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ON THE UPPER LIMIT OF THE RADIUS OF ELEMENTARY PARTICLES

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In the present paper we consider some consequences of the analyticity of the formfactor F(t) in the complex t-plane with a cut from $t = 4\mu^2$ to ∞. Concerning the behaviour of the modulus of F(t) on the cut and in the physical region of the scattering channel we make some assumptions which can be checked experimentally. Under these assumptions we shall get upper limits of the meansquared radius $\langle r^2 \rangle$ which is connected with the derivation of the formfactor by the relation

$$\langle r^2 \rangle = 6\, F'(0) = 6 \left. \frac{dF(t)}{dt} \right|_{t=0}$$

We shall prove the two following theorems:

THEOREM 1. Let the integral

$$\int_{4\mu^2}^{\infty} \bigl| \ln |F(t)| \bigr|^{p} \frac{dt}{t\sqrt{t-4\mu^2}} \tag{1}$$

exist for some $p \geqslant 1$. Then

$$\int_{4\mu^2}^{\infty} \ln |F(t)| \frac{dt}{t\sqrt{t-4\mu^2}} \geqslant 0 . \tag{2}$$

If the left-hand side of this relation gets its minimum, i.e.

$$\int_{4\mu^2}^{\infty} \ln |F(t)| \frac{dt}{t\sqrt{t-4\mu^2}} = 0 , \tag{3}$$

then the derivative $F'(0)$ is

$$F'(0) = \frac{1}{4\pi\mu^2} \int_{1}^{\infty} \ln \left| F(4\mu^2 x) \right| \frac{dx}{x^2\sqrt{x-1}} . \tag{4}$$

THEOREM 2. Let the formfactor F(t) be bounded on the cut and have the modulus smaller than unity in the physical region of the scattering channel

$$|F(t)| \leq \begin{cases} M & \text{for } t \geqslant 4\mu^2 \\ 1 & \text{for } t \leqslant 0 . \end{cases} \tag{5}$$

Then the derivative $F'(0)$ satisfies the following inequalities

$$|F'(0)| \leq \begin{cases} \dfrac{\sqrt{M}\,\ln M}{2\mu^2} & \text{for } \ln M \leqslant 2 \\[2ex] \dfrac{e(\ln M)^2}{4\mu^2} & \text{for } \ln M \geqslant 2 . \end{cases} \tag{6}$$

Moreover, if the integral (1) exists for some p ≥ 1 and the condition (3) is satisfied, then

$$|F'(0)| \le \frac{\ln M}{8\mu^2} \qquad (7)$$

<u>Proof of Theorem 1</u>

By mean of the substitutions of the variables

$$z = \frac{t}{4\mu^2}, \qquad \xi = \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}} \qquad (8)$$

we firstly realize the conformal mapping of the complex t-plane with a cut from t= $4\mu^2$ to ∞ into an unit circle, and we denote F(t) = f(ξ). We form the Blaschke product (see [1], chap.V)

$$B(\xi) = \prod_{n=1}^{\infty} \frac{\xi_n^*}{|\xi_n|}\,\frac{\xi_n-\xi}{1-\xi_n^*\xi}, \qquad (9)$$

where ξ_n are the zeros of the function f (ξ), and we put

$$g(\xi) = \frac{f(\xi)}{B(\xi)}. \qquad (10)$$

$g(\xi)$ is an analytic function without zeros in the unit circle and therefore $\ln|g(\xi)|$ is a harmonic function in this circle. Since on the boundary of the unit circle the modulus of $B(\xi)$ is equal to 1

$$|B(e^{i\varphi})| = 1,$$

then the existence of the integral (1) means the existence of the integral

$$\int_0^{2\pi} \left|\ln|f(e^{i\varphi})|\right|^p d\varphi = \int_0^{2\pi} \left|\ln|g(e^{i\varphi})|\right|^p d\varphi$$

for some p ≥ 1. In other words the function $\ln|g(\xi)|$ harmonic in the circle $|\xi| \le 1$ is bounded by the L^p-norm on the boundary of this circle for some p ≥ 1 . It is well-known (see [1], chapter III) that for such a function we can apply the Poisson formula

$$\ln|g(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} \ln|g(e^{i\varphi})|\,\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}\,d\varphi = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(e^{i\varphi})|\,\frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}\,d\varphi. \qquad (11)$$

In particular

$$\ln|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(e^{i\varphi})|\,d\varphi. \qquad (12)$$

On the other hand f(o) = 1 and

$$|B(0)| = \prod_{n=1}^{\infty} |\xi_n| \le 1, \qquad \ln|B(0)| \le 0.$$

B(o) is equal to 1 only if the function f (ξ) has no zeros in the unit circle. Therefore from the relation (12) it follows immediately that

$$\frac{1}{2\pi}\int_0^{2\pi} \ln|f(e^{i\varphi})|\,d\varphi = -\ln|B(0)| \ge 0.$$

Transforming into the old variable t, we get the formula (2).

Suppose that the condition (3) is satisfied. This means that f (ξ) has no zeros, and instead of the formula (11) we have

$$\ln|f(re^{i\theta})| = \frac{1}{2\pi}\int_0^{2\pi} \ln|f(e^{i\varphi})|\,\frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}\,d\varphi.$$

From this relation we get the equation

$$f'(0) = \left.\frac{df(r)}{dr}\right|_{r=0} = \frac{1}{\pi}\int_0^{2\pi} \ln|f(e^{i\varphi})|\cos\varphi\,d\varphi$$

which is completely equivalent to the formula (4). Thus theorem 1 has been proved.

