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SOLUTION OF THE  $t$ -MATRIX EQUATION  
IN CLOSED-SHELL NUCLEI

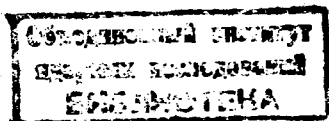
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**SOLUTION OF THE t-MATRIX EQUATION  
IN CLOSED-SHELL NUCLEI**



## 1. Introduction

The problem of solving the  $t$ -matrix equation in finite nuclei with realistic forces is still of great interest. Most of the papers dealing with this problem<sup>1-5</sup> can be characterized by the adaption of the methods which have been developed for the nuclear matter (separation method, reference-spectrum method). Another approach has been proposed by Eden and Emery<sup>6</sup>, and is based on a consistent use of the harmonic oscillator representation which is motivated both physically (shell-model calculations for light nuclei) and mathematically (great simplification of the equations).

On the other hand, the use of this approach practically restricts us to the study of the light closed-shell nuclei  $\text{He}^4$ ,  $\text{O}^{16}$  and  $\text{Ca}^{40}$  since for any other nucleus the problem of degeneracy arises which represents still a principal difficulty.

The calculation of the ground state characteristics of the above mentioned nuclei can be performed by means of the Goldstone expansion in  $t$ -interaction. However in actual calculations<sup>6-8</sup> a number of approximations was introduced. These approximations refer on the one hand to the solution of the  $t$ -matrix equation, on the other hand to the treatment of the higher order diagrams in the Goldstone expansion (approximate self-consistence etc.). In the present work attention is paid to the first group of them.

Probably the most important is the approximation of the exclusion operator  $Q$  which neglects the influence of the motion of the centre of mass of a given pair of nucleons. This neglect is either complete <sup>6</sup> or a correction is introduced in the selfconsistent part of Hamiltonian but the accuracy is not estimated <sup>7</sup>. Similarly the paper of Day and Kallio <sup>8</sup>, which otherwise seems to represent even in its preliminary results a considerable progress, does not contain an estimation of the accuracy of the used approximation of  $Q$ . A more complete attempt to handle exactly this operator has been made by Wong <sup>2</sup>. However, his estimation of accuracy is closely connected with the reference-spectrum method and cannot be applied in the harmonic oscillator formalism.

In the present work we start with the usual assumptions:

- 1) a spherical harmonic oscillator representation is used;
- 2) the unperturbed wave function of the nucleus is nondegenerate i.e. we consider only He<sup>4</sup>, O<sup>16</sup> and Ca<sup>40</sup>;
- 3) the nucleon-nucleon interaction is supposed to be described by a general static potential with a hard-core (Hamada-Johnson <sup>10</sup>) or a soft-core (Bressel et al. <sup>11</sup>) repulsion.

The t-matrix equation is transformed to an infinite algebraic system for the t-matrix elements in LS-coupling, the exclusion operator being treated exactly. Further, generalizing the results obtained for a simplified one-dimensional case <sup>12</sup>, we obtain an approximate exclusion operator  $Q_M$  which depends only on the truncation constant  $M$  and converges rapidly to the exact  $Q$  with increasing  $M$ . (For the lowest value of  $M$  which is 0 for He<sup>4</sup>, 1 for O<sup>16</sup> and 2 for Ca<sup>40</sup>,  $Q_M$  is identical with the approximation of Eden-Emery).

With a fixed finite value of  $M$ , we can transform the algebraic system into a finite system of coupled integral or integrodifferential equations for the correlated two-particle wave functions plus a finite algebraic system. However, solving such a system would result in great computational complexity.

We therefore propose an approximate method, the main feature of which is the replacement of the kernels of the coupled integral equations by finite sums. The convergence of this procedure is proved <sup>13</sup>. Then we can

return to the algebraic system for the t-matrix elements which is now finite. Choosing the dimension of the system sufficiently large, we obtain the solution which will be very close to the solution of the exact infinite system. Hence the results of the proposed algebraic method can be made arbitrarily accurate. Moreover, the method is convenient for calculating higher-order diagrams of the Goldstone-expansion since all "on" and "off the energy-shell" t-matrix elements, which are necessary for calculating energy up to the second order and the mean values of such quantities as r.m.s. radius and nucleon density to the first order, are expressed directly by means of the solution of the algebraic system.

## 2. The Exclusion Operator

First of all, we shall write down equations for those t-matrix elements which are necessary for calculating the first and second order diagrams of the Goldstone expansion for energy and the first-order diagrams which occur in the expression of the mean value of one-particle operators.

For this purpose we need only matrix elements between  $\Phi_0$ ,  $\Phi_1$  or  $\Phi_0$ ,  $\Phi_1^{(1)}$  or  $\Phi_0$ ,  $\Phi_{1j}^{(2)}$ , where  $\Phi_0$  denotes the unperturbed ground state,  $\Phi_1^{(1)}$  and  $\Phi_{1j}^{(2)}$  the unperturbed states with one- and two-particle excitation, respectively. Thus on the one side of the considered matrix elements there are always only occupied states (hereafter denoted by the upper index 0), while on the other side both occupied and unoccupied states can occur. Further, the energy denominator  $e$  is for such matrix elements always of the form

$$e = e_1^{(0)} + e_2^{(0)} - H_0, \quad (1)$$

where  $e_1^{(0)}$ ,  $e_2^{(0)}$  are the single-particle energies of the corresponding occupied states and  $H_0$  is the unperturbed Hamiltonian.

This form of  $e$  and the whole formulation in the present paper

admits the self-consistence procedure proposed by Day-Kallo. This procedure seems to be very convenient for our method and hence we shall not treat this point in more detail.

In LST-coupling the equation for the t-matrix elements reads

$$\langle \underline{1,2} | t | \underline{1}^{(0)}, \underline{2}^{(0)} \rangle = \langle \underline{1,2} | v | \underline{1}^{(0)}, \underline{2}^{(0)} \rangle + \langle \underline{1,2} | v \frac{Q}{e} t | \underline{1}^{(0)}, \underline{2}^{(0)} \rangle, \quad (2)$$

where 1,2 means 10 quantum numbers of a pair of nucleons:

$$(2')$$

$$\underline{1,2} = n_1, l_1, n_2, l_2, l_{12}, m_{12}, S, S_z, T, T_z.$$

Here  $n_i$  is the oscillator radial quantum number,  $l_i$  is the orbital angular momentum of the  $i$ -th nucleon ( $i=1,2$ ),  $\vec{l}_{12} = \vec{l}_1 + \vec{l}_2$ ,  $m_{12}$  is the  $z$ -component of  $\vec{l}_{12}$ ;  $S, S_z$  is the total isospin of the pair and  $z$ -component and similarly  $T, T_z$  refer to the total isospin.

