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RELATIVISTICALLY COVARIANT EQUATIONS FOR TWO PARTICLES IN QUANTUM FIELD THEORY

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RELATIVISTICALLY COVARIANT EQUATIONS FOR TWO PARTICLES IN QUANTUM FIELD THEORY

In the present paper we consider the problem a relativistically covariant description of a system of two interacting particles in the framework of quantum field theory. A generally accepted approach in this field bases on the Bethe-Salpeter equation . As is well known the B.S. amplitudes depend on the two space-time points x_1 and x_2 and thus relativistic covariance is achieved due to the introduction of two times. This leads one to some difficulties in clearing up the physical meaning of the B.S. amplitudes (for example, it is not clear how to interprete the relative time, and how to normalize the B.S. amplitude, etc.). The problem we are concerned with in the present paper is to obtain the relativistically covariant equations for two interacting particles (with spin 0 or 1/2) the solutions of which would allow a quantum-mechanical probability interpretation.

It can be shown that the physical quantities such as scattering matrix on the mass shell and energy spectrum of bound states coincide with those ones obtained by means of the B.S. equation . As will be seen below, our method of the solution of this problem is directly connected with the quasipotential approach in quantum field theory developed in papers [2].

I. Wave Equations for Two Scalar Particles

1. Free Particles

Taking two free spinless particles as an example we demonstrate the possibility of a relativistically covariant one-time description of a system of two particles.

It is well known that in quantum field theory the two-particle system is described by the Bethe-Salpeter amplitude

$$\chi_p(x_1, x_2) = \langle 0 | T\left(\varphi_1(x_1)\, \varphi_2(x_2)\right) | p \rangle , \tag{1.1}$$

where $\varphi_{1,2}(x)$ are the Heisenberg fields of two scalar particles of equal masses and $|p\rangle$ is the state vector with a definite value of the four-momentum p.

In the case of absence of interaction the Bethe-Salpeter amplitude (1.1) satisfies the equations

$$\begin{aligned} (\Box_{x_1} - m^2)\, \chi_p(x_1, x_2) &= 0 ; \\ (\Box_{x_2} - m^2)\, \chi_p(x_1, x_2) &= 0 , \end{aligned} \tag{1.2}$$

where $\Box_x = -\partial_x^2 = -\partial_t^2 + \vec{\nabla}^2$

Using translation invariance

$$\chi_p(x_1, x_2) = e^{-iPX} \cdot \chi_p(z) ; \tag{1.3}$$

$$X = \frac{1}{2}(x_1 + x_2) ; \quad z = (x_1 - x_2)$$

and going over to the momentum representation

$$\chi_p(z) = \int dq\, e^{-iqz} \cdot \chi_p(q) , \tag{1.4}$$

we get the following equations

$$\left[\left(\frac{P}{2} + q\right)^2 - m^2\right] \cdot \chi_p(q) = 0 ; \tag{1.5}$$

$$\left[\left(\frac{P}{2} - q\right)^2 - m^2\right] \cdot \chi_p(q) = 0 .$$

Due to the equality of the masses of the two particles, from (1.5) we obtain

$$\left(\frac{1}{4} P^2 + q^2 - m^2\right) \chi_p(q) = 0 , \tag{1.6a}$$

$$(p \cdot q)\, \chi_p(q) = 0 . \tag{1.6b}$$

Hence it follows that the amplitude $\chi_p(q)$ can be represented in the form

$$\chi_p(q) = \delta(n \cdot q)\, \varphi_p(q) ; \quad n_\mu = \frac{p_\mu}{\sqrt{p^2}} , \quad p^2 > 0 ; \tag{1.7}$$

the function $\varphi_p(q)$ being determined only for those values of relative momentum q which are related by the condition $p \cdot q = 0$.

Now we determine the one-time wave function for scalar particles in the centre-of-mass system $(\vec{p} = 0)$:

$$\chi_0(t, \vec{x}_1 ; t, \vec{x}_2) = e^{-iEt} \cdot \chi_0(\vec{x}_1 - \vec{x}_2 ; 0) . \tag{1.8}$$

From eq. (1.4) it is easy to see that the function (1.8) may be expressed in terms of the Fourier transform of the B.S. amplitude which is integrated over the relative energy in the c.m.s.

$$\chi_0(\vec{z}, 0) = \int d\vec{q}\, dq_0\, e^{i\vec{q}\cdot\vec{z}} \cdot \chi_0(\vec{q}, q_0) \tag{1.9}$$

or, using the representation (1.7)

$$\chi_o(\vec{x},o) = \int d\vec{q}\, e^{i\vec{q}\vec{x}} \varphi_o(\vec{q}) . \tag{1.10}$$

From eqs. (1.5) and (1.6) it follows that the wave function in the c.m.s. obeys the equation

$$\left(\frac{1}{4}E^2 - \vec{q}^{\,2} - m^2\right)\varphi_o(\vec{q}) = 0 , \tag{1.11}$$

and has two solutions

$$\varphi_o(\vec{q}) = a_E \cdot \delta(E \mp 2W) ; \qquad W = \sqrt{m^2 + \vec{q}^{\,2}} . \tag{1.12}$$

The solution with positive total energy $E = 2W$ describes the state of two particles "1" and "2" and the solution with negative energy $E = -2W$ can be related to the state of two antiparticles "$\bar{1}$" and "$\bar{2}$" by means of charge conjugation, i.e.

