of the Lorentz group[4,9]).

a) The Spherical System (S-system)

The S-system in which a four-velocity is parametrized as

$$u = (u_0, u_1, u_2, u_3) = (\mathrm{ch}a , \mathrm{sh}a \sin\theta \cos\phi, \quad \mathrm{sh}a \sin\theta \sin\phi , \mathrm{sh}a \cos\theta) \tag{1}$$

$$0 \leq a < \infty \qquad 0 \leq \theta \leq \pi \qquad 0 \leq \phi < 2\pi$$

is specially suitable for the consideration of scattering in the centre-of-mass system. Really, let us consider the s-channel process

$$1 + 2 \longrightarrow 3 + 4 \tag{2}$$

and choose the coordinate system such that $0xz$ is the reaction plane (i.e. $\phi = 0$ for all particles), that the three-momentum $\vec{p}_1$ is along the axis $0z$ and the four-velocity of the centre-of-mass

$$u_s = \frac{p_1 + p_2}{\sqrt{s}}$$

has the components $u_s = (1,0,0,0)$ (see fig. 1a, 2a) The momenta of the particles now are

$$\begin{aligned}
p_1 &= m_1(\mathrm{ch}a_1, \quad 0 \quad , \quad 0 \quad , \mathrm{sh}a_1 \quad) \\
p_2 &= m_2(\mathrm{ch}a_2, \quad 0 \quad , \quad 0 \quad , -\mathrm{sh}a_2 \quad) \\
p_3 &= m_3(\mathrm{ch}a_3 , \mathrm{sh}a_3 \sin\theta_s , \quad 0 \quad , \mathrm{sh}a_3 \cos\theta_s) \\
p_4 &= m_4(\mathrm{ch}a_4 , -\mathrm{sh}a_4 \sin\theta_s , \quad 0 \quad , -\mathrm{sh}a_4 \cos\theta_s) .
\end{aligned} \tag{3}$$

Momentum conservation and the choice of the c.m.s. in which $\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 = 0$ implies that

$$m_1 \mathrm{sh}a_1 = m_2 \mathrm{sh}a_2 \qquad m_1 \mathrm{ch}a_1 + m_2 \mathrm{ch}a_2 = m_3 \mathrm{ch}a_3 + m_4 \mathrm{ch}a_4 . \tag{4}$$
$$m_3 \mathrm{sh}a_3 = m_4 \mathrm{sh}a_4 .$$

Let us choose $a_3 = a_s$ and θ_s as the independent variables. In terms of $s = (p_1 + p_2)^2$ and $t = (p_1 - p_3)^2$ we now have

$$m_3 \mathrm{ch}a_s = \frac{s + m_3^2 - m_4^2}{2\sqrt{s}}$$

$$\cos\theta_s = \frac{2s(t - m_1^2 - m_3^2) + (s + m_1^2 - m_2^2)(s + m_3^2 - m_4^2)}{\sqrt{[s-(m_1-m_2)^2][s-(m_1+m_2)^2][s-(m_3-m_4)^2][s-(m_3+m_4)^2]}} \tag{5}$$

The analogous formulae for the physical region of the t-channel

$$2 + \bar{4} \longrightarrow \bar{1} + 3 \tag{6}$$

in which the 4-velocity

$$u_t = \frac{p_2 - p_4}{\sqrt{t}}$$

is placed in the vertex of the hyperboloid $u_t = (1,0,0,0)$ will be

$$m_1 \operatorname{ch} a_t = \frac{t + m_1^2 - m_3^2}{2\sqrt{t}}$$

$$\cos\theta_t = -\frac{2t(s - m_1^2 - m_2^2) + (t + m_1^2 - m_3^2)(t + m_2^2 - m_4^2)}{\sqrt{[t-(m_1+m_3)^2][t-(m_1-m_3)^2][t-(m_2+m_4)^2][t-(m_2-m_4)^2]}} . \tag{7}$$

In terms of these variables the expansion of the scattering amplitude is

$$f(a,\theta) = \sum_{\ell=0}^{\infty} \int_0^{\infty} \rho^2 \, d\rho \, A_\ell(\rho) \frac{1}{\sqrt{\operatorname{sh} a}} P^{-(\ell+1/2)}_{-1/2+i\rho}(\operatorname{ch} a) P_\ell(\cos\theta) . \tag{8}$$

The inverse formula to (8), as well as to the expansion formulae in all other coordinate systems, is given in SV. $P^{\mu}_{\nu}(z)$ are Legendre functions.

In approach II the choice of the c.m. system just leads to the conventional phase-shift analysis. Namely, the little group is chosen by fixing the vector $p_1 + p_2$, thus also fixing the essential parameter a. The expansion is simply

$$f(a,\theta) = \sum_{\ell=0}^{\infty} B_\ell(a) P_\ell(\cos\theta) . \tag{9}$$

Thus, in this case the SV expansion can be obtained independently of group theoretical considerations by expanding the coefficient in (9)

$$B_\ell(a) = \int_0^\infty \rho^2 \, d\rho \; A_\ell(\rho) \, \frac{1}{\sqrt{\operatorname{sh} a}} \, P^{-(\ell+1/2)}_{-1/2+i\rho}(\operatorname{ch} a) . \tag{10}$$

The more subtle question, concerning the conditions, under which (10) is convergent, the extraction of non-convergent parts of the amplitude and the extension to processes with spin, will be treated separately.

b) The Hyperbolic System (H-system)