In this coupling the exclusion operator  $Q$  together with eq.(1) gives

$$\frac{Q}{e} = \sum_{\substack{\epsilon'_1 = \epsilon_0 \\ \epsilon'_2 = \epsilon_0}}^{\infty} \frac{1}{e_1^{(0)} + e_2^{(0)} - e_1' - e_2'} \sum_{\substack{2n'_1 + l'_1 = \epsilon'_1 \\ 2n'_2 + l'_2 = \epsilon'_2}} \sum_{\lambda\mu} \sum_{i_1 i_2} |n'_1 l'_1 n'_2 l'_2 \lambda\mu\rangle \langle n'_1 l'_1 n'_2 l'_2 \lambda\mu| \quad (3)$$

where  $\epsilon_1$  denotes the oscillator energetic number i.e.  $\epsilon_1 = 2n_1 + l_1$  and  $\epsilon_0$  is the energetic quantum number of the first unoccupied shell ( $\epsilon_0 = 1,2,3$  for  $\text{He}^4, \text{O}^{16}, \text{Ca}^{40}$ , respectively).<sup>x)</sup>

Now  $Q/e$  must be transformed in the centre-of-mass (c.m.) system. For this purpose it is convenient to introduce new summation indices

$$\nu = \epsilon'_1 + \epsilon'_2 \quad \text{i.e.} \quad 2\epsilon_0 \leq \nu < \infty$$

$$\nu' = \epsilon'_2 \quad \text{i.e.} \quad \epsilon_0 \leq \nu' < \nu - \epsilon_0.$$

The sum over  $\nu'$  can be rewritten

<sup>x)</sup> Note that in the spin and isospin space  $Q$  is simply unity and we therefore omit writing explicitly the spin and isospin state-vectors in the expressions for  $Q$ .

$$\sum_{\nu'=\epsilon_0}^{\nu-\epsilon_0} = \sum_{\nu'=0}^{\nu} - \sum_{\nu'=0}^{\epsilon_0-1} - \sum_{\nu'=\nu-\epsilon_0+1}^{\nu} \quad (4)$$

Substituting into eq. (3) and using unitary properties of the Moshinsky transformation<sup>14</sup>, we obtain  $\delta$ -functions in all indices in the first sum on the right-hand side of eq. (4), while the other two sums cannot be simplified. They are responsible for the fact that  $Q/e$  is not diagonal in the c.m. system:

$$\frac{Q}{e} = \sum_{\nu=2\epsilon_0}^{\infty} \frac{1}{e_1^{(0)} + e_2^{(0)} - \hbar\omega(\nu+3)} \left\{ \sum_{2n+\ell+2N+L=\nu} \sum_{\lambda\mu} |n\ell NL \lambda\mu\rangle \langle n\ell NL \lambda\mu| - \right. \quad (5)$$

$$\left. \sum_{2n'+\ell'+2N'+L'=\nu} \sum_{\lambda\mu} |n'\ell' N'L' \lambda\mu\rangle \langle n'\ell' N'L' \lambda\mu| \times f_{n\ell NL, n'\ell' N'L', \lambda}^{(\nu, \epsilon_0)} \right\},$$

where

$$f_{n\ell NL, n'\ell' N'L', \lambda}^{(\nu, \epsilon_0)} = [1 + (-1)^{L+L'}] \times$$

$$\times \sum_{\nu'=0}^{\epsilon_0-1} \sum_{2n_1+\ell_1=\nu-\nu'} \sum_{2n_2+\ell_2=\nu'} \langle n\ell NL \lambda | n_1 \ell_1 n_2 \ell_2 \lambda \rangle \langle n'\ell' N'L' \lambda | n_1 \ell_1 n_2 \ell_2 \lambda \rangle.$$

For obtaining this expression for  $f$  we have used the following property of the Moshinsky-coefficients:

$$\langle n\ell NL \lambda | n_1 \ell_1 n_2 \ell_2 \lambda \rangle = (-1)^{L+\lambda} \langle n\ell NL \lambda | n_2 \ell_2 n_1 \ell_1 \lambda \rangle. \quad (5')$$

The fact that  $Q/e$  is not diagonal introduces considerable complications in the equation (2). Therefore all the authors who have treated this problem used different diagonal approximations. The simplest one introduced by Eden-Emery neglects the second term in eq.(5) at all. It is clearly a serious simplification and it does not give any possibility how to estimate the induced error.

The "diagonalization" may be achieved in another way by neglecting all the coefficients  $f$  which are not diagonal with respect to all quantum numbers  $n\ell NL$ . This has been done by Day-Kallio and earlier,

together with other approximations, by Wong. Again the procedure gives no possibility to estimate the accuracy. The  $f$ -coefficient is an uncomplete scalar product of the elements of an orthogonal matrix and this suggests the rapid decrease of the magnitude of  $f$  with increasing energy  $\nu$  (the number of energy shells in the sum is constant). This was in fact quantitatively confirmed for a slightly different representation<sup>12</sup>. However, from the fact that the complete sum gives unity for diagonal and zero for nondiagonal cases, we can hardly conclude that the nondiagonal uncomplete sums are negligible with respect to the diagonal ones since the Moshinsky-coefficients have both positive and negative signs and their dependence on parameters is very complicated.

It is of course possible that the diagonalization of the  $f$ -coefficients is a good approximation. But the only argument for it would be a good agreement of the calculated nuclear properties with experiment. On the other hand, if this agreement were not sufficiently convincing we could not exclude the possibility that the discrepancies are due to the uncorrect treatment of the exclusion operator.

This suggests to introduce such an approximation of  $Q$  which uses only the proved rapid decrease of the  $f$ -coefficients with  $\nu$ . This can be done as follows: we divide the expression (5) into the diagonal and nondiagonal parts leaving the first one unchanged, while in the second part the infinite upper bound in the sum is replaced by a finite number  $M$ . This gives an approximate operator  $(Q/e)_M$  which, in view of the dependence of the  $f$ -coefficients on  $\nu$ , can replace the exact  $Q/e$  with arbitrary accuracy. Consequently we must choose such a value of  $M$  for which the change  $M \rightarrow M + 1$  will not practically influence the calculated quantities.

Such a treatment of  $Q/e$  clearly leads to considerable complications in calculation. In the following sections we shall try to show that even with these complications eq.(2) can be numerically solved.



### 3. The Algebraic System for the t-Matrix Elements

Now we shall turn our attention to eq. (2). We must transform it to such a form which will make it possible to use the approximate operator  $(Q/e)_M$  as derived in the preceding section. For this purpose we introduce a mixed representation which does not change the state vectors on the right-hand side of the t-matrix elements, while on the left-hand side we pass to the c.m. system (cf. eq. (5)). For further simplifications it is necessary to make use of the properties of the nucleon-nucleon potential  $v$ .

