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EQUAL TIME CURRENT-CURRENT COMMUTATOR IN QUANTUM ELECTRODYNAMICS

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In this note we give the calculation of

$$\delta(x^0) < [\, j^i(x), j^0(0)\,] >_0 \quad , \quad i = 1, 2, 3, \tag{1}$$

where j^μ is given as

$$j^\mu(x) = e : \bar{\psi}(x) \gamma^\mu \psi(x) : \tag{2}$$

that corresponds to the first order of the perturbation theory of quantum electrodynamics. As it has bean pointed out by Schwinger /1/, in the theories with nontrivial S -matrix (1) should not vanish. At the same time the statement is widespread that if the current is given as (2), then (1) should vanish. Indeed direct commutation (2) gives:

$$<[:\bar{\psi}(x)\gamma^\mu \bar{\psi}(x)::\bar{\psi}(y)\gamma^\nu \psi(y):]>_0 = \frac{1}{i} < \bar{\psi}(x)\gamma^\mu S(x-y)\gamma^\nu \psi(y) - \tag{3}$$

$$- \bar{\psi}(y)\gamma^\nu S(y-x)\gamma^\mu \psi(x) >_0 \ .$$

For equal times it follows:

$$\delta(x^0-y^0) < [\, j^i(x)\, j^0(y)\,] >_0 = e^2\, \delta(x^0-y^0)\delta(\vec{x}-\vec{y}) < \bar{\psi}(x)\gamma^i \psi(y) - \bar{\psi}(y)\gamma^i \psi(x) >$$

and if $\bar{\psi}(x), \psi(y)$ had c -numbers, one would indeed obtain zero. However, as $\bar{\psi}(x), \psi(y)$ are operator-valued distributions, instead of zero one obtains:

$$\delta(x^0-y^0)<[j^1(x)j^0(y)]>_0 = -e^2\delta(x-y)\partial_1 D^1(x-y),$$

(where $iD^1(x) = D^+(x) - D^-(x)$), which is not defined. Indeed $D^1(x) = -\frac{2}{(2\pi)^3} P\frac{1}{x^2}$ but the function $\delta(x) P\frac{1}{x^2}$ is not defined as a generalized function. Modification of the current (2) was a natural response to the situation which takes place. Necessity of the appearance of the Schwinger term was connected with the necessity to define local bilinear operator product as a limit of the product defined at points spatially separated from each other by small distances:

$$\tilde{j}^{\mu}(x) = \lim_{\epsilon\to 0} e : \bar{\psi}(x+\frac{\epsilon}{2})\gamma^{\mu}\psi(x-\frac{\epsilon}{2}) : , \qquad (5)$$

where $\epsilon = \vec{\epsilon}$.

Then one may conclude [1]:

$$\delta(x^0-y^0)<[j^0(x)\tilde{j}^1(y)]>_0 = \delta(x^0-y^0)\partial_1\delta(\vec{x}-\vec{y})K^2 , \qquad (6)$$

where
$$K^2 = \lim_{\epsilon\to 0}\frac{2e^2}{i3\pi^2\vec{\epsilon}^2} .$$

However, the appearance of such ambiguities and divergences is not surprising. Indeed from the theory of the Lorentz invariant generalized functions [2,3] it is well known, how much the Lorentz invariant generalized functions are sensitive to the transition to the limit to the origin of the light cone, performed not sufficiently accurately. Let us try, following [2,3] to carry out a transition to equal times in the expression $<[j^1(x)j^0(y)]>_0$ more accurately. The expression (3) is equivalent to:

$$<[:\psi(x)\gamma^{\mu}\psi(x):, :\psi(y)\gamma^{\nu}\psi(y):]> =$$
$$= \frac{1}{(2\pi)^6}\int d\vec{k}_1 d\vec{k}_2 \{ e^{-i(k_1+k_2)(x-y)} Sp\gamma^{\mu}\frac{\hat{k}_1+m}{2k_1^0}\gamma^{\nu}\frac{\hat{k}_2-m}{2k_2^0} - e^{i(k_1+k_2)(x-y)} Sp\gamma^{\nu}\frac{\hat{k}_1+m}{2k_1^0}\times$$
$$\times\gamma^{\mu}\frac{\hat{k}_2-m}{2k_2^0}\}. \qquad (7)$$