The H-system, in which we have

$$u = (\operatorname{ch} \alpha \operatorname{ch} \beta , \operatorname{ch} \alpha \operatorname{sh} \beta \cos \phi , \operatorname{ch} \alpha \operatorname{sh} \beta \sin \phi , \operatorname{sh} \alpha) \tag{11}$$

$$-\infty < \alpha < \infty , \; 0 \leq \beta < \infty \qquad 0 \leq \phi < 2\pi$$

is specially useful in connection with the so-called brick-wall (or Breit) system. We define the b.w. system by placing the 4-velocity

$$u_t = \frac{p_4 - p_2}{\sqrt{-t}} = \frac{p_1 - p_3}{\sqrt{-t}}$$

into the point (0,0,0,1) on the one-sheet hyperboloid $u^2 = -1$. In the equal mass case $m_1 = m_3$, $m_2 = m_4$, for which the b.w. system is usually defined, the 4-velocity $(p_2 + p_4)[(p_2+p_4)^2]^{-1/2}$ then obtaines the coordinates (1, 0,0,0). Fixing the remaining unessential parameters by choosing the reaction plane as $0xz$ and the 3-vectors $\vec{p}_2$ (and $\vec{p}_4$) along the axis $0z$, we obtain

$$\begin{aligned}
p_1 &= m_1(\operatorname{ch} \alpha_1 \operatorname{ch} \beta, \operatorname{ch} \alpha_1 \operatorname{sh} \beta , \; 0 , \operatorname{sh} \alpha_1) \\
p_2 &= m_2(\operatorname{ch} \alpha_2 , \; 0 , \; 0 , \operatorname{sh} \alpha_2) \\
p_3 &= m_3(\operatorname{ch} \alpha_3 \operatorname{ch} \beta , \operatorname{ch} \alpha_3 \operatorname{sh} \beta, \; 0 , \operatorname{sh} \alpha_3) \\
p_4 &= m_4(\operatorname{ch} \alpha_4 , \; 0 , \; 0 , \operatorname{sh} \alpha_4)
\end{aligned} \tag{12}$$

with

$$m_1 \operatorname{ch} \alpha_1 = m_3 \operatorname{ch} \alpha_3 \qquad m_2 \operatorname{ch} \alpha_2 = m_4 \operatorname{ch} \alpha_4$$
$$m_1 \operatorname{sh} \alpha_1 + m_2 \operatorname{sh} \alpha_2 = m_3 \operatorname{sh} \alpha_3 + m_4 \operatorname{sh} \alpha_4 . \tag{13}$$

The kinematical diagrams are given on figs. 1b, 2b. They are drawn for the case

$$t \leq -\max\left(| m_1^2 - m_3^2 | , | m_2^2 - m_4^2 | \right)$$

and will be somewhat different in all other cases (e.g. for $-|m_4^2 - m_2^2| < t < 0$ the point B will lie on the Oz axis to the left of both 1 and 2) but this does not influence the subsequent formulae.

Choosing $\alpha_1 = \alpha_s$ and β_s as the independent variables in the s-channel, we find

$$m_1 \operatorname{sh} \alpha_s = \frac{t + m_1^2 - m_3^2}{2\sqrt{-t}}$$
$$\operatorname{ch} \beta_s = -\frac{2t(s - m_1^2 - m_2^2) + (t + m_1^2 - m_3^2)(t + m_2^2 - m_4^2)}{\sqrt{[t-(m_1-m_3)^2][t-(m_1+m_3)^2]}\,\sqrt{[t-(m_2-m_4)^2][t-(m_2+m_4)^2]}} . \tag{14}$$

Analogous formulae for $m_2 \operatorname{sh} \alpha_t$ and $\operatorname{ch} \beta_t$ i.e. the b.w. system in the t-channel are obtained by exchanging s and t and replacing all indices in the following manner $1 \to 2$, $2 \to 4$, $3 \to 1$, and $4 \to 3$.

The Lorentz invariant expansion according to SV now is

$$f(\alpha, \beta) = \int_0^\infty \rho^2 d\rho \int_0^\infty q \operatorname{th} \pi q \, dq \, \Big\{ A^+(\rho, q) \frac{1}{\sqrt{\operatorname{ch} \alpha}} P^{iq}_{-1/2+i\rho}(-i \operatorname{sh} \alpha) + \tag{15}$$
$$+ A^-(\rho, q) \frac{1}{\sqrt{\operatorname{ch} \alpha}} P^{iq}_{-1/2+i\rho}(i \operatorname{sh} \alpha) \Big\} P_{-1/2+iq}(\operatorname{ch} \beta) .$$

In the approach II we fix the momentum $p_1 - p_3$ i.e. the variable α (or t according to (14)) and thus obtain the little group O(2,1) and an expansion in terms of $P_{-1/2+iq}(\mathrm{ch}\,\beta)$ which coincides with (15) if we put

$$B_q(\alpha) = \frac{1}{\sqrt{\mathrm{ch}\,\alpha}} \int_0^\infty \rho^2\, d\rho \left\{ A^+(\rho q)\, P^{iq}_{-1/2+i\rho}(-i\,\mathrm{sh}\,\alpha) + A^-(\rho q)\, P^{iq}_{-1/2+i\rho}(i\,\mathrm{sh}\,\alpha) \right\} \tag{16}$$

c) <u>The Horospherical System (0-system)</u>

The four-velocity in the 0-system is given as

$$u = \left(\mathrm{ch}\,\gamma + \frac{1}{2} r^2 e^{-\gamma},\; r e^{-\gamma} \cos\phi,\; r e^{-\gamma} \sin\phi,\; \mathrm{sh}\,\gamma + \frac{1}{2} r^2 e^{-\gamma} \right) \tag{17}$$

with $\qquad -\infty < \gamma < \infty \qquad 0 \leq r < \infty$.