$$\varphi_o^{*}(\vec{q}, E<0) = \varphi_o^{c}(-\vec{q}, -E>0) . \tag{1.13}$$

The normalization and orthogonality conditions of the solution of eq. (1.11) with different values of positive total energy are given by

$$\int \varphi_o^{*}(\vec{q},E')\varphi_o(\vec{q},E)\, d\vec{q} = \delta_{E',E} \quad ; \qquad (E',E>0) . \tag{1.14}$$

Eq. (1.11) is a quasipotential equation describing two free scalar particles in the c.m.s. An important merit of the quasipotential approach is the fact that the two-particle wave function $\varphi_o(\vec{q})$ depends only on the three-dimensional relative momentum $\vec{q}$ and can be normalized, i.e. allows a probability quantum-mechanical interpretation [2].

The relativistic generalization of eq. (1.11) is of the form

$$\left[P^2 - M^2\right] \cdot \varphi_p(q) = 0 , \tag{1.15}$$

under the additional condition

$$p \cdot q = 0 ; \tag{1.16}$$

in this case

$$M = 2\sqrt{m^2 - q^2} \tag{1.17}$$

is the operator of the effective mass of the system of two free scalar particles.

The conditions of normalization and orthogonality of states with different values of the total mass is relativistically generalized in the following way:

$$\int \varphi^*_{p'}(q)\, \varphi_p(q)\, \delta(n\cdot q)\, dq = \delta_{M';M} \quad ; \quad M' = \sqrt{p'^2}\, ; \quad M = \sqrt{p^2}\, . \tag{1.18}$$

2. Interacting Scalar Particles

In the presence of the interaction the B.S. amplitude (1.1) of two scalar particles satisfies the relativistically covariant equation which in the momentum representation is of the form[1]:

$$\left[\left(\frac{p}{2}+q\right)^2 - m^2\right]\left[\left(\frac{p}{2}-q\right)^2 - m^2\right] \chi_p(q) = \int K_p(q,q')\, \chi_p(q')\, dq' . \tag{2.1}$$

The kernel of the equation $K_p(q,q')$ is found, using the perturbation theory, as a sum of all the irreducible diagrams defining the two-particle scattering matrix. The wave function does not satisfy the normalization condition of the type (1.18) and, consequently, do not allow the usual probability interpretation.

In order to conserve the normalization condition of the type (1.18) we consider the possibility of describing the interaction of two particles on the basis of the set of equations (1.6) . We do not change eq. (1.6b) and include the interaction into eq. (1.6a) in the following manner:

$$\left(\frac{1}{4}p^2 + q^2 - m^2\right) \chi_p(q) = \int W_p(q,q')\, \chi_p(q')\, dq' ; \tag{2.2a}$$

$$(p\cdot q)\cdot \chi_p(q) = 0 . \tag{2.2b}$$

For these equations to be compatible it is necessary that the potential should obey the condition $(p\cdot q)\, W_p(q,q') = 0$. From where

$$W_p(q,q') = \delta(n\cdot q)\cdot V_p(q,q') . \tag{2.3}$$

Bearing in mind (2.2b) it is convenient to introduce the function $\varphi_p(q)$:

$$\chi_p(q) = \delta(n\cdot q)\, \varphi_p(q) \; ; \quad n_\mu = \frac{p_\mu}{\sqrt{p^2}} , \tag{2.4}$$

which as can be seen satisfies the equation

$$\left(\frac{1}{4}p^2 + q^2 - m^2\right) \varphi_p(q) = \int V_p(q,q')\, \delta(n\cdot q')\, \varphi_p(q')\, dq' ; \qquad p\cdot q = 0 . \tag{2.5}$$

The wave functions $\varphi_p(q)$ obey the relativistically invariant orthonormalization condition of the following form:

$$\int \varphi^{*}_{p'}(q)\, \varphi_p(q)\, \delta(n\cdot q)\, dq = \delta_{M',M} \; ; \qquad M' = \sqrt{p'^2}\, ; \quad M = \sqrt{p^2} . \tag{2.6}$$

In the c.m.s. $(\vec{p}=0)$ eqs. (2.5) (2.6) have the form:

$$\left(\tfrac{1}{4}E^2 - \vec{q}^{\,2} - m^2\right)\varphi_E(\vec{q}) = \int V_E(\vec{q},\vec{q}\,')\,\varphi_E(\vec{q}\,')\, d\vec{q}\,' ; \tag{2.7}$$

$$\int \varphi^{*}_{E'}(\vec{q})\, \varphi_E(\vec{q})\, d\vec{q} = \delta_{E',E} . \tag{2.8}$$

For a suitable choice of the interaction potential $V_E(\vec{q},\vec{q}\,')$ eq. (2.7) coincides with the quasipotential equation in quantum field theory suggested in papers [2]. In this connection it is appropriate to recall some basic statements of the quasipotential approach