<u>Proof of Theorem 2</u>

We notice firstly that F(t) increases in the complex t plane more slowly than any linear exponential [2]. Therefore owing to the generalized maximum principle from the condition (5) it follows that

$$|F(t)| \le M \qquad \text{for every } t .$$

We introduce a new variable

$$w = 1 + \frac{t}{A},$$

where $A \gg 4\mu^2$, and we put $F(t) = \varphi(w)$. Denote by E_a the ellipse with the foci at the points ± 1 of the w-plane and with the major semi-axis $a = 1 + 4\mu^2/A$. According to the conditions of the theorem $|\varphi(w)| \le 1$ for $w \le 1$ and $|\varphi(w)| \le M$ on the boundary of E_a. By means of the mapping

$$\eta = w + \sqrt{w^2 - 1}$$

we transform the ellipse E_a into a circle with the radius $R = a + \sqrt{a^2 - 1}$.
In this mapping the interval $-1 \le w \le 1$ is transformed into the unit circle, and the ellipse E_a with a real cut from -1 to +1 is transformed into a ring with the exterior and interior radius R and 1, respectively. The function $\psi(\eta) = \varphi(w)$ is analytic in this ring. Applying to $\psi(\eta)$ the Handamard theorem on three circles (see [3], chapter V) we can prove the following inequality

$$|\psi(\eta)| \le e^{\frac{\ln|\eta|}{\ln R}\ln M}. \qquad (13)$$

We choose the number A to be large enough such that

$$R = 1 + \frac{4\mu^2}{A} + \sqrt{\left(2 + \frac{4\mu^2}{A}\right)\frac{4\mu^2}{A}} \approx 1 + \sqrt{\frac{8\mu^2}{A}},$$

$$\ln R \approx \sqrt{\frac{8\mu^2}{A}}.$$

Then

$$\ln|\eta| \approx \frac{\sqrt{2|t|}}{A},$$

and we can rewrite the formula (13) in the form

$$|F(t)| \le e^{\sqrt{\frac{|t|}{4\mu^2}}\ln M}. \qquad (14)$$

Now we apply to F(t) the following lemma proved by Bessis [4].

<u>Lemma.</u> If the function F(z) is analytic in the circle $|z| \le \rho$ and equals to unity at z=0, then it is different from zeros everywhere in the circle $|z| \le \rho_0$ with the radius

$$\rho_0 = \frac{\rho}{\max\limits_{|z| = \rho} |F(z)|}$$

Since the formfactor F(t) is analytic in the circle $|t| \le \rho$ for any $\rho \le 4\mu^2$, then according to the lemma and the inequality (14) it is not equal to zero everywhere

in the circle $|t| \le \rho_0$, where

$$\rho_0 = \rho\, e^{-\sqrt{\frac{\rho}{4\mu^2}}\ln M}, \quad \rho \le 4\mu^2. \qquad (15)$$

It is not difficult to prove that the function in the right-hand side of the formula (15) reaches its maximum at the point

$$\rho = 4\mu^2 \min\left[\frac{4}{(\ln M)^2},\ 1\right]. \qquad (16)$$

Thus F(t) has no zeros in the circle $|t| \le t_0$ with the radius

$$t_0 = \begin{cases} \dfrac{4\mu^2}{M} & \text{if } \ln M \le 2 \\ \dfrac{16\mu^2}{e^2(\ln M)^2} & \text{if } \ln M \geqslant 2, \end{cases} \qquad (17)$$

and the function

$$h(t) = \frac{\ln F(t)}{t}$$

is analytic in this circle. Owing to the maximum principle

$$|F'(0)| = |h(0)| \le \frac{1}{t'} \max_{|t| = t'} |\ln F(t)|$$

for any $t' \le t_0$. Applying the Carathedory theorem (see [3], chapter V) we get

$$|F'(0)| \le \frac{2}{t_0 - t'} \max_{|t| = t_0} \ln|F(t)|.$$

Since t' can be chosen to be arbitrarily small, then from this inequality it follows immediately that

$$|F'(0)| \le \frac{2}{t_0} \max_{|t| = t_0} \ln|F(t)|. \qquad (18)$$

From the relation (14), (17) and (18) we obtain

$$|F'(0)| \le \frac{\ln M}{\mu\sqrt{t_0}} = \begin{cases} \dfrac{\sqrt{M}\ln M}{2\mu^2} & \text{if } \ln M \le 2 \\ \dfrac{e(\ln M)^2}{4\mu^2} & \text{if } \ln M \geqslant 2. \end{cases}$$

This proves the first part of theorem 2.

If the integral (1) exists for some $p \geqslant 1$ and the condition (3) is satisfied, then we have the formula (4). From this formula we get

$$F'(0) \leq \frac{1}{4\pi\mu^2} \ln M \int_1^{\infty} \frac{dx}{x^2\sqrt{x-1}} = \frac{\ln M}{8\mu^2}.$$

Theorem 2 has been proved.

We notice that if we identify F(t) with the formfactor of π-meson then theorems 1 and 2 concern only the physical quantities which can be determined imediately in the experiment. They are the values of the modulus of F(t) at $t \geqslant 4\mu^2$ and $t \leqslant 0$. To determine F(t) in the region $t \geqslant 4\mu^2$ it is sufficient to measure the cross section of the annihilation process

$$e^+ + e^- \longrightarrow \pi^+ + \pi^-,$$

and F(t) at $t \leqslant 0$ can be determined by studying the scattering

$$\pi^\pm + e \longrightarrow \pi^\pm + e$$

The check of the relations (2), (4), (6) and (7) when the corresponding conditions are satisfied would be the experimental check of the assumption about the analyticity of the formfactor - one of the general assumptions of the existing theory of elementary particles.

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