This system, according to the approach I, can be used to describe the scattering of particles with arbitrary masses and arbitrary values of the kinematical variables. The corresponding Lorentz frame is chosen by fixing an "inessential" light-like vector $K(s,t)$. To clarify the relation between the 0-system and expansions in terms of the E_2 little group, corresponding to $t=(p_4-p_2)^2=0$ it is convenient to choose the vector $K(s,t)$ as

$$K(s,t) = p_4 \frac{m_2}{m_4}\, e^{-A(s,t)} - p_2 \tag{18}$$

with

$$\mathrm{ch}\, A(s,t) = \frac{(p_2 p_4)}{m_2 m_4} = \frac{m_2^2 + m_4^2 - t}{2 m_2 m_4} . \tag{19}$$

It is easy to verify that

$$K^2(s,t) = 0 \tag{20}$$

$$K(s,0) = (p_4 - p_2)\big|_{t=0} .$$

The coordinate system is specified as usually choosing $0xz$ as the reaction plane, the momenta $\vec{p}_2$ (and $\vec{p}_4$) along $0z$ and the vector $K(s,t)$ in the standard form

$$K(s,t) = (\omega,\ 0,\ 0,\ \omega)\ . \tag{21}$$

The four-momenta of the particles now are:

$$\begin{aligned}
p_1 &= m_1\left(\mathrm{ch}\,\gamma_1 + \frac{1}{2} r_1^2 e^{-\gamma_1},\quad r_1 e^{-\gamma_1},\ 0,\ +\mathrm{sh}\,\gamma_1 + \frac{r_1^2}{2} e^{-\gamma_1}\right)\\
p_2 &= m_2\left(\mathrm{ch}\,\gamma_2,\quad 0,\ 0,\ +\mathrm{sh}\,\gamma_2\right)\\
p_3 &= m_3\left(\mathrm{ch}\,\gamma_3 + \frac{1}{2} r_3^2 e^{-\gamma_3},\quad r_3 e^{-\gamma_3},\ 0,\ +\mathrm{sh}\,\gamma_3 + \frac{r_3^2}{2} e^{-\gamma_3}\right)\\
p_4 &= m_4\left(\mathrm{ch}\,\gamma_4,\quad 0,\ 0,\ +\mathrm{sh}\,\gamma_4\right).
\end{aligned} \tag{22}$$

with

$$\begin{aligned}
m_1(e^{+\gamma_1} + r_1^2 e^{-\gamma_1}) + m_2 e^{+\gamma_2} &= m_3(e^{+\gamma_3} + r_3^2 e^{-\gamma_3}) + m_4 e^{+\gamma_4}\\
m_1 e^{-\gamma_1} + m_2 e^{-\gamma_2} &= m_3 e^{-\gamma_3} + m_4 e^{-\gamma_4}\\
m_1 r_1 e^{-\gamma_1} &= m_3 r_3 e^{-\gamma_3}\\
\omega &= m_2 e^{+\gamma_4}\,\mathrm{sh}\,(\gamma_4 - \gamma_2)\ .
\end{aligned} \tag{23}$$

The corresponding kinematical graphs are given on figs. 1c,2c.

Note that if we introduce

$$\omega_1 \equiv \frac{(K\,p_1)}{m_1} = \omega\, e^{-\gamma_1}\,, \tag{24}$$

where ω_1 is clearly the frequency (energy) of the "photon" K as seen by an observer in the rest frame of particle 1, we obtain $\gamma_1 = -\ln \frac{\omega_1}{\omega}$. Therefore ω serves as a scale factor for the frequency and $e^{-\gamma_1}$ determines the Doppler effect for the "photon" K, connected with the rest system 1, which clearly determines the velocity of the first particle. This is a manifestation of an interesting property of the 0-system (and of the geometry of a horosphere in Lobachevsky space) namely that ω can be chosen arbitrarily.

With our choice of $(21)'$ and (22) ω is subject to the condition

$$\epsilon = \mathrm{sgn}\,\omega = \mathrm{sgn}\,(m_4 - m_2) = \mathrm{sgn}\,(\gamma_4 - \gamma_2)\,.$$

Choosing $\gamma_1 = \gamma_s$ and $r_1 = r_s$ as the new variables, we have

$$m_1\,\omega\, e^{-\gamma_s} = \frac{(s - m_1^2 - m_2^2)(m_2^2 - m_4^2 + t + \epsilon R) + 2m_2^2(t + m_1^2 - m_3^2)}{2[m_2^2 + m_4^2 - t - \epsilon R]} \tag{25}$$

$$\left(\frac{r_s}{\omega}\right)^2 = \frac{(m_2^2 + m_4^2 - t - \epsilon R)^2\{R^2[(s - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2] - [(s - m_1^2 - m_2^2)(m_2^2 - m_4^2 + t) + 2m_2^2(t + m_1^2 - m_3^2)]^2\}}{m_2^2 R^2[(s - m_1^2 - m_2^2)(m_2^2 - m_4^2 + t + \epsilon R) + 2m_2^2(t + m_1^2 - m_3^2)]^2}$$

with

$$R = \sqrt{[t - (m_2 - m_4)^2][t - (m_2 + m_4)^2]}\,.$$

Comparing figs. 2b and 2c we see that the Lorentz frame of reference which we are using in the 0-system is very similar to the brick-wall system. A different Lorentz frame in the same channel is obtained by standardizing a different isotropic vector, namely $K' = p_3 \frac{m_4}{m_3} e^{-A'} + p_4$. This system is similar to the c.m. system and it has no direct connection with the little group corresponding to isotropic momentum transfer.

Two analogous "horospherical" systems can be constructed in the t-channel and under the crossing transformation the "b.w.-like system" in one channel goes over into the "c.m.-like system" in the other.

The Lorentz invariant expansions in terms of these variables are (we put $\omega = 1$):

$$f(\gamma, r) = \int_0^\infty \rho^2 d\rho \int_0^\infty k\, dk\, A(\rho, k)\, e^{\gamma} K_{i\rho}(k e^{+\gamma}) J_0(kr), \tag{26}$$

where $K_\nu(z)$ and $J_m(z)$ are the MacDonald and Bessel cylindrical functions, respectively. They are applicable for all values of s and t.