The quasipotential equation was obtained on the basis of the B.S. equation for the Fourier transform of the one-time wave function of two particles in the c.m.s. and has the form:

$$\left(\tfrac{1}{4}E^2 - \vec{q}^{\,2} - m^2\right)\widetilde{\varphi}_E(\vec{q}) = \frac{1}{\sqrt{m^2+\vec{q}^{\,2}}} \cdot \int \widetilde{V}_E(\vec{q},\vec{q}\,')\,\widetilde{\varphi}_E(\vec{q}\,')\, d\vec{q}\,' ;$$

where

$$\widetilde{\varphi}_E(\vec{q}) = \int_{-\infty}^{+\infty} dq_0\, \chi_E(\vec{q}, q_0) . \tag{2.10}$$

In the same papers a method for constructing the quasipotential $\widetilde{V}_E(\vec{q},\vec{q}\,')$ by means of perturbation theory was suggested. In this case the scattering amplitude calculated by means of such a potential coincides on the mass shell with the scattering amplitude obtained on the basis of the B.S. equation. Since it is always possible to choose the relativistically invariant potential (2.3) which coincides in the c.m.s. with the quasipotential:

$$V_E(\vec{q},\vec{q}\,') = \frac{1}{\sqrt{m^2+\vec{q}^{\,2}}} \cdot \widetilde{V}_E(\vec{q},\vec{q}\,') , \tag{2.11}$$

then eq. (2.2) may be considered as a relativistic generalization of the quasipotential equation.

In conclusion of this section we note that while the B.S. equation allows one to determine the Fourier transform of the four-point Green function $G_p(q,q')$ over the whole region of change of the variables p, q and q', the system of equations (2.5) makes it possible to determine the same quantity only on the mass shell.

II. Quasipotential Method for Particles with Spin 1/2

1. Free Particles

The B.S. amplitude of two spin particles having equal masses is determined by the expression:

$$\chi_p(x_1,x_2) = \langle 0 | T\left(\psi_1(x_1)\,\psi_2(x_2)\right) | p \rangle \; ; \tag{1.1}$$

where $\psi_{1,2}(x)$ are the Heisenberg fields of particles with spin 1/2, and $|p\rangle$ is the state with a definite four momentum p. When the interaction is absent the amplitude (1.1) satisfies the system of two equations x) :

$$(i\gamma^{(1)}\cdot\partial_{x_1} - m)\,\chi_p(x_1,x_2) = 0 \; ; \tag{1.2}$$

$$(i\gamma^{(2)}\cdot\partial_{x_2} - m)\,\chi_p(x_1,x_2) = 0 \, .$$

Using translation invariance

$$\chi_p(x_1,x_2) = e^{-iPX}\cdot\chi_p(x) \; ; \tag{1.3}$$

$$X = \frac{1}{2}(x_1+x_2) \; ; \qquad x = (x_1 - x_2) \, ,$$

and going over to the momentum representation

$$\chi_p(x) = \int dq\, e^{-iqx}\cdot\chi_p(q) \, , \tag{1.4}$$

we obtain the following equations for the function $\chi_p(q)$ of two free particles

$$\left[\gamma^{(1)}\cdot\left(\frac{p}{2}+q\right) - m\right]\cdot\chi_p(q) = 0 \; ; \tag{1.5}$$

$$\left[\gamma^{(2)}\cdot\left(\frac{p}{2}-q\right) - m\right]\cdot\chi_p(q) = 0 \, .$$

Owing to the equality of the masses of the particles under consideration the function $\chi_p(q)$ satisfies the condition

$$(p\cdot q)\cdot\chi_p(q) = 0 \, , \tag{1.6}$$

from where it follows that $\chi_p(q)$ may be represented in the form

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x) Particles "1" and "2" of equal masses may differ from one another by the signs of the charges. If, for instance, particle "2" is an antiparticle for matrices $\gamma^{(2)}$ one should use the charge conjugate representation $\gamma^c = -\gamma_T$ where „T" - denotes transposition.

$$\chi_p(q) = \delta(n\cdot q)\cdot\widetilde{\chi}_p(q)\,; \quad n_\mu = \frac{p_\mu}{\sqrt{p^2}}\,; \quad p^2 > 0\,. \tag{1.7}$$

It should be noted that the funotion $\widetilde{\chi}_p(q)$ is determined only for those values of the total and relative momenta which are conneoted by the oondition $pq = 0$.

Now we determine the one-time wave funotion of two spin partioles in the o.m.s.

$$\chi_0(t,\vec{x}_1;t,\vec{x}_2) = e^{-iEt}\cdot\chi_0(\vec{x}_1-\vec{x}_2;0)\,. \tag{1.8}$$

From eq. (1.4) it is not diffioult to see that the function (1.8) is expressed in terms of the Fourier oomponent of the B.S. amplitude whioh is integrated over the relative energy in the o.m.s.

$$\chi_0(\vec{x};0) = \int d\vec{q}\, e^{i\vec{q}\vec{x}}\cdot\int dq_0\, \chi_0(\vec{q},q_0) \tag{1.9}$$

or, using the representation (1.7)

$$\chi_0(\vec{x};0) = \int d\vec{q}\, e^{i\vec{q}\vec{x}}\cdot\widetilde{\chi}_0(\vec{q})\,. \tag{1.10}$$

From eqs. (1.5) it follows that in the c.m.s. the function satisfies the system of equations

$$\begin{aligned}&\left[\gamma_0^{(1)}\cdot\frac{E}{2} - \vec{\gamma}^{(1)}\cdot\vec{q} - m\right]\widetilde{\chi}_0(\vec{q}) = 0\,;\\ &\left[\gamma_0^{(2)}\cdot\frac{E}{2} + \vec{\gamma}^{(2)}\cdot\vec{q} - m\right]\widetilde{\chi}_0(\vec{q}) = 0\,.\end{aligned} \tag{1.11}$$