It is well known that the most convenient representation for  $v$  is that in which  $\ell$  and  $S$  are coupled to the resulting  $J$ . On the other hand, passing to the c.m. system we must couple  $\ell$  with  $L$  to  $\lambda$ . For connecting these two couplings we introduce a "total" angular momentum  $J$  by  $\vec{J} = \vec{\ell} + \vec{L} + \vec{S}$ . There are two representations in which  $J$  is diagonal, the following unitary transformation connecting them<sup>9</sup> (we omit writing the radial quantum numbers  $n, N$ ):

$$|(\ell L)\lambda, S, JJ_z\rangle = (-1)^{\ell+L+\lambda} |(L\ell)\lambda, S, JJ_z\rangle \quad (6)$$

$$(-1)^{\lambda+S+J} \sum_j \begin{Bmatrix} L & \ell & \lambda \\ S & J & j \end{Bmatrix} \sqrt{((2\lambda+1)(2j+1))} |L, (\ell S), JJ_z\rangle.$$

Passing in the expression (5) for  $Q/e$  and on the left-hand side of the t-matrix elements to the state vectors  $|(\ell L)\lambda, S, JJ_z\rangle$  we obtain from eq. (2):

$$\begin{aligned} &\langle (n\ell, NL)\lambda, S, JJ_z | t^{(r)} | \underbrace{1^{(0)}, 2^{(0)}} \rangle = \langle (n\ell, NL)\lambda, S, JJ_z | v^{(r)} | \underbrace{1^{(0)}, 2^{(0)}} \rangle + \\ &+ \sum_{\nu=2}^{\infty} \frac{1}{e_1^{(0)} + e_2^{(0)} - \hbar\omega(\nu+3)} \sum_{2n'+\ell'+2N'+L'=\nu} \sum_{s\lambda'j'j'_z} \quad (7) \end{aligned}$$

$$\begin{aligned} &\langle (n\ell, NL)\lambda, S, JJ_z | v^{(r)} | (n'\ell', N'L')\lambda', S', J'J'_z \rangle \times \langle (n'\ell', N'L')\lambda', S', J'J'_z | t^{(r)} | \underbrace{1^{(0)}, 2^{(0)}} \rangle - \\ &- \sum_{2n''+\ell''+2N''+L''=\nu} \langle (n''\ell'', N''L'')\lambda'', S'', J''J''_z | t^{(r)} | \underbrace{1^{(0)}, 2^{(0)}} \rangle \times f_{n''\ell'', N''L'', n''\ell'', N''L'', \lambda''}^{(\nu, E_0)}. \end{aligned}$$

Here  $r = \frac{1}{2} T_{\mathbf{x}}(T_{\mathbf{x}} + 1)$  i.e.  $v^{(1)}$  denotes the nuclear potential plus Coulomb interaction while  $v^{(0)}$  is the pure nuclear interaction, and the same refers to  $t^{(r)}$ .

For calculating the matrix elements of  $v$  we use (6) and then pass to the state vectors  $|\ell S j j_{\mathbf{x}}\rangle$  by

$$|L, (\ell S) j, J j_{\mathbf{x}}\rangle = \sum_{M+j_{\mathbf{x}}=J_{\mathbf{x}}} (LM, j j_{\mathbf{x}} | L j, J j_{\mathbf{x}}) |LM\rangle | \ell S j j_{\mathbf{x}} \rangle.$$

In these representation the matrix elements of  $v$  are diagonal with respect to  $S, j, j_{\mathbf{x}}$  and do not depend on  $j_{\mathbf{x}}$ . Moreover, they are diagonal with respect to  $\ell$ , except of the tensor part which has non-vanishing matrix elements for  $|\ell - \ell'| = 2$  as well. Hence

$$\langle \ell S j j_{\mathbf{x}} | v^{(r)} | \ell' S' j' j'_{\mathbf{x}} \rangle = \delta_{SS'} \delta_{jj'} \delta_{j_{\mathbf{x}}j'_{\mathbf{x}}} v_{\ell S \ell' S'}^{(r)}.$$

Using further the unitary properties of the Clebsh-Gordan coefficients we obtain

$$\begin{aligned} & \langle (n \ell, N L) \lambda, S, J j_{\mathbf{x}} | v^{(r)} | (n' \ell', N' L') \lambda', S', J' j'_{\mathbf{x}} \rangle = \\ & = \delta_{NN'} \delta_{LL'} \delta_{j_{\mathbf{x}}j'_{\mathbf{x}}} \delta_{SS'} \delta_{JJ'} \langle n \ell | v_{\ell \lambda, \ell' \lambda'}^{r, J, S, L} | n' \ell' \rangle, \end{aligned} \quad (8)$$

where

$$\begin{aligned} \langle n \ell | v_{\ell \lambda, \ell' \lambda'}^{r, J, S, L} | n' \ell' \rangle & = (-1)^{\lambda + \lambda'} \sqrt{(2\lambda + 1)(2\lambda' + 1)} \times \\ & \times \sum_j (2j + 1) \begin{Bmatrix} \ell & \lambda \\ S & J & j \end{Bmatrix} \begin{Bmatrix} \ell' & \lambda' \\ S & J & j \end{Bmatrix} \langle n \ell | v_{\ell S \ell' S'}^{(r)} | n' \ell' \rangle. \end{aligned}$$

Similarly the inhomogeneous term in eq.(7) becomes

$$\langle (n\ell, NL)\lambda, S, JJ_z | v^{(r)} | 1^{(0)}, 2^{(0)} \rangle = \delta_{J_z, m_{12}^{(0)} + S_z^{(0)}} \delta_{SS^{(0)}} \times$$

$$\times (\rho_{12}^{(0)}, m_{12}^{(0)}, S_z^{(0)} | \rho_{12}^{(0)} S_z^{(0)} JJ_z ) \sum_{2n'+\ell'=\epsilon_1^{(0)} + \epsilon_2^{(0)} - 2N-L} \langle n'\ell', NL, \rho_{12}^{(0)} | n_1^{(0)} \ell_1^{(0)}, n_2^{(0)} \ell_2^{(0)}, \rho_{12}^{(0)} \rangle \times$$

$$\times \langle n \rho | v_{\rho\lambda, \rho \cdot \rho_{12}}^{(r, \ell, S, L)} | n' \rho' \rangle.$$

Substituting (8) and (8') into eq.(7) we see that in our mixed representation there is no coupling with respect to  $S, J, J_z$ . Further, since the matrix elements of  $v$  connect only the states with the same parity of  $\ell$  and so do the coefficients  $f$  with respect to  $L$  (cf. eq. (5)), it is obvious that eq. (7) couples only the  $t$ -matrix elements with the same parity of  $\ell$ . Hence for a given set of quantum numbers  $1^{(0)}, 2^{(0)}, S, J, J_z$  (hereafter we shall call such a set simply a "case"), eq.(7) represents a system for the  $t$ -matrix elements coupled over all admissible combinations of indices  $n, \ell, N, L, \lambda$ .

The form of the inhomogeneous term (8') leads to further simplifications when we note that the solution for a given "case" is nonvanishing only if (8') does not vanish at least for one combination of the coupled indices. Consequently, if it does not hold simultaneously

$$J_z = m_{12}^{(0)} + S_z^{(0)} \quad \text{and} \quad (\rho_{12}^{(0)}, m_{12}^{(0)}, S_z^{(0)} | \rho_{12}^{(0)} S_z^{(0)} JJ_z ) \neq 0.$$

then the "case" is trivial. If both these conditions are fulfilled then we can divide eq. (7) by the Clebsch-Gordan coefficient of (8') and obtain a system in which nothing depends on  $J_z, m_{12}^{(0)}, S_z^{(0)}$ . In this way, we are led to "reduced"  $t$ -matrix elements (denoted by a bar):

x) We suppose the single-particle energies  $e_1^{(0)} + e_2^{(0)}$  to have the form  $e_1^{(0)} + e_2^{(0)} = \hbar\omega [\epsilon_1^{(0)} + \epsilon_2^{(0)} + 3 + c_{12}]$ , where  $c_{12}$  can depend on  $n_1^{(0)}, \ell_1^{(0)}, n_2^{(0)}, \ell_2^{(0)}$ , and  $\rho_{12}^{(0)}$  but not on  $m_{12}^{(0)}$  and  $S_z^{(0)}$ .

$$\langle (n \ell, NL) \lambda, S, J J_z | t^{(r)} | 1^{(\bullet)}, 2^{(\bullet)} \rangle = \delta_{J_z, m_{12}^{(0)} + S^{(\bullet)}} \times \quad (9)$$

$$\times \langle \rho_{12}^{(0)} m_{12}^{(0)}, S^{(0)} S_z^{(0)} | \rho_{12}^{(0)} S_z^{(0)} J J_z \rangle \langle n \ell, NL, \lambda | \bar{t} | A_{12}, S, r, p \rho \rangle.$$

Here  $p \rho$  denotes the parity of  $\rho$  ( $p \rho = 0$  for even parity and  $p \rho = 1$  for the odd one),  $A_{12}$  denotes a "reduced case"  $A_{12} = n_1^{(0)}, \rho_1^{(0)}, n_2^{(0)}, \rho_2^{(0)}, \rho_{12}^{(0)}, J$  and on the left-hand side of the  $t$ -matrix element there remain only the indices over which the system is coupled.