In approach II expansions with respect to the little group E_2 are obtained only for t=0. In this case formulae (25) simlify to

$$m_1 \omega e^{-\gamma_s} = \frac{m_1^2 - m_3^2}{2}$$
$$\left(\frac{r_s}{\omega}\right)^2 = \frac{4}{(m_1^2 - m_3^2)(m_4^2 - m_2^2)}\left[s - m_1^2\left(1 + \frac{m_4^2 - m_2^2}{m_1^2 - m_3^2}\right) - m_2^2\left(1 + \frac{m_1^2 - m_3^2}{m_4^2 - m_2^2}\right)\right] \tag{27}$$

(in complete agreement with /14/).

In this special case, putting

$$B_k(\gamma) = \int_0^\infty \rho^2 d\rho\, A(\rho, k) K_{i\rho}(k e^{+\gamma})\, e^{\gamma}$$

we obtain from (26) the usual expansion with respect to E_2.

d) <u>The Little Group 0(3,1)</u>

In the approach I the scattering amplitude is always decomposed with respect to the irreducible representations of the Lorentz group 0(3,1) and this can be done for arbitrary values of the kinematical variables.

In approach II the group 0(3,1) is a little group corresponding to a fixed null vector, so that the corresponding expansions only occur in the specific case when the momentum transfer $p_1 - p_3$ is a null-vector. This corresponds to elastic scattering in the forward direction i.e.

$$m_1 = m_3\,,\quad m_2 = m_4 \qquad t = 0\,. \tag{28}$$

Salam et al.[15] have suggested a generalization in which the 0(3,1) expansion is applied for arbitrary t and arbitrary spin amplitudes. Contrary to SV they only expand in terms of a single variable (namely $\mathrm{ch}\,\beta_s$ of the H-system), keeping the dependence on t (or α_s) inside the expansion coefficient.

The expansions of approach I can of course also be applied in the case (28) and also reduce to one-dimensional expansions. In the S-system they can be obtained directly by putting $\theta = 0$, in the H and 0 system a limiting procedure is necessary to turn the chosen space-like or isotropic vector into a null-vector. Specifically for the H-system (28) implies $\alpha = 0$, $\mathrm{ch}\,\beta = \frac{(s - m_1^2 - m_2^2)}{2 m_1 m_2}$ and from (15) we obtain the expansion

$$f(0, \beta) = \int_0^\infty q\, \mathrm{th}\, \pi q\, dq\, B_q(0)\, P_{-1/2+iq}(\mathrm{ch}\,\beta)\,. \tag{29}$$

The corresponding expansion in the 0-system can easily be obtained and we shall not go into the details here.

e) Invariant Expansions and Crossing Transformations

The question of crossing transformations from one reaction channel to another acquires special significance in connection with Lorentz invariant expansions. If certain analytical properties of the amplitude are assumed, then the crossing transformation can be considered to be an analytical continuation from the physical region of one channel into the physical region of the other. Performing such transformations for fixed values of the inessential parameters and for a fixed choice of the essential ones, we obtain a transformation from a standard Lorentz frame in one channel to a generally speaking different frame in the other channel, related to the first one by a complex Lorentz transformation[25].

Specifically, comparing the formulae (7) and (14) we see that the parameters of the brick-wall system in the s-channel are related to those of the c.m.s. in the t-channel

$$\alpha_s + a_t = i\,\frac{\pi}{2} \qquad (30)$$

$$\beta_s = i\,\theta_t$$

These substitutions have a simple geometrical interpretation which is discussed in the Appendix[x].

In[16] Boyce considered the expansion in β_s corresponding to the little group $0(2,1)$ and proved that if the amplitude satisfies a one-dimensional dispersion relation for fixed t, this expansion is an analytical continuation of the t-channel $0(3)$ expansion in terms of $\cos\theta_t$.

We postpone a discussion of the analytical continuation and crossing transformation problem for the two-dimensional expansion of approach I to a future publication. Here we would only like to stress that the crossing properties of these expansions should be simple, since the parameter ρ figuring in them is related to the Casimir operators of the Lorentz group and is thus also invariant under the complex Lorentz group, realizing the transformation between the channels. However, we expect that to prove the actual analytical continuation from (15) to (8) it will be necessary to postulate double dispersion relations of the Mandelstam type in s and t.

Conclusions

Let us summarize the main conclusions of the preceding sections.

1. The methods of SV make it possible to write Lorentz invariant expansions in terms of the basis functions of the irreducible representations of $0(3,1)$ for arbitrary values of the kinematical variables s and t. The coordinate systems S,H and 0 correspond to different Lorentz frames of

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[x] Formula (30) may be called a substitution law; it gives the connection between coordinates seen by two observers (one at the point s and another at the unphysical point t) using the same families of coordinate surfaces.

reference on one hand and to reductions of the $O(3,1)$ group with respect to its $O(3)$, $O(2,1)$ and E_2 subgroups respectively, on the other.

2. The expansions in the S and H systems can be obtained from the Poincaré invariant expansions with respect to the $O(3)$ and $O(2,1)$ little groups, by expanding the coefficients of the little group expansions in terms of the remaining essential parameter, in a definite manner, prescribed by the structure of $O(3,1)$ representations.

3. The 0-system expansion for arbitrary t is a generalization of the E_2 little group expansion applicable only for $t=0$.

For $t=0$ the Lorentz invariant expansions in the 0-system are related to the E_2 little group expansions in the same manner, as the S and H-system expansions are related to the $O(3)$ and $O(2,1)$ little group ones.

4. It follows from the whole discussion that the considered expansions (of approach I) are not only Poincaré invariant in the usual sense, but also satisfy the condition of "extended" Lorentz invariance, defined in section 2.

5. We have already stressed that "extended Lorentz invariance" is an assumption concerning the convergence of the integrals and series figuring in the expansions. The formulae, as written in this paper, assume square integrability, since the amplitude is expanded only in terms of unitary representations of the principal series. However, this is by no means necessary and the formulae of SV are equally applicable for the supplementary series and for non-unitary representations (as given in /3/). Thus if we know the asymptotic behaviour of the amplitude we can find more general expansions involving integrals over certain contours in the complex ρ -plane and afterwards move the contour towards the real axis, collecting contributions from singularities in the ρ -plane (e.g. "Lorentz poles" /12,13/).