We perform the Foldy-Wouthuysen transformation [5] on the wave funotion $\widetilde{\chi}_0(\vec{q})$ for the case of two free spin partioles in the c.m.s.

$$\widetilde{\chi}_0(\vec{q}) = T_0(\vec{q})\cdot\widetilde{\Psi}_0(\vec{q})\,; \tag{1.12}$$

where

$$\begin{aligned}&T_0(\vec{q}) = \frac{(m+W-\vec{\gamma}^{(1)}\cdot\vec{q})(m+W+\vec{\gamma}^{(2)}\cdot\vec{q})}{2W(m+W)}\,; \quad W = \sqrt{m^2+\vec{q}^{\,2}}\,;\\ &T_0^{+}(\vec{q})\cdot T_0(\vec{q}) = 1\,.\end{aligned} \tag{1.13}$$

Eqs. (1.11) in the Foldy-Wouthuysen representation takes on the following form:

$$\left[\gamma_0^{(1)}\cdot\frac{E}{2}-W\right]\cdot\tilde{\Psi}_0(\vec{q})=0\,; \qquad (1.14)$$

$$\left[\gamma_0^{(2)}\cdot\frac{E}{2}-W\right]\cdot\tilde{\Psi}_0(\vec{q})=0\,.$$

and have two solutions:

$$E=\pm 2W\,;\quad \gamma_0^{(1)}=\gamma_0^{(2)}=\pm 1\,. \qquad (1.15)$$

The solution with positive energy corresponds to particles "1" and "2" while the solution with negative energy may be connected by means of the operation of charge conjugation with the state of antiparticles "$\bar{1}$" and "$\bar{2}$":

$$C^{(1)}\cdot C^{(2)}\cdot\tilde{\Psi}_0(\vec{q},E<0)=\tilde{\Psi}_0^{c}(-\vec{q},-E>0)\,; \qquad (1.16)$$

where $C=\gamma_0\cdot\gamma_2$ is the charge conjugation matrix. The general solution for the set of equations (1.14) is of the form

$$\tilde{\Psi}_0(\vec{q})=\delta\left(\frac{E^2}{4}-W^2\right)\cdot\Lambda^{(+)}\varphi\,; \qquad (1.17)$$

where φ is an arbitrary 16-component spinor (undor); $\Lambda^{(+)}=\left(\frac{1+\gamma_0^{(1)}\cdot\gamma_0^{(2)}}{2}\right)$ is the projection operator . Note that

$$\Lambda^{(+)}\cdot\varphi=\varphi^{(++)}+\varphi^{(--)}\ , \qquad (1.18)$$

where

$$\varphi^{(\pm\pm)}=\left(\frac{1\pm\gamma_0^{(1)}}{2}\right)\left(\frac{1\pm\gamma_0^{(2)}}{2}\right)\varphi.$$

Eqs. (1.14) describe two free spin particles in the c.m.s. by means of the wave function $\tilde{\Psi}_0(\vec{q})$ depending on the three-dimensional relative momentum. The normalization and orthogonality conditions of the states with different total energy have the form:

$$\int d\vec{q}\,\tilde{\Psi}_0^{+}(\vec{q},E')\,\tilde{\Psi}_0(\vec{q},E)=\delta_{E',E}\qquad (E',E>0)\,. \qquad (1.19\ )$$

The relativistic generalization of eqs. (1.14) are the equations:

$$\left[\gamma^{(1)}p-M\right]\tilde{\Psi}_p(q)=0\,, \qquad (1.20)$$

$$\left[\gamma^{(2)}p-M\right]\tilde{\Psi}_p(q)=0\,.$$

under the additional condition

$$(p\cdot q)=0; \tag{1.21}$$

where $M=2\sqrt{m^2+\vec{q}^{\,2}}$ is the effective mass operator of the system of two free particles. Eqs. (1.20) with the additional condition (1.21) and the arbitrary mass operator M are known as the equations of the Yukawa bilocal theory [4], which earlier were also investigated by M.Markov [3].

The function $\tilde{\Psi}_p(q)$, coinciding in the c.m.s. with the function $\tilde{\Psi}_0(\vec{q})$, is connected with the amplitude $\tilde{\chi}_p(q)$ in the arbitrary system by means of the generalized Foldy- Wouthuysen transformation:

$$\tilde{\chi}_p(q)=T_p(q)\cdot\tilde{\Psi}_p(q), \tag{1.22}$$

where

$$T_p(q)=\frac{(m+W-\gamma^{(1)}\cdot q)(m+W-\gamma^{(2)}\cdot q)}{2W(m+W)}; \qquad W=\sqrt{m^2-q^2} \ . \tag{1.23}$$

Notice that the transformation (1.22) is not unitary but satisfies the condition:

$$T_p(-q)\cdot T_p(q)=1. \tag{1.24}$$

The relativistic generalization of the normalization and orthogonality conditions (1.19) for the states with different total masses is of the form:

$$\int\bar{\tilde{\Psi}}_{p'}(q)\,\tilde{\Psi}_p(q)\,\delta(nq)\,dq=\delta_{M'M} \tag{1.25}$$

where

$$\bar{\tilde{\Psi}}_p(q)=\tilde{\Psi}_p^{+}(q)\cdot\gamma_0^{(1)}\gamma_0^{(2)}. \tag{1.26}$$