Further, interchanging  $n_1^{(0)}, \rho_1^{(0)}$  with  $n_2^{(0)}, \rho_2^{(0)}$ , using the symmetry relation (5') and condition  $(-1)^{\rho_1^{(0)} + \rho_2^{(0)}} = (-1)^{p \rho + L}$  which satisfies the Moshinsky-coefficient in (8'), we find

$$\langle n \ell, NL, \lambda | \bar{t} | A_{12}, S, r, p \rho \rangle = (-1)^{\rho_1^{(\bullet)} + \rho_2^{(\bullet)} + p \rho + \rho_{12}^{(\bullet)}} \langle n \ell, NL, \lambda | \bar{t} | A_{21}, S, r, p \rho \rangle.$$

Hence it is sufficient to consider for given  $S, r, p \rho$  only such "cases" for which either  $2n_1^{(0)} + \rho_1^{(0)} > 2n_2^{(0)} + \rho_2^{(0)}$  or  $2n_1^{(\bullet)} + \rho_1^{(0)} = 2n_2^{(0)} + \rho_2^{(\bullet)}$  and  $\rho_1^{(\bullet)} \geq \rho_2^{(\bullet)}$ .

Finally, the Moshinsky-coefficients impose two more conditions

$$|\rho_1^{(0)} - \rho_2^{(0)}| \leq \rho_{12}^{(0)} \leq \rho_1^{(0)} + \rho_2^{(0)} \quad \text{and} \quad \rho_1^{(\bullet)} + \rho_2^{(0)} - p \rho \geq 0.$$

As to the dependence of the  $t$ -matrix elements on  $r$ , it can, of course, be neglected in comparison with the proper nuclear force. On the other hand, it is very simple to treat the influence of the Coulomb force exactly. After calculating all the "cases" with  $r = 0$  we must perform the calculation with  $r = 1$  for those "cases" for which  $S + r + p \rho$  is odd. Then the Coulomb energy of a given nucleus is the difference of the binding energy calculated with the Coulomb force and the binding energy calculated without it.

Summarizing all the conditions for the creation of "cases" (the exact treatment of the Coulomb force being included) we obtain for a given nucleus (occupied energy shells  $0, 1, \dots, \epsilon_0 - 1$ ):

(i)  $S = 0, 1; p_\rho = 0, 1; r = 0$  or  $r = 1$  if  $S + r + p_\rho$  is odd;

$$2n_1^{(0)} + l_1^{(0)} = 0, 1, \dots, \in_0 - 1;$$

(ii)  $2n_2^{(0)} + l_2^{(0)} = 0, 1, \dots, 2n_1^{(0)} + l_1^{(0)} - 1$  or  $2n_2^{(0)} + l_2^{(0)} = 2n_1^{(0)} + l_1^{(0)}$  and  $l_1^{(0)} \geq l_2^{(0)}$ ;

(iii) each pair  $l_1^{(0)}, l_2^{(0)}$  must satisfy  $l_1^{(0)} + l_2^{(0)} - p_\rho \geq 0$ ;

(iv) for given  $l_1^{(0)}, l_2^{(0)}$ ;

$$|l_1^{(0)} - l_2^{(0)}| \leq l_{12}^{(0)} \leq l_1^{(0)} + l_2^{(0)}; \quad (11)$$

(v) for a given  $l_{12}^{(0)}: |l_{12}^{(0)} - S| \leq J \leq l_{12}^{(0)} + S$ .

As an example all "cases" for  $\text{He}^4$  are listed in table 1 using notation (11). Note that for  $\text{He}^4$  only even values of  $l$  may occur since for odd  $l$  ( $p_\rho = 1$ ) the condition (11 - iii) cannot be satisfied for any "case".

Now we shall consider the coupled indices. For a given "case" it is convenient to put together the angular indices  $l, L, \lambda$  creating all admissible combinations of them and numerating these combinations ( $i=1, 2, \dots$ ). Each  $i$  represents thus three numbers  $l_i, L_i, \lambda_i$ . Now the values of  $\lambda$  are limited to  $|S - J| \leq \lambda \leq S + J$ . As to the values of  $l$ , they correspond to the partial-wave expansion. It is well established that in nuclear problems only several lowest values must be taken into account. Hence we shall suppose the values of  $l$  not to exceed some upper bound  $l_m$ . Because of the triangular condition between  $l, L, \lambda$  we see that the number of combinations is finite. We denote it by  $Z$  and list the conditions for the creation of combinations  $(l, L, \lambda)$  for a given "case":

(i) 
$$l = p_\rho, p_\rho + 2, p_\rho + 4, \dots, l_m$$

(ii) 
$$|S - J| \leq \lambda \leq S + J \quad (12)$$

(iii) 
$$|l - \lambda| \leq L \leq l + \lambda$$

(iv) 
$$(-1)^{l+L} = (-1)^{\tau_A} \text{ where } \tau_A = \frac{1}{2} [1 - (-1)^{l_1^{(0)} + l_2^{(0)}}]$$

(v) at least for one combination it must hold simultaneously  $\lambda_i = l_{12}^{(0)}$  and  $k_i \geq 0$

where 
$$k_i = n_1^{(0)} + n_2^{(0)} + \frac{1}{2} (l_1^{(0)} + l_2^{(0)} - L_i - l_i).$$

The last two conditions are due to the requirement that  $(\delta^s)$  must not vanish.

Since the matrix elements of  $v$  are diagonal with respect to  $L$ , it is convenient to arrange the combinations  $(\ell, L, \lambda)$  into subgroups according to increasing  $L$ , and inside of each subgroup ( $L$  is fixed) according to increasing  $\ell$ . Thus if there are  $Z_L$  different values of  $L$  we have  $L_1 < L_2 < \dots < L_{Z_L}$ , and the  $s$ -th subgroup ( $s = 1, 2, \dots, Z_L$ ) consists of combinations, with the common value  $L = L_s$ , the indices of these combinations ranging from  $i = d_s$  to  $i = h_s$  (obviously  $d_1 = 1$  and  $h_{Z_L} = Z$ ). In table 2 all combinations for the  $\text{He}^4$  - "cases" are listed for  $\ell_m = 2$ .