6. Expansions in the H-system are directly related to the problem of daughter trajectories /22,23/, since in this case $O(3,1)$ is reduced with respect to $O(2,1)$. However, in this general formalism no definite conclusions can be made, even for elastic scattering in the forward direction. Indeed, it can be seen from (29) that the expansion coefficient $B_q(0)$ ($t=0$) depends upon the "complex angular momentum" q in an unknown manner (since the coefficients A^+ and A^- in (15) depend upon ρ and q) so that the existence of a dominating Lorentz pole in ρ has no definite

implications for poles in q. However, the hypothesis of equally spaced daughter Regge trajectories can be built into the theory by making definite dynamical assumptions as to the behaviour of $A^{+}(\rho, q)$ and $A^{-}(\rho, q)$ as functions of q. On the other hand, physical arguments against the existence of daughter trajectories have been given by various authors /23/ and the question of correct dynamical assumptions is far from settled. Thus the existence of daughter trajectories is equivalent to assuming some type of "dynamical" symmetry.

7. Let us remark that Lorentz invariant expansions need not be tied up with little group expansions, but are more general, since many other coordinate systems exist in which the Laplace operator on a hyperboloid allows the separation of variables /24/.

8. There is one more property of invariant expansions, which will be discussed in a later publication. The coefficients of the expansions of amplitudes may be further decomposed into terms which transform under irreducible representations of the group of permutations of the three arguments s,t, and u $(s + t + u = \text{const.})$. In this way we get additional quantum numbers.

In future publications we plan to return to the problem of Lorentz invariant expansions for arbitrary spins, to discuss their relation to the analytical properties of amplitudes, specify their properties with respect to crossing transformation and apply them to particular physical processes.

In conclusion we thank R.Mir-Kasimov and M. Uhlíř for helpful discussions.

APPENDIX

Coordinate Systems in Lobachevsky Space

We shall briefly recapitulate the main properties of the coordinate systems used in this paper. A detailed discussion of the related kinematics was already published in /5,26/. We would like to stress, that although the knowledge of Lobachevsky geometry is by no means essential for an understanding of Lorentz invariant expansions, it does simplify the kinematics and makes it possible to replace complicated algebraical calculations by simple geometrical considerations.

The kinematics of the process are illustrated by a "kinematical graph" i.e. by four points representing the 4-velocities $u = \frac{1}{m} p$ on the upper sheet of the hyperboloid $u^2 = 1$ in the Minkovsky 4-momentum space. In this language the parametrization of the process $1 + 2 \to 3 + 4$ consists of the three steps:

1) Choice of a definite type of coordinate system on the hyperboloid.

2) Specific localization of the kinematical graph on the hyperboloid (i.e. identification of a certain velocity, e.g. that of the c.m. system, with the origin of the coordinate system and of certain directions in velocity space with the coordinate axes).

3) Expressing the momentum components of one of the initial or final particles in terms of the chosen coordinates.

Let us discuss these steps separately.

Ad 1. In this paper we only consider three types of coordinate systems, namely the S,H and O- systems (see figs. 1a,b,c).Group theoretically these systems correspond to the reductions $O(3,1) \supset O(3) \supset O(2)$, $O(3,1) \supset O(2,1) \supset O(2)$ and $O(3,1) \supset E_2 \supset O(2)$ respectively. Geometrically these are systems possessing a single centre and axial symmetry. This means that in each of these systems one family of coordinate curves is a family of concentrical spheres (hyperspheres, horospheres,), the second is obtained by rotating a bundle of lines, intersecting in the corresponding centre, about the axis Oz and the third family is a bundle of planes, intersecting along the axis Oz (for rigorous definitions cf. /24/).

Let us describe the individual systems. We always choose $V = (1,0,0,0)$ as the origin of the coordinate system on the hyperboloid and use the axial symmetry to choose an angle ϕ in the Oxy plane as one of the coordinates. This angle is not drawn on the three-dimensional figs. 1.

In the <u>S-system</u> we place the centre in the point V thus obtaining a family of spheres, one of which is indicated by the circle on fig. 1a and a family of straight lines, intersecting in the vertex V. Point A is now parametrized by the "distance" $a = VD$ (more precisely by the area under the hyperbola on fig. 1a) to the sphere on which it lies and by the angle θ (on the sphere).

In the H-system we place the centre in the point $(0,0,0,1)$ on the one-sheet hyperboloid (imaginary Lobachevsky space) obtaining a family of hyperspheres (hyperboloids obtained on fig. 1b by cutting the original hyperboloid by a plane, perpendicular to Oz) and a family of divergent lines (one of which passes through V). Point A is now parametrized by the distance $\alpha = VD$ to the corresponding two-dimensional hyperboloid and the distance $\beta = DA$ along this hyperboloid.

In the 0-system we place centre in the point $(\omega, 0, 0, \omega)$ of the light cone, obtaining a family of horospheres (parabolas on fig. 1c), obtained by cutting the hyperboloid by planes perpendicular to the generating line $(\omega, 0, 0, -\omega)$, and a family of parallel lines, one of which passes through V. Point A is parametrized by the distance $\gamma = VD$ to the horosphere and by r, where $r e^{-\gamma}$ is the distance along the horosphere. These geometrical properties are best seen by introducing the new coordinates

$$\omega_1 = r e^{-\gamma}, \; \omega_2 = u_0 + u_3 = e^{\gamma} + r^2 e^{-\gamma}, \; \omega_3 = u_0 - u_3 = e^{-\gamma}.$$

Thus γ fixes the section $u_0 - u_3 = \mathrm{const}$(the horosphere) and ω_1, ω_2 are coordinates on the horosphere (parabola) $\omega_1^2 + 1 = e^{-\gamma} \omega_2$. Thus $r = \mathrm{const}$ corresponds to a bundle of parallel lines and the paraboloid carries an euclidean geometry with a scale factor $e^{-\gamma}$ determined by the distance of the section from the origin.