It is not difficult to check also, passing to the c.m.s. and using eqs. (1.12) and (1.13) that the normalization and orthogonality conditions may be expressed in terms of the functions $\tilde{\chi}_p(q)$:

$$\int\bar{\tilde{\chi}}_{p'}(q)\,\tilde{\chi}_p(q)\,\delta(nq)\,dq=\delta_{M'M}. \tag{1.27}$$

Thus, we have shown that eqs. (1.14) describing two spin particles in the c.m.s. in the absence of the interaction by means of the function $\tilde{\Psi}_0(\vec{q})$, connected with the one--time Bethe-Salpeter amplitude by the expressions (1.10) and (1.12), allow the relativistically covariant generalization (see e.g. eqs. (1.20) (1.21) (1.25)) .

In this case the relativistic amplitude $\tilde{\Psi}_p(q)$ is connected with the quasipotential function $\tilde{\Psi}_0(\vec{q})$ by the Lorentz transformation L :

$$\tilde{\Psi}_p(q) = \frac{[\gamma^{(1)}\cdot p + M][\gamma^{(2)}\cdot p + M]}{2M(M+p_0)} \cdot \tilde{\Psi}_0(L^{-1}q) \tag{1.28}$$

so that

$$L^{-1}p = \{M, 0\}; \quad L^{-1}q = \{0, \vec{q}\}; \tag{1.29}$$

where $\vec{q}$ is the space relative momentum, determined in the c.m.s. of two particles.

The question arises: Is it possible to describe the system of two interacting spin particles with the aid of the relativistically covariant equations of the type (1.20) with the additional condition (1.21) and a certain mass operator M , which in the c.m.s. would coincide with the equation of the quasipotential method?

The quasipotential equations for two interacting particles with spin 1/2 were investigated in detail in the papers by R.N.Faustov [6] and G.Desimirov and D.Stoyanov [7].

In the next section we present a somewhat modified derivation of the quasipotential equations for two spin particles by means of the generalized Foldy-Wouthuysen transformations.

2. Quasipotential Equations for Spin Particles

In this section we shall start from the equations which define in quantum field theory the 4-time Green function and the two-particle B.S. amplitude and introduce the equations for the two-time Green function and the one-time wave function of two particles.

The four-time Green function is determined by the following expression:

$$G(xy; x'y') = \langle 0| T\left(\Psi_1(x)\Psi_2(y)\bar{\Psi}_1(x')\bar{\Psi}_2(y')\right)|0\rangle , \tag{2.1}$$

where $\Psi_{1,2}(x)$ are the Heisenberg fields of spin particles. As is known, the four-time Green function obeys the B.S. equation:

$$G(xy; x'y') = S_1(x-x')\cdot S_2(y-y') + \tag{2.2}$$
$$+ \int S_1(x-x_1) S_2(y-y_1) K(x_1y_1; x_2y_2) G(x_2y_2; x'y') dx_1 dx_2 dy_1 dy_2 ;$$

where $S_{1,2}(x)$ is the Green function of free particles

$$S_{1,2}(x-x') = \langle 0|T(\psi_{1,2}(x)\bar{\psi}_{1,2}(x'))|0\rangle = \frac{i}{(2\pi)^4}\int \frac{e^{ip(x-x')}}{\gamma^{(1,2)}\cdot p - m + i0}\,dp\,. \tag{2.3}$$

The two-particle B.S. amplitude $\chi_p(xy)$ determined by eq. (1.1) obeys the appropriate homogeneous equation

$$\chi_p(x,y) = \int S_1(x-x_1) S_2(y-y_1) K(x_1 y_1; x_2 y_2)\, \chi_p(x_2 y_2)\, dx_1 dx_2 dy_1 dy_2\,. \tag{2.4}$$

Notice that the kernel of these equations is found by perturbation theory as a sum of irreducible diagrams, determining the two-particle scattering matrix. Further it is convenient to introduce the c.m.s. variables X, x and X', x' :

$$X = \frac{1}{2}(x+y)\,; \qquad X' = \frac{1}{2}(x'+y')\,;$$
$$x = (x-y)\,; \qquad x' = (x'-y')\,. \tag{2.5}$$

Using the translation invariance we determine the Fourier transforms of the quantities entering eq. (2.2) in the following way:

$$G(xy;x'y') = G(X-X';x,x') = \frac{1}{(2\pi)^8}\int G_p(q,q')\, e^{ip(X-X')+iqx-iq'x'}\, dp\,dq\,dq'\,;$$
$$K(xy;x'y') = K(X-X';x,x') = \frac{1}{(2\pi)^8}\int K_p(q,q')\, e^{ip(X-X')+iqx-iq'x'}\, dp\,dq\,dq'\,; \tag{2.6}$$
$$S_1(x-x')\, S_2(y-y') = \frac{1}{(2\pi)^8}\cdot\int dp\,dq\,dq'\, F_p(q,q')\, e^{ip(X-X')+iqx-iq'x'}\,;$$

where

$$F_p(q,q') = -\frac{\delta(q-q')}{[\gamma^{(1)}\cdot(\frac{p}{2}+q)-m+i0][\gamma^{(2)}\cdot(\frac{p}{2}-q)-m+i0]} \tag{2.7}$$