We introduce further for each  $i$  an integer quantity  $E_i = \text{entier} \left( \frac{L_i + \ell_i}{2} \right) = \left( \frac{L_i + \ell_i - 1}{2} \right)$  (cf. (12-IV)) and a new index  $\nu$  instead of  $n$  in (7):  $\nu = n + N + E_i$ . Denoting

$$\frac{2}{\hbar \omega} \langle n \ell, NL, \lambda | \bar{t} | A_{12}, S, r, F, f \rangle = \epsilon_{iN\nu}^{(s)} \quad (9')$$

where, according to our notation  $\ell = \ell_i, \lambda = \lambda_i, L = L_i = L_s$  ( $s$ -th subgroup i.e.  $i$  satisfies  $d_s \leq i \leq h_s$ ), the explicit writing of the "case" indices being omitted, we obtain from eqs. (7), (8'), (9)

$$\begin{aligned} \epsilon_{iN\nu}^{(s)} &= \sum_{j=d_s}^{h_s} \delta_{\lambda_j} \rho_{12}^{(0)} \langle k_j - N, \ell_j, NL_j, \lambda_j | n_1^{(0)}, \rho_1^{(0)}, n_2^{(0)}, \rho_2^{(0)}, \rho_{12}^{(0)} \rangle \times \\ &\times \langle \nu - N - E_i, \ell_i | v_{ij}^{(s)} | k_j - N, \ell_j \rangle + \sum_{j=d_s}^{h_s} \sum_{\mu=\max(\epsilon_0, N+E_j)}^{\infty} \epsilon_{iN\mu}^{(s)} \times \\ &\times \frac{\langle \nu - N - E_i, \ell_i | v_{ij}^{(s)} | \mu - N - E_j, \ell_j \rangle}{4(k_j + E_j - \mu) + 2c_{12}} - \sum_{j=d_s}^{h_s} \sum_{\mu=\max(\epsilon_0, N+E_j)}^{\infty} \frac{\langle \nu - N - E_i, \ell_i | v_{ij}^{(s)} | \mu - N - E_j, \ell_j \rangle}{4(k_j + E_j - \mu) + 2c_{12}} \times \\ &\times \sum_{s'=1}^{Z_L} \sum_{j'=d_{s'}}^{h_{s'}} \sum_{N'=0}^{\mu-E_j'} \delta_{\lambda_j, \lambda_{j'}} f_{iN, j'N'}^{-(\mu, \epsilon_0)} \epsilon_{j'N'\mu}^{(s')} \quad \begin{matrix} s=1, 2, \dots, Z_L \\ i=d_s, d_s+1, \dots, h_s \\ N=0, 1, \dots \\ \nu=\max(\epsilon_0, N+E_i), \dots \end{matrix} \end{aligned} \quad (13)$$

Here we have denoted

$$\langle n, \ell_1 | v_{1j}^{(n)} | n', \ell_j \rangle = \frac{2}{\hbar \omega} \langle n \ell_1 | v_{1j}^{(r, j, s, L_s)} | n' \ell_j \rangle,$$

$$f_{jN, j'N'}^{(\mu, \epsilon_0)} = f_{\mu - N - E_1, \ell_1, NL_1, \mu - N' - E_1, \ell_j, N' L_j, \lambda_1}$$

For the definition of  $c_{12}$  see the footnote on p. 11, and  $k_1$  is defined by (12-V). Finally, the coordinate representation of the state vectors  $|n, \ell\rangle$  is

$$R_{n\ell}(r) = \sqrt{\left(\frac{2n!}{\Gamma(n + \ell + 3/2)}\right)} r^{\ell+1} e^{-\frac{r^2}{2}} L_n^{\left(\ell + \frac{1}{2}\right)}(r^2),,$$

where  $L$  is a Laguerre polynomial and  $r$  is related to the internucleon distance  $x$  by  $r = \sqrt{\left(\frac{m\omega}{2\hbar}\right)}x$  ( $m$  is the nucleon mass,  $\omega$  - the fixed oscillator frequency), the normalization being

$$\int_0^{+\infty} [R_{n\ell}(r)]^2 dr = 1.$$

This system is infinite with respect to  $N$  and  $\nu$ . The infinity in  $N$  can be removed introducing the approximate exclusion operator as proposed in sec.2. Hence we replace in the term containing the  $f$ -coefficients the infinite upper bound of the sum over  $\mu$  by a finite constant  $M$  and this automatically leads to the following limit of  $N$  (for a given  $s$ ).

$$N = 0, 1, \dots, m_s \quad \text{where} \quad m_s = \max_{d_s < l < h_s} (M - E_1). \quad (14)$$

The system (13) is, of course, a generalization of both of the diagonal approximation discussed in sec.2; that of Eden-Emery is obtained putting  $M = \epsilon_0 - 1$ , while the other one means that we retain in the sum over  $s', j', N'$  only the term with  $s = s', j = j'$  and  $N = N'$ . In both cases, besides the coupling over  $\nu$ , the system remains coupled only over  $i$  within the given subgroup  $s$  (this is due to the noncentral forces).

As to the infinity in  $\nu$ , we can remove it passing from the algebraic system to equivalent integral or integrodifferential equations for the correlated two-particle state vector  $|\psi_{1(0), 2(0)}\rangle$  which is related to the unperturbed state vector  $|1^{(0)}, 2^{(0)}\rangle$  by

$$\nu |\psi_{1(0), 2(0)}\rangle = t |1^{(0)}, 2^{(0)}\rangle.$$

In our representation we obtain

$$\epsilon_{iN\nu}^{(s)} = \int_0^{+\infty} R_{\nu - N - E_i} \sum_{j=d_s}^{h_s} v_{ij}^{(s)}(r) \psi_{jN}^{(s)}(r) dr. \quad (15)$$

Using this relation we can transform (13) to a finite system of integral or integrodifferential equations for  $\psi_{iN}^{(s)}$  the coupling extending over  $s$ ,  $i$  and  $N$  ( $s = 1, 2, \dots, Z_L$ ;  $i = d_s, d_s + 1, \dots, h_s$ ;  $N = 0, 1, \dots, m_s$ ).

However, solving this coupled system would result in very tedious calculations. We assume the algebraic form to be much more convenient, provided that (13) can be approximated sufficiently accurately by a finite system with respect to  $\nu$ . The only possibility for this is replacing operator

$$\hat{G}_\lambda^{(\ell)} = \sum_{\substack{n=0 \\ n \neq \lambda}}^{\infty} \frac{|n\rangle \langle n|}{4(\lambda - n)} \quad (\lambda \text{ real})$$

which is implicitly contained in (13), by a finite sum. This replacement is theoretically correct since the expansion of  $\hat{G}_\lambda^{(\ell)}$  which occurs in (13) is convergent <sup>13</sup>

$$\int_0^{+\infty} G_\lambda^{(\ell)}(r, r') f(r') df' = \sum_{\substack{n=0 \\ n \neq \lambda}}^{\infty} \frac{R_n \rho(r) \int_0^{+\infty} R_n \rho(r') f(r') dr'}{4(\lambda - n)},$$

where  $f$  is an arbitrary quadratic integrable function.