The 0-system in velocity space has an interesting physical interpretation. The (relative) velocity of an observer can be measured by determining the ratio of the frequency ω' of a real photon as seen by the observer and some standard frequency ω (confront formula (24) and the following text).

Thus an additional point has to be chosen in the 0-system, namely the system in which K has the frequency ω. This is the ambiguity in the definition of K mentioned in the text and it is due to the fact, that the "distance" to K is infinite in every coordinate system and only the differences between two such distances have a physical meaning. From this point of view one set of coordinate lines corresponds to observers, who all measure the same frequency and the other set to those, who measure the same direction of propagation.

Ad 2 and 3. We shall now discuss the localization of the kinematical graph on the hyperboloid, i.e. discuss the choice of special Lorentz frames of reference. It is convenient to illustrate these by kinematic graphs drawn using the Beltrami model of the Lobachevsky space. In this paper we always set $\phi = 0$ so that we can use a two-dimensional model, in which the light-cone is represented by a unit circle, the interior of which is the real Lobachevsky space, corresponding to $u^2 = 1$ and the exterior – the imaginary space, corresponding to $u^2 = -1$. The connection between the kinematical graphs on the hyperboloid and the Beltrami model is given by introducing the inhomogeneous coordinates

$$\operatorname{th} z_1 = \frac{p_1}{p_0} \tag{A.1}$$

so that the distance between two points is equal to the hyperbolic tangent of their relative velocity. In the Beltrami model straight lines are projected into straight lines, but angles are distorted.

The kinematical graphs on the Beltrami plane are given on figs. 2a, b, c for the S, H and 0 systems respectively. The points X and Z are always the inhomogeneous coordinates of the parametrized velocity (i.e. they are obtained by drawing perpendiculars from the corresponding point to the z_1 and z_3 axes). The point D is the intersection of z_3 and the sphere (hypersphere, horosphere) on which the parametrized point is located, so that its distance from the origin (the vertex of the hyperboloid) is respectively $\operatorname{th} a$, $\operatorname{th} \alpha$ and $\operatorname{th} \gamma$.

The line P1(P2) on figs. 2c, (2c′) is parallel to z_3.

The distances along the z_1, z_3 axes are actually measured as $\operatorname{th} z_1$ ($\operatorname{th} z_3$).

Using figs. 2 and elementary trigonometric formulae of the Lobachevsky geometry it is easy to check that the inhomogeneous coordinates of the parametrized point are exactly those, which we would obtain from formula (A.1).

The S-system is specially suitable for a description of scattering in the c.m. system. We identify the velocity S of the c.m.s. with the vertex V (the origin of the coordinate system on fig. 2a) and direct the axis z_3 along $\vec{u}_1$ in the c.m.s.

It is convenient to associate the <u>H-system</u> with the brick-wall system and to identify the velocity B of this system with the vertex V and direct the axis z_3 along $\vec{u}_4$ in the b.w.s.

As was mentioned in the text, it is possible to introduce two different types of <u>0-systems.</u> Fig. 2c corresponds to the "Breit-like" 0-system, in which $\vec{u}_4$ is along z_3 and the point B' corresponding to the standartization of K (cf. (21)) is indentified with the vertex V. The "c.m.-like" 0-system is illustrated by. fig. 2c$'$ in which z_3 is along $\vec{u}_3$ and the velocity S' obtained by standartizing $K' = p_3 \frac{m_4}{m_3} e^{-A'} + p_4$. is identified with the origin.

The figures, as described above, always correspond to the physical region of the s channel. We can however, also use them to describe t-channel processes and thus obtain geometrically the substitution laws (30). Indeed, consider the point t on fig. 2b (the point $(0,0,0,1)$ on fig.1b) as the origin of the S-system in the (unphysical) region of the t-channel (cf. fig.3). The distance between B_s (velocity of the b.w.s. in the s-channel) and S_t (c.m.s. in the t-channel) is $\operatorname{th} B_s S_t = \infty$ thus we have $B_s S_t = \frac{i\pi}{2}$. Directly from fig. 3 we can now read off the relations (30): $\beta_s = i\,\theta_t$, $\alpha_s + a_t = i\,\frac{\pi}{2}$.

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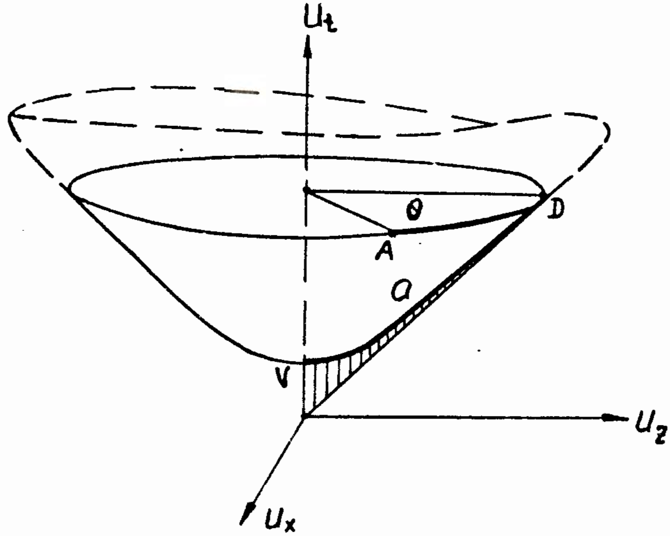

Fig. 1a. The spherical coordinate system S.