Inserting eqs. (2.6) to eq. (2.2) we get

$$G_p(q,q') = F_p(q,q') + \int F_p(q,q_1)\, K_p(q_1,q_2)\, G_p(q_2,q')\, dq_1 dq_2\,. \tag{2.8}$$

Now we determine the two-time Green function

$$G(t,\vec{x},\vec{y};t',\vec{x}',\vec{y}') = G(xy;x'y')\Big|_{\substack{x_0=y_0=t \\ x_0'=y_0'=t'}} \tag{2.9}$$

Let $t > t'$. Then, using the completeness of the system of stationary states and the definition of the one-time B.S. amplitude (1.8) we get

$$G(t,\vec{x},\vec{y};t',\vec{x}',\vec{y}') = \sum_{np} \chi_{np}(t,\vec{x};t,\vec{y}) \bar{\bar{\chi}}_{np}(t',\vec{x}';t',\vec{y}') ; \qquad (t > t') \tag{2.10}$$

or using eq. (1.3)

$$G(t,\vec{x},\vec{y};t',\vec{x}',\vec{y}') = \sum_{np} \chi_{np}(\vec{x}-\vec{y};0) \bar{\bar{\chi}}_{np}(\vec{x}'-\vec{y}';0) e^{-iP_0(t-t')+i\vec{P}\left(\frac{\vec{x}+\vec{y}}{2} - \frac{\vec{x}'+\vec{y}'}{2}\right)} . \tag{2.11}$$

In eqs. (2.10) and (2.11) we have used the notation $\bar{\bar{\chi}}_{np} = \chi^+_{np} \gamma_0^{(1)} \gamma_0^{(2)}$.
Thus, if we find an equation which is satisfied by the two-time Green function (2.9) then the corresponding homogeneous equation is satisfied by the one-time wave function $\chi_{np}(t,\vec{x};t,\vec{y})$. Let us determine the Fourier transform of the two-time Green function

$$G(t,\vec{x},\vec{y};t,\vec{x}',\vec{y}') = \frac{1}{(2\pi)^8} \int \tilde{G}_p(\vec{q},\vec{q}') e^{iP_0(t-t') - i\vec{P}(\vec{X}-\vec{X}') - i\frac{\vec{q}}{2}(\vec{x}-\vec{y}) + i\frac{\vec{q}'}{2}(\vec{x}'-\vec{y}')} dp \, d\vec{q} \, d\vec{q}' ; \tag{2.12}$$

where $\tilde{G}_p(\vec{q},\vec{q}')$ is connected with the Fourier transform of the four-time Green function $G_p(q,q')$ as follows

$$\tilde{G}_p(\vec{q},\vec{q}') = \int dq_0 \, dq_0' \, G_p(\vec{q},q_0;\vec{q}',q_0') . \tag{2.13}$$

Note that the definition of the two-time Green function (2.9) is, generally speaking, relativistically non-covariant if the frame of reference is not fixed.

Below we construct equations which will be satisfied by the Fourier transform of the two-time Green function in the c.m.s. of two particles:

$$\tilde{G}(\vec{q},\vec{q}') \equiv \tilde{G}_{\vec{p}=0}(\vec{q},\vec{q}') \tag{2.14}$$

We write the B.S. equation (2.8) in a symbolic form

$$G = F + FKG , \tag{2.15}$$

and solve it by the iteration method with respect to G:

$$G = F + FKF + FKFKF + \cdots \tag{2.16}$$

Inserting (2.16) into eq. (2.14) we get the following expansion for the two-time Green function

$$\widetilde{G} = \widetilde{F} + \widetilde{FKF} + \dots \tag{2.17}$$

Here the sign ,$\sim$' denotes the operation of integration over the relative energies q_0, q_0' performed by the formula (2.14) in the s.m.s. $(\vec{p}=0)$.

For example, the free term in the expression (2.17) is of the form

$$\widetilde{F} = \int dq_0 dq_0' F_{\vec{p}=0}(q,q') = -\delta(\vec{q}-\vec{q}') \int_{-\infty}^{+\infty} \frac{dq_0}{\left[\gamma_0^{(1)}(\frac{p_0}{2}+q_0) - \vec{\gamma}^{(1)}\vec{q} - m + i0\right]\left[\gamma_0^{(2)}(\frac{p_0}{2}-q_0) + \vec{\gamma}^{(2)}\vec{q} - m + i0\right]} . \tag{2.18}$$

In what follows it is convenient to go over in eq. (2.17)to the Foldy–Wouthuysen representation:

$$\widetilde{G} \to \widetilde{G}_F = T_0(\vec{q})\, \widetilde{G}(\vec{q},\vec{q}')\, T_0(\vec{q}') ; \tag{2.19}$$

where the unitary operator $T_0(\vec{q})$ is determined by the formula (1.15). The free term (2.18) in the Foldy–Wouthuysen representation is of the form

$$\widetilde{F}_F = -\delta(\vec{q}-\vec{q}') \int \frac{dq_0}{\left[\gamma_0^{(1)}(\frac{E}{2}+q_0) - W + i0\right]\left[\gamma_0^{(2)}(\frac{E}{2}-q_0) - W + i0\right]} \tag{2.20}$$

where $W = \sqrt{m^2 + \vec{q}^{\,2}}$; $E = p_0$.