In practice, we must choose some upper bound  $D$ , this choice being correct only if the results do not considerably change when  $D \rightarrow D + 1$ ,  $D + 2$  etc. Since the dimension of the system increases very rapidly with  $D$  and, on the other hand, the correct value of  $D$  may be rather great, it would be hardly possible to solve the complete system. The solution can



be found in a simpler way if we start with one of the diagonal approximations and consider the nondiagonal part as a perturbation. For this purpose we rewrite (13)

$$\epsilon_{iN\nu}^{(s)} = \sum_{j=d_s}^{h_s} \sum_{\mu=\max(\epsilon_0, N+E_j)}^D \epsilon_{iN\mu}^{(s)} \frac{\langle \nu-N-E_1, \ell_1 | v_{1j}^{(s)} | \mu-N-E_1, \ell_1 \rangle}{4(k_j + E_j - \mu) + 2c_{12}} = \sum_{j=d_s}^{h_s} \delta_{\ell_2}^{(s)} \lambda_j \times \quad (16)$$

$$\times \langle k_j, -N, \ell_j, N L_j, \lambda_j | n_1^{(0)} \ell_1^{(0)}, n_2^{(0)} \ell_2^{(0)}, \lambda_j \rangle \langle \nu-N-E_1, \ell_1 | v_{1j}^{(s)} | k_j, -N, \ell_j \rangle -$$

$$- \sum_{j=d_s}^{h_s} \sum_{\mu=\max(\epsilon_0, N+E_j)}^M \gamma_{iN\mu}^{(s)} \frac{\langle \nu-N-E_1, \ell_1 | v_{1j}^{(s)} | \mu-N-E_1, \ell_1 \rangle}{4(k_j + E_j - \mu) + 2c_{12}} \quad \begin{array}{l} s=1, 2, \dots, Z_L \\ N=0, 1, \dots, m_s \\ i=d_s, d_s+1, \dots, h_s \\ \nu=\max(\epsilon_0, N+E_j), \dots, D \end{array}$$

where

$$\gamma_{iN\mu}^{(s)} = \sum_{s'=1}^{Z_L} \sum_{j=d_{s'}}^{h_{s'}} \sum_{N'=0}^{\nu-E_j} \delta_{\lambda_1} \lambda_j \frac{f_{iNjN'}^{(\nu, \epsilon_0)} \epsilon_{jN'\nu}^{(s')}}{f_{iNjN'}^{(\nu, \epsilon_0)} \epsilon_{jN'\nu}^{(s')}} \quad (17)$$

For solving (16) the following iterative procedure is proposed: let the  $n$ -th approximation of  $\gamma$  be known (in the zeroth approximation we put  $\gamma=0$  - the diagonal approximation of Eder-Emery). Then for each pair of indices  $s, N$  we solve (16), the coupling referring now only to  $i$  and  $\nu$ . This gives us the  $n$ -th approximation of  $\epsilon$ . Putting these quantities into the formula for  $\gamma$  (17) we obtain the  $(n+1)$ -th approximation of  $\gamma$ . In this way our iterative procedure is uniquely defined.

The convergence of this iterative procedure seems to be guaranteed from the following considerations:

- (i) the diagonal approximation is undoubtedly a good zero-order approximation
- (ii) the form of the  $f$ -coefficients (5) suggests that the nondiagonal part of the exclusion operator  $Q$  can be considered as a relatively small correction.

For a numerical treatment this procedure will be obviously very convenient even if the dimensions of the algebraic systems are large, since

we simply multiply the inverse matrices (which do not change by iterations) by the right-hand side vectors. The only numerically tedious part is the calculation of the matrix elements of  $\nu$  which must be performed before starting iterations. Here we want to underline that the above method is directly applicable only for the soft-core potentials for which the matrix elements are finite. For the hard-core potentials a modification is necessary.

Regarding the hard-core as a limit of a finite rectangular repulsion  $v_0$ , the dimensionless hard-core radius being  $a$ , we obtain for a given  $v_0$  from (15), (16), (17), for  $0 \leq r \leq a$ :

$$\left[ \frac{d^2}{dr^2} - \nu^2 - \frac{\ell_1(\ell_1+1)}{r^2} + 4(k_1 - N) + 2\ell_1 + 3 + 2c_{12} \right] \psi_{1N}^{(*)}(r) = v_0 \psi_{1N}^{(*)}(r) - \sum_{\nu=N+E_1}^{\epsilon_0-1} \epsilon_{1N\nu}^{(*)} \times$$

$$\times R_{\nu-N, -E_1, \ell_1}(r) - \sum_{\nu=\max(\epsilon_0, N+E_1)}^M \gamma_{1N\nu} R_{\nu-N, -E_1, \ell_1}(r) + 2c_{12} \delta_{\lambda_1} \rho_{12}^{(0)} \langle k_1 - N, \ell_1, N L_1, \lambda_1 |$$

$$| n_1^{(0)} \rho_1^{(0)}, n_2^{(0)} \rho_2^{(0)} \lambda_1 \rangle \times R_{k_1 - N, \ell_1}(r).$$

Multiplying by any function with integrable second derivative on  $[0, a]$  and integrating over this interval, we find

$$\lim_{v_0 \rightarrow +\infty} \int_0^a f(r) v_0 \psi_{1N}^{(*)}(r) dr = B_{1N}^{(*)} f(a) + \sum_{\nu=N+E_1}^{\epsilon_0-1} \epsilon_{1N\nu}^{(*)} \int_0^a R_{\nu-N, -E_1, \ell_1}(r) f(r) dr +$$

$$+ \sum_{\nu=\max(\epsilon_0, N+E_1)}^M \gamma_{1N\nu} \int_0^a R_{\nu-N, -E_1, \ell_1}(r) f(r) dr - 2c_{12} \delta_{\lambda_1} \rho_{12}^{(0)} \int_0^a R_{k_1 - N, \ell_1}(r) f(r) dr \quad (18)$$

$$\times \langle k_1 - N, \ell_1, N L_1, \lambda_1 | n_1^{(0)} \rho_1^{(0)}, n_2^{(0)} \rho_2^{(0)}, \lambda_1 \rangle$$

where

$$B_{1N}^{(*)} = \lim_{v_0 \rightarrow +\infty} \left( \frac{d}{dr} \psi_{1N}^{(*)}(r) \Big|_{r=a+} \right).$$

For  $a \leq r < +\infty$  the functions  $\psi$  satisfy

$$\psi_{1N}^{(s)}(r) = \delta \ell_{12}^{(0)} \lambda_1 \langle k_1 - N, \ell_1, NL_1, \lambda_1 | n_1^{(0)} \ell_1^{(0)}, n_2^{(0)} \ell_2^{(0)}, \lambda_1 \rangle R_{k_1 - N, \ell_1}(r) +$$

$$+ \sum_{\nu=\max(\epsilon_0, N+E_1)}^D \epsilon_{1N\nu}^{(s)} \frac{R_{\nu-N-E_1, \ell_1}(r)}{4(k_1 + E_1 - \nu) + 2c_{12}} - \sum_{\nu=\max(\epsilon_0, N+E_1)}^M \gamma_{1N\nu}^{(s)} \frac{R_{\nu-N-E_1, \ell_1}(r)}{4(k_1 + E_1 - \nu) + 2c_{12}} \quad (19)$$

Further we have from (15)

$$\epsilon_{1N\nu}^{(s)} = \lim_{\alpha \rightarrow +\infty} \int_0^\alpha R_{\nu-N-E_1, \ell_1}(r) v_0 \psi_{1N}^{(s)}(r) dr + \sum_{j=d_s}^{h_s} \int_\alpha^{+\infty} R_{\nu-N-E_1, \ell_1}(r) v_{1j}^{(s)}(r) \psi_{1N}^{(s)}(r) dr.$$

For calculating the first term, we use (18), and for the second term (19).