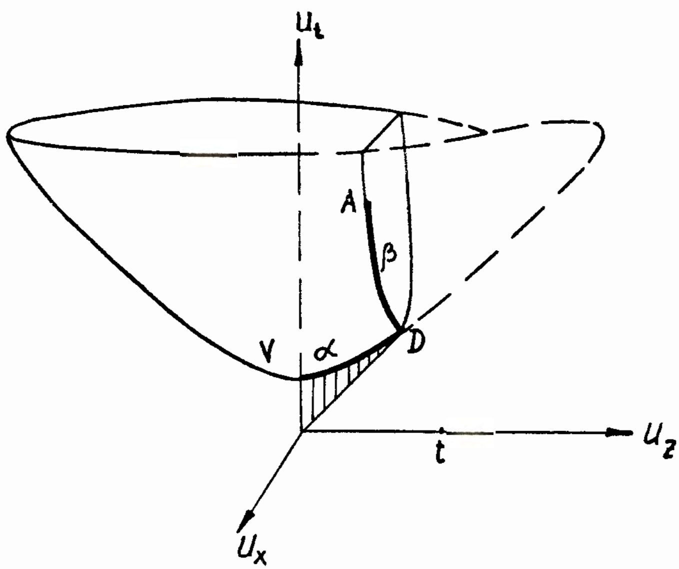

Fig. 1b. The hyperbolical coordinate system H.

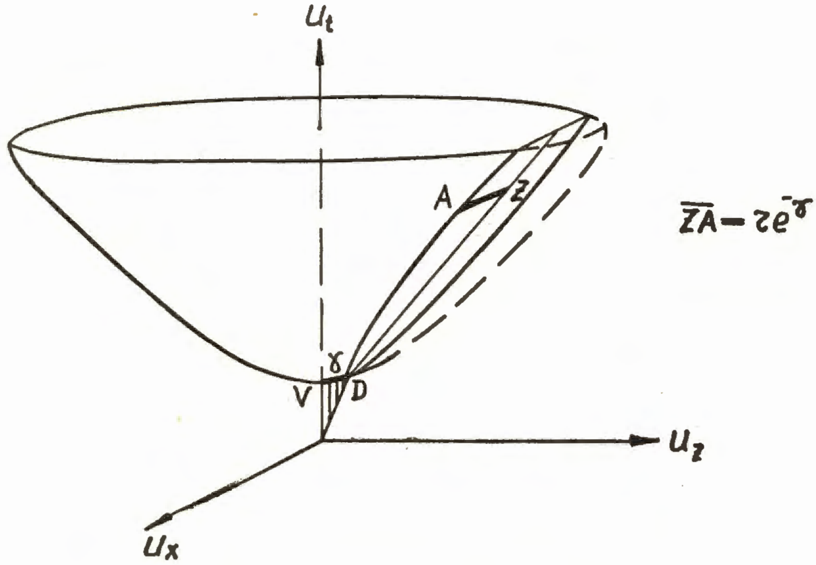

Fig. 1c. The horospherical coordinate system 0.

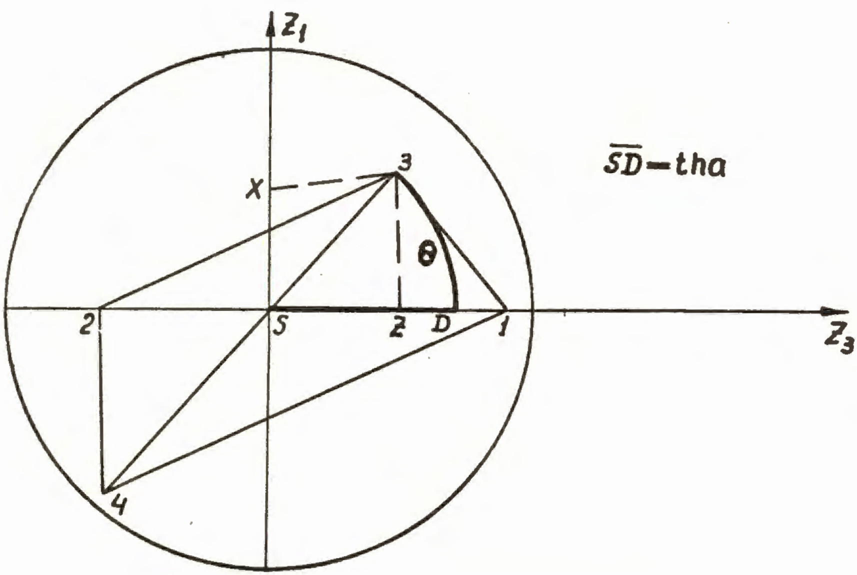

Fig. 2a. Process 1 + 2 → 3 + 4 in the S-system.

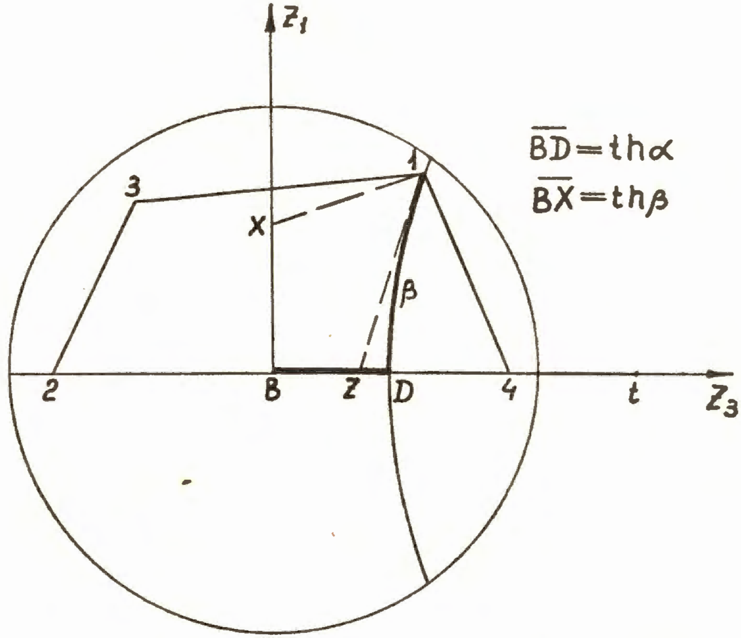

Fig. 2b. Process 1 + 2 → 3 + 4 in the H-system.