Calculating the integral in (2.20) we get

$$\widetilde{F}_F = \delta(\vec{q}-\vec{q}') \frac{\pi i}{E} \left\{ \frac{\left[\gamma_0^{(1)}(E-W)+W\right](1+\gamma_0^{(2)})}{E-2W} + \frac{\left[\gamma_0^{(2)}(E+W)+W\right](1-\gamma_0^{(1)})}{E+2W} \right\} . \tag{2.21}$$

In the case of spinless particles the quasipotential is determined by the following expression [2]

$$\left[\widetilde{G}\right]^{-1} = \left[\widetilde{F}\right]^{-1} - \frac{1}{2\pi i} V , \tag{2.22}$$

where the multiplier $-\frac{1}{2\pi i}$ is introduced for the sake of oonvenience, the imaginary part of the determined potential being a negative determined quantity. Further we shall bear in mind that the inverse operator is determined by the following expression

$$\int d\vec{q}'' \left[\widetilde{G}(\vec{q},\vec{q}'')\right]^{-1} \cdot \widetilde{G}(\vec{q}'',\vec{q}') = \delta(\vec{q}-\vec{q}') \tag{2.23}$$

It can be shown, however, that the operator (2.21) has no inverse and the determination of the quasipotential by means of (2.22) is meaningless. The above mentioned trouble is caused by the following. Unlike the case of the scalar particles the Green functions of the spin particles G , F and others are the matrix operators acting in space 16-component spinors φ.

Let us break down all the spinor space φ into two subspaces by means of the projection operators $\Lambda^{(\pm)}$:

$$\Lambda^{(\pm)} = \left(\frac{1 \pm \gamma_0^{(1)}\gamma_0^{(2)}}{2}\right); \qquad \Lambda^{(+)} + \Lambda^{(-)} = 1. \tag{2.24}$$

It is not difficult to see that

$$\Lambda^{(+)} = \left(\frac{1+\gamma_0^{(1)}}{2}\right)\left(\frac{1+\gamma_0^{(2)}}{2}\right) + \left(\frac{1-\gamma_0^{(1)}}{2}\right)\left(\frac{1-\gamma_0^{(2)}}{2}\right); \tag{2.25}$$

$$\Lambda^{(-)} = \left(\frac{1+\gamma_0^{(1)}}{2}\right)\left(\frac{1-\gamma_0^{(2)}}{2}\right) + \left(\frac{1-\gamma_0^{(1)}}{2}\right)\left(\frac{1+\gamma_0^{(2)}}{2}\right). \tag{2.26}$$

Thus, from eqs. (2.25) and (2.26) it follows that either only "upper" or only "lower" components of the spinors $\Lambda^{(+)}\varphi$ differ from zero $(\gamma_0^{(1)} = \gamma_0^{(2)} = \pm 1)$.

Any operator A acting in the space of the spinors may be divided into four components and written in a symbolic matrix form:

$$A = \begin{bmatrix} A^{++} & A^{+-} \\ A^{-+} & A^{--} \end{bmatrix}, \tag{2.27}$$

where the operators A^{++} and A^{--} act only on the subspaces of the spinors $\Lambda^{(+)}\varphi$ and $\Lambda^{(-)}\varphi$, respectively, while the operators A^{+-} and A^{-+} transfer the spinors from one subspace to the other.

The operator $\widetilde{F}_F$ (2.21) can be represented in the form (2.27)

$$\widetilde{F}_F = \begin{bmatrix} \widetilde{F}_F^{++} & 0 \\ 0 & 0 \end{bmatrix}; \tag{2.28}$$

where

$$\widetilde{F}_F^{++} = 2\pi i\, \delta(\vec{q}-\vec{q}\,')\frac{1}{\gamma_0^{(1,2)}E - 2W + i0}. \tag{2.29}$$

and, as is easily seen, has no inverse operator. If we restrict ourselves to the subspace $\Lambda^{(+)}_{\varphi}$ of the spinors the two-time Green function of free particles has an inverse operator equal to

$$\left[\tilde{F}_F^{++}\right]^{-1} = \frac{1}{2\pi i}\delta(\vec{q}-\vec{q}')\left[\gamma_0^{(1,2)}E-2W\right] \tag{2.30}$$

This is related to the fact that in the case of two noninteracting ; spin particles the solutions of eqs. (1.14) belong to the subspace $\Lambda^{(+)}_{\varphi}$. Below we deduce the quasi-potential equations for that part of the one-time wave function of two interacting particles with spin which in the Foldy-Wouthuysen representation belongs to the subspace $\Lambda^{(+)}_{\varphi}$:

$$\tilde{\Psi}_0(\vec{q}) = \Lambda^{(+)}\int dq_0\, \Psi_{\vec{p}=0}(\vec{q},q_0)\,, \tag{2.31}$$

where $\Psi_{\vec{p}=0}(q)$ is connected with the Fourier transform of the one-time B.S. function $\chi_{\vec{p}=0}(q)$ by the Foldy-Wouthuysen unitary transformation (1.13):

$$\chi_{\vec{p}=0}(\vec{q},q_0) = T_0(\vec{q})\,\Psi_{\vec{p}=0}(\vec{q},q_0)\,. \tag{2.32}$$