Hence for each pair  $s, N$  ( $s=1, 2, \dots, Z_L; N=0, 1, \dots, m_s$ ) we have

$$\epsilon_{1N\nu}^{(s)} = \sum_{\mu=N+E_1}^{\epsilon_0-1} \epsilon_{1N\mu}^{(s)} J_{\nu-N-E_1, \mu-N-E_1}^{(\ell_1)}(a) - \sum_{j=d_s}^{h_s} \sum_{\mu=\max(\epsilon_0, N+E_1)}^D \epsilon_{1N\mu}^{(s)} \times$$

$$\frac{\langle \nu-N-E_1, \ell_1 | \bar{v}_{1j}^{(s)} | \mu-N-E_1, \ell_1 \rangle}{4(k_1 + E_1 - \mu) + 2c_{12}} - B_{1N}^{(s)} R_{\nu-N-E_1, \ell_1}(a) = \sum_{j=d_s}^{h_s} \delta_{12}^{(0)} \lambda_j \times$$

$$\times \langle k_1 - N, \ell_1, NL_1, \lambda_j | n_1^{(0)} \ell_1^{(0)}, n_2^{(0)} \ell_2^{(0)}, \lambda_j \rangle [\langle \nu-N-E_1, \ell_1 | \bar{v}_{1j}^{(s)} | k_1 - N, \ell_1 \rangle - 2\delta_{1j} c_{12} \times$$

$$\times J_{\nu-N-E_1, k_1-N}^{(\ell_1)}(a)] - \sum_{j=d_s}^{h_s} \sum_{\mu=\max(\epsilon_0, N+E_1)}^M \gamma_{1N\mu}^{(s)} \left[ \frac{\langle \nu-N-E_1, \ell_1 | \bar{v}_{1j}^{(s)} | \mu-N-E_1, \ell_1 \rangle}{4(k_1 + E_1 - \mu) + 2c_{12}} - \right.$$

$$\left. - \delta_{1j} \times J_{\nu-N-E_1, \mu-N-E_1}^{(\ell_1)}(a) \right]$$

where  $\bar{v}$  denotes the regular part of  $v$  (for  $a \leq r < +\infty$ ) and  $J_{mn}^{(\ell)}(a) = \int_0^a R_n \ell(r) R_m \ell(r) dr$ . For the HJ -potential the matrix elements of  $\bar{v}$  are available<sup>15</sup>. In (20) there are, besides the quantities  $\epsilon_{1N\nu}^{(s)}$ , another unknown quantities  $B_{1N}^{(s)}$  ( $i = d_s, d_s + 1, \dots, h_s$ ).

The equations by which (20) must be supplemented in order to define all the unknown quantities uniquely, follow from the boundary conditions

$$\lim_{\nu_0 \rightarrow +\infty} \psi_{iN}^{(\ast)}(a) = 0 \quad i = d_s, d_s + 1, \dots, h_s.$$

Putting in (19)  $r = a$  we find

$$\sum_{\nu = \max(\epsilon_0, N + E_1)}^D \frac{R_{\nu - N - E_1, \ell_1}(a)}{4(k_1 + E_1 - \nu) + 2c_{12}} \epsilon_{iN\nu}^{(\ast)} = \sum_{\nu = \max(\epsilon_0, N + E_1)}^M \frac{R_{\nu - N - E_1, \ell_1}(a)}{4(k_1 + E_1 - \nu) + 2c_{12}} \gamma_{iN\nu}^{(\ast)} -$$

$$(20)$$

$$- \delta_{12} \rho_{12}^{(0)} \lambda_1 < k_1 - N, \ell_1, N L_1, \lambda_1 | n_1^{(0)} \rho_1^{(0)}, n_2^{(0)} \rho_2^{(0)} \lambda_1 > R_{k_1 - N, \ell_1}(a).$$

The algebraic system (20), (20') for the unknown quantities

$\epsilon_{iN\nu}^{(\ast)}$  ( $i = d_s, d_s + 1, \dots, h_s; \nu = N + E_1, N + E_1 + 1, \dots, D$ ) and  $B_{iN}^{(\ast)}$  ( $i = d_s, d_s + 1, \dots, h_s$ ) represents the required modification of (16) for the hard-core potentials and can be solved by the same iterative method.

#### 4. Discussion

We have presented in eqs. (16), (17), (20), (20') a numerical procedure for calculating the t-matrix elements between the ground state and arbitrary excited or unexcited state (cf. the beginning of sec.2). With these matrix elements we can calculate immediately the first - and second - order term in the Goldstone expansion for energy and the first-order term of the one-particle-excitation part of the wave function. The latter quantity is necessary for evaluating the mean value of one-particle operators (r.m.s. radius, density) up to the first order in  $t$ .

The evaluation of higher-order terms requires the t-matrix elements between excited states, the numerical calculation of these terms being

very tedious. The evaluation of other diagrams requires the  $t$ -matrix elements between excited states, the propagator  $Q/e$  being much more complicated in this case and, consequently, the numerical treatment of the corresponding equations being very difficult. Because of this difficulty, we assume that the Goldstone expansion is practically applicable only if a self-consistent calculation of the first and second order gives a good agreement with experiment. Otherwise this expansion, even if it were convergent, could not be used at the present state of numerical techniques for actual calculations of nuclear properties.

It is obvious that if we want to decide the question of the practical applicability of the Goldstone expansion, we must be sure that our method for calculating the numerically accessible part of the expansion is sufficiently accurate and that possible discrepancies with experimental data are not due to uncorrect approximations. We hope that the method of the present paper which works only with approximation permitting to obtain an arbitrary accuracy of the results, can provide such a sufficiently accurate solution. (The results of preliminary calculations for  $\text{He}^4$  according to (20), (20') with the HJ -potential will be soon published).

In conclusion, we shall show how the  $t$ -matrix elements occurring in the first- and second - order diagrams can be expressed by means of the solution of the algebraic systems (16) or (20).