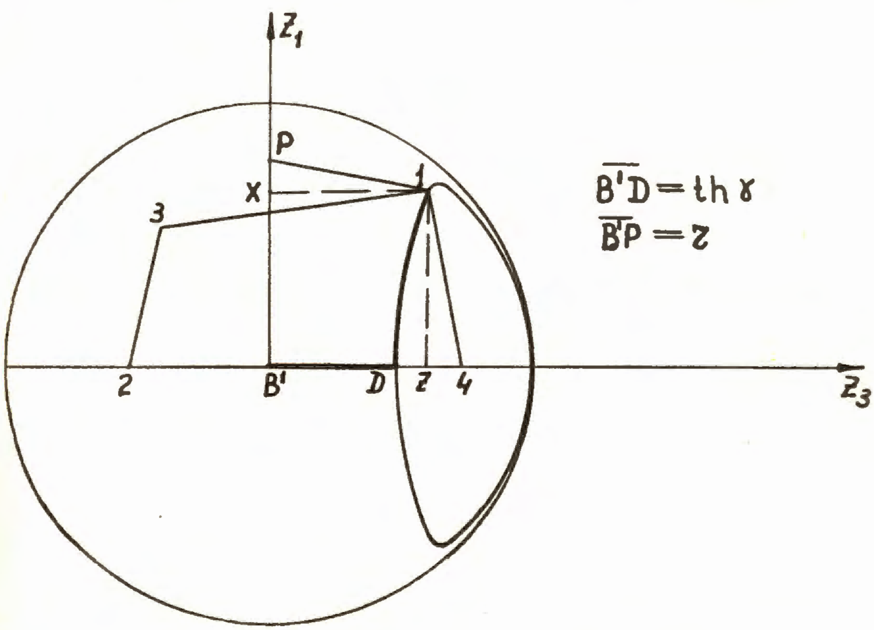

Fig. 2c. Process 1 + 2 → 3 + 4 in the "Breit-like" 0-system.

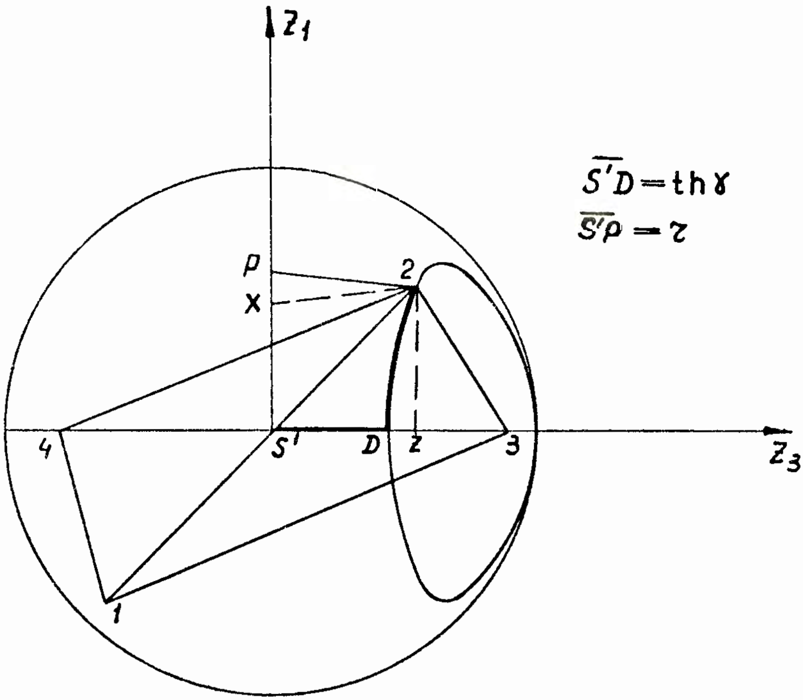

Fig. 2c.' Process 1 + 2 → 3 + 4 in the "c.m.-like" system.

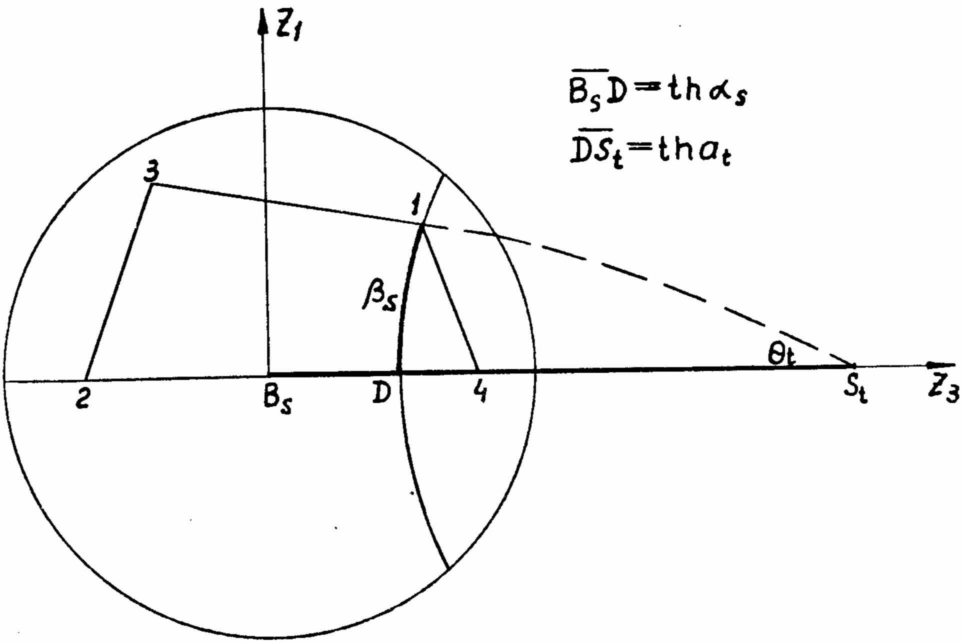

Fig. 3. Relation between s and t channel scattering in the H and ξ systems.