Taking into account the gauge invariance one finds:

$$F^{\mu\nu}(k)=\frac{1}{(2\pi)^4}\int d^4x\, e^{-ikx}<[:\bar{\psi}(x)\gamma^{\mu}\psi(x):,:\bar{\psi}(0)\gamma^{\nu}\psi(0):]>_0 = \quad (8)$$

$$=\frac{2}{3(2\pi)^6}\left(g^{\mu\nu}-\frac{k^{\mu}k^{\nu}}{k^2}\right)\int\frac{d\vec{k}_1}{k_1^0k_2^0}(2m^2+k_1k_2)\{\theta(k^0)\delta(-k^0+k_1^0+k_2^0)-$$

$$-\theta(-k^0)\delta(k^0+k_1^0+k_2^0)\}|_{\vec{k}=\vec{k}_1+\vec{k}_2}.$$

It follows

$$F^{\mu\nu}(k)=\frac{2}{3}\left(g^{\mu\nu}-\frac{k^{\mu}k^{\nu}}{k^2}\right)\frac{1}{(2\pi)^5}\int_{(2m)^2}^{\infty}d\kappa^2\left(m^2+\frac{\kappa^2}{2}\right)\sqrt{1-\frac{4m^2}{\kappa^2}}\,\epsilon(k^0)\delta(k^2-\kappa^2) \quad (9)$$

and

$$<[j^0(x)j^1(y)]>_0=e^2\int d^4k\, e^{ik(x-y)}F^{01}(k)= \quad (10)$$

$$=\partial_0\partial_1\int_{4m^2}^{\infty}d\kappa^2\rho(\kappa^2)D(x-y,\kappa^2),$$

where

$$\rho(\kappa^2)=\frac{ie^2}{3(2\pi)^2}\left(1+\frac{2m^2}{\kappa^2}\right)\sqrt{1-\frac{4m^2}{\kappa^2}}\,. \quad (11)$$

It should be noted now, that if we carry out differentiation under the sign of spectral integral, then we get (see also /4/):

$$\delta(x^0)<[j^0(x)j^1(0)]>_0=\delta(x^0)\partial_1\delta(\vec{x})K^2,\ K^2=\int_{4m^2}^{\infty}d\kappa^2\rho(\kappa^2)\,. \quad (12)$$

Divergence of the integral for K^2 shows that it is impossible to make such a differentiation under the sign of the improper integral. Let us break now $\rho(\kappa^2)$ into two parts:

$$\rho(\kappa^2)=\rho_1(\kappa^2)+\rho_2(\kappa^2) \quad (13)$$

$$\rho_1(\kappa^2)=\frac{ie^2}{3(2\pi)^2}$$

$$\rho_2(\kappa^2) = \frac{i e^2}{3(2\pi)^2} \{ (1 + \frac{2m^2}{\kappa^2}) \sqrt{1 - \frac{4m^2}{\kappa^2}} - 1 \} .$$

In $\partial_0 \partial_1 \int_{4m^2}^{\infty} d\kappa^2 \rho_2(\kappa^2)\, D(x-y, \kappa^2)$ we can already differentiate under the sign of the integral. Then we find:

$$\delta(x^0 - y^0)\, \partial_1 \partial_0 \int_{4m^2}^{\infty} d\kappa^2\, \rho_2(\kappa^2)\, D(x-y, \kappa^2) = \partial_1\, \delta(x-y) \frac{(-i)e^2 2m^2}{3(2\pi)^2} \qquad (14)$$

Moreover

$$\delta(x^0-y^0)\, \partial_1 \partial_0 \int_0^{4m^2} d\kappa^2 \rho_1(\kappa^2)\, D(x-y, \kappa^2) = \partial_1\, \delta(x-y) \frac{i e^2 4m^2}{3(2\pi)^2} \qquad (15)$$

$$\delta(x^0-y^0)\partial_1 \partial_0 \int_0^{\infty} d\kappa^2 \rho_1(\kappa^2)\, D(x-y, \kappa^2) = \frac{i e^2}{3(2\pi)^2}\, \partial_1 \{ \delta(x^0-y^0)\partial_0 \int_0^{\infty} d\kappa^2\, D(x-y, \kappa^2)\} . \qquad (16)$$

Detailed analysis of the quantity $\delta(x_0)\frac{\partial}{\partial x_0} \int_0^{\infty} d\kappa^2\, D(x, \kappa^2)$ was carried out in /3/:

$$\int_0^{\infty} d\kappa^2\, D(x, \kappa^2) = \frac{1}{(2\pi)^3 i} \int d^4k\, e^{ikx}\, \epsilon(k^0)\, \theta(k^2) =$$

$$= -\frac{2i}{\pi^2} \{ (x^2 + i\epsilon x_0)^{-2} - (x^2 - i\epsilon x_0)^{-2} \} = \frac{4}{\pi} \epsilon(x^0)\, \delta'(x^2) .$$

Then

$$\delta(x^0) \frac{\partial}{\partial x^0} \{ \epsilon(x^0)\, \delta'(x^2) \} = -\frac{3}{4} \pi \Box \delta(x) .$$

Thus

$$\delta(x^0) \frac{\partial}{\partial x^0} \int_0^{\infty} d\kappa^2\, D(x, \kappa^2) = -3 \Box\, \delta(x) . \qquad (17)$$

Finally combining (14)-(17) we obtain

$$\delta(x^0-y^0)\langle[j^0(x)\,j^1(y)]\rangle_0 = -\frac{ie^2}{(2\pi)^2}\{2m^2\delta(x^0-y^0)\partial_1\delta(\vec{x}-\vec{y})+\partial_1\Box\,\delta(x-y)\}. \quad (18)$$

Analogous reasonings are valid for each order of the perturbation theory.

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