For the first order we can directly express the energy correction

$$T_1 = \frac{1}{2} \sum_{i^{(0)}, j^{(0)}} \langle i^{(0)}, j^{(0)} | t | i^{(0)}, j^{(0)} \rangle_A,$$

while for the second - order energy correction and for the first-order correction to the above mentioned mean values we need:

$$T_2^{(i, j^{(0)})} = \sum_{k^{(0)}} \langle k^{(0)}, i | t | k^{(0)}, j^{(0)} \rangle_A.$$

Here the subscript  $A$  reminds that we are dealing with antisymmetrized matrix elements, and each letter  $i, j, k$  denotes a one-particle state characterized by 5 oscillator, spin and isospin quantum numbers in usual notation, e.g.

$$i \equiv n_i, \ell_i, m_i, s_z^{(i)}, t_z^{(i)}, \quad j^{(0)} \equiv n_j^{(0)}, \ell_j^{(0)}, m_j^{(0)}, s_z^{(j)}, t_z^{(j)}$$

the upper index 0 denoting occupied states. Introducing the LST-coupling (2'), passing in the left-hand side state vectors to the c.m. system and using symmetry properties of the Clebsh-Gordan and Moshinsky coefficients, we find

$$T_1 = \frac{1}{2} \sum_{n_i^{(0)} \ell_i^{(0)} m_i^{(0)} s_z^{(i)} t_z^{(i)} n_j^{(0)} \ell_j^{(0)} m_j^{(0)} s_z^{(j)} t_z^{(j)}} \langle i^{(0)}, j^{(0)} | t | i^{(0)}, j^{(0)} \rangle_A$$

$$T_2^{(i,j^{(0)})} = \sum_{k^{(0)} s_z^{(k)} T} \sum_{\ell_k, \ell_j} \left( \frac{1}{2} s_z^{(k)}, \frac{1}{2} s_z^{(i)} \middle| \frac{1}{2} \frac{1}{2}, S S_z^{(ki)} \right) \left( \frac{1}{2} s_z^{(k)} \frac{1}{2} s_z^{(j)} \middle| \frac{1}{2} \frac{1}{2} S S_z^{(kj)} \right) \times$$

$$\times \delta \left( t_z^{(i)} t_z^{(j)} \left( \frac{1}{2} t_z^{(k)}, \frac{1}{2} t_z^{(i)} \middle| \frac{1}{2} \frac{1}{2}, T T_z \right)^2 (\ell_k m_k, \ell_i m_i) | \ell_k \ell_i, \ell_{ki} m_{ki} \right) \times$$

$$\times (\ell_k m_k, \ell_j m_j | \ell_k \ell_j, \ell_{kj} m_{kj}) \langle k^{(0)}, i | t | k^{(0)}, j^{(0)} \rangle_A,$$

where

$$S_z^{(ki)} = s_z^{(k)} + s_z^{(i)}, \quad S_z^{(kj)} = s_z^{(k)} + s_z^{(j)}, \quad m_{ki} = m_k + m_i, \quad m_{kj} = m_k + m_j,$$

$$T_z = t_z^{(k)} + t_z^{(i)}, \quad S^{(ki)} = S^{(kj)} = S, \quad T^{(ki)} = T^{(kj)} = T,$$

and

$$\langle a, b | t | c^{(0)}, d^{(0)} \rangle_A = \sum_{2n+\ell+2N+L=\epsilon_a+\epsilon_b} \langle n\ell, NL, \ell_{ab} | n_a \ell_a, n_b \ell_b, \ell_{ab} \rangle \times$$

$$\times [1 - (-1)^{s+\tau+\ell}] \langle n\ell, NL, \ell_{ab} m_{ab}, S S_z^{(ab)}, T T_z | t | c^{(0)}, d^{(0)} \rangle.$$

Further we make use of the coupled representation  $|(n\ell, NL)\lambda, S, J J_z\rangle$  and of eqs. (9), (9'), applying the notation (12). Then

$$\begin{aligned}
 \langle a, b | t | c^{(0)}, d^{(0)} \rangle &= \frac{\hbar \omega}{4} (1 + (-1)^{\ell_a + \ell_b + \ell_c^{(0)} + \ell_d^{(0)}}) \sum_{p, q=0,1} [1 - (-1)^{S+T+p, q}] \times \\
 &\times \sum_{J=|\ell_c^{(0)} - S|}^{\ell_c^{(0)} + S} \delta_{m_{ab} + S_z^{(ab)}, m_{cd}^{(0)} + S_z^{(cd)}} \langle \ell_{ab} m_{ab}, S S_z^{(ab)} | \ell_{ab} S, J m_{ab} + S_z^{(ab)} \rangle \times \\
 &\times \langle \ell_{cd}^{(0)} m_{cd}^{(0)}, S S_z^{(cd)} | \ell_{cd}^{(0)} S, J m_{cd}^{(0)} + S_z^{(cd)} \rangle \times \quad (22)
 \end{aligned}$$

$$\times \sum_{a=1}^{Z_L} \sum_{j=d_a}^h \delta_{\lambda_j} \langle \nu - E_j - N, \ell_j, NL_j, \lambda_j | n_a \ell_a n_b \ell_b, \ell_{ab} \rangle \epsilon_{jN\nu}^{(0)}(r, A_{cd}),$$

where  $r = \frac{1}{2} T_z (T_z + 1)$  and  $\nu = n_a + n_b + \text{entier}(\frac{\ell_a + \ell_b}{2})$ .

Note that the quantum numbers  $a, b$  in (22) must satisfy  $\nu \leq D$  since the systems (16) or (20) do not give  $\epsilon$  for higher values of  $\nu$ , these quantities being neglected by our D-approximation. From (22) we see that this means putting zero all the t-matrix elements  $\langle a, b | t | c^{(0)}, d^{(0)} \rangle$  with the excitation energy greater than  $\hbar \omega (2D - \epsilon_c^{(0)} - \epsilon_d^{(0)})$ .

This is a quite natural restriction if we note that these matrix elements occur in diagrams only in sums which extend to infinity with respect to all excited states. Consequently, they must tend to zero with increasing excitation energy in order to ensure convergence of these infinite sums.

The expression for  $T_1$  can be further simplified putting (22) into (21) and performing summations over  $m_{1j}^{(0)}$  and  $S_z$

$$T = \frac{\hbar \omega}{4} \sum_{S, T, p, q} [1 - (-1)^{S+T+p, q}] \sum_{\Lambda_{12}} (2J+1) \sum_{a=1}^{Z_L} \sum_{j=d_a}^h \delta_{\lambda_j} \rho_{12}^{(0)} \times \quad (23)$$

$$\times \sum_{N=0}^{\min(\nu - E_j, m_a)} \langle \nu - E_j - N, \ell_j, NL_j, \lambda_j | n_1^{(0)} \rho_1^{(0)}, n_2^{(0)} \rho_2^{(0)}, \rho_{12}^{(0)} \rangle \sum_{T_z=-T}^T \epsilon_{jN\nu}^{(0)} \times \\
 \times (\frac{1}{2} T_z (T_z + 1), \Lambda_{12})$$

where  $\sum$  denotes summation over all "reduced cases" (cf. (9)) and the dependence of  $\epsilon$  on  $r$  and  $\Lambda_{12}$  is explicitly written.

The second-order correction for energy and the first-order correction for the mean values of the r.m.s. radius and density can be immediately expressed by means of  $T^{(i,j^{(0)})}$ , the possible summations over projections leading in some cases to considerable simplifications of the resulting formulae.

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Table 1

Case No	s	r	$P_{\ell}$	$n_1^{(0)}$	$n_2^{(0)}$	$n_{-2}^{(0)}$	$\ell_2^{(0)}$	$\ell_{12}^{(0)}$	J
1	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	1

Table 2

Case No1  
( $Z_L=2, Z=2$ )

Cases No 2,3,  
( $Z_L=3, Z=7$ )

s	1	2	1	1	2	2	2	2	3
i	1	2	1	2	3	4	5	6	7
$\ell_1$	0	2	0	2	0	2	2	2	2
$L_1$	0	2	0	0	2	2	2	2	4
$\lambda_1$	0	0	0	2	2	0	1	2	2
$E_1$	0	2	0	1	1	2	2	2	3
$k_1$	0	-2	0	-1	-1	-2	-2	-2	-3