We use eq. (2.22) for the determination of the quasipotential on the subspace $\Lambda^{(+)}_{\varphi}$:

$$\left[\tilde{G}_F^{++}\right]^{-1} = \left[\tilde{F}_F^{++}\right]^{-1} - \frac{1}{2\pi i}V_F \tag{2.33}$$

The quasipotential can be found starting from the iteration expansion (2.17)

$$\frac{1}{2\pi i}V_F = \left[\tilde{F}_F^{++}\right]^{-1}\cdot\left[\widetilde{FKF}\right]_F^{++}\cdot\left[\tilde{F}_F^{++}\right]^{-1} + \ldots \tag{2.34}$$

Using eqs. (2.2) and (2.33) we get an equation for the Fourier transform of the two-time Green function of spin particles in the Foldy-Wouthuysen representation

$$\left[\gamma_0^{(1,2)}E-2W\right]\tilde{G}_F^{++}(\vec{q},\vec{q}') = \int d\vec{q}''\, V_F(\vec{q},\vec{q}'')\,\tilde{G}_F^{++}(\vec{q}'',\vec{q}') + 2\pi i\,\delta(\vec{q}-\vec{q}') \tag{2.35}$$

The wave function of two particles (2.31) will satisfy the appropriate homogeneous equation

$$\left[\gamma_0^{(1,2)}E-2W\right]\tilde{\Psi}_0(\vec{q}) = \int d\vec{q}'\, V_F(\vec{q},\vec{q}')\,\tilde{\Psi}_0(\vec{q}')\,. \tag{2.36}$$

Eqs. (2.35) and (2.36) are the basic equations of the quasipotential method for spin particles. Determining the effective mass operator

$$M\cdot\tilde{\Psi}_0(\vec{q}) = 2W\tilde{\Psi}_0(\vec{q}) + \int \tilde{V}_F(\vec{q},\vec{q}\,')\tilde{\Psi}_0(\vec{q}\,')d\vec{q}\,' \qquad (2.37)$$

we can write eqs. (2.36) in the following form

$$\left[\gamma_0^{(1,2)}\cdot E - M\right]\tilde{\Psi}_0(\vec{q}) = 0, \qquad (2.38)$$

which generalizes eq. (1.14) for free particles in the presence of the interaotion. The normalization and orthogonality conditions of the states with different values of the total energies E and E' are of the form

$$\int \tilde{\Psi}_0^{+}(\vec{q},E')\,\gamma_0^{(1,2)}\tilde{\Psi}_0(\vec{q},E)\,d\vec{q} = \delta_{E',E}\,. \qquad (2.39)$$

Let us make the two important remarks concerning the mass operator (2.37).

First of all, the quasipotential V_F, determined by the expression (2.33) and, consequently, the mass operator are, generally speaking, the complex functions of the energy E.

The antihermitian part of the potential are characterized by possible inelastic processes in the interaction of two particles and defines the width of the bound state levels.

For the unitarity condition, which implies that the sum of the probabilities of all the possible processes does not exceed unity, to be fulfilled it is necessary that the antihermitian part of the mass operator should be negative definite quantity.

Indeed, remembering that the time-dependent wave function of the bound state with energy E is of the form

$$\tilde{\Psi}_0(\vec{q},t) = e^{-iEt}\cdot\tilde{\Psi}_0(\vec{q},E) \qquad (2.40)$$

we get the following expression for the change of the norm of the state depending on time:

$$\frac{\partial}{\partial t}\int\tilde{\Psi}_0^{+}(\vec{q},t)\,\gamma_0^{(1,2)}\tilde{\Psi}_0(\vec{q},t)\,d\vec{q} = 2\int\tilde{\Psi}_0^{+}(\vec{q},t)\,D(\vec{q},\vec{q}\,')\,\tilde{\Psi}_0(\vec{q}\,',t)\,d\vec{q}\,d\vec{q}\,' < 0\,; \qquad (2.41)$$

where

$$D(\vec{q},\vec{q}\,') = \frac{1}{2i}\left(M - M^{+}\right). \qquad (2.42)$$

The negative definiteness of the antihermitean part of the mass operator (2.42) can be established by studying the analytical properties of the two-time Green function in a way similar to that used in the case of scalar particles [2]. The second remark concerning the mass operator is the following. Unlike the case of free particles the solutions of eqs. (2.38) are not, generally speaking, the eigenfunctions of the operators $\gamma_0^{(1,2)}$ and contain both upper $\varphi^{(++)}$ and lower $\varphi^{(--)}$ components

$$\tilde{\Psi}_0 = \begin{pmatrix} \varphi^{(++)} \\ \varphi^{(--)} \end{pmatrix}; \qquad (2.43)$$

where

$$\varphi^{(\pm\pm)} = \left(\frac{1 \pm \gamma_0^{(1)}}{2}\right)\left(\frac{1 \pm \gamma_0^{(2)}}{2}\right)\tilde{\Psi}_0 . \qquad (2.44)$$

This is a consequence of the fact that the arbitrary mass operator (2.37) does not commute, generally speaking, with the matrices $\gamma_0^{(1,2)}$ and mix the components $\varphi^{(++)}$ and $\varphi^{(--)}$, when acting on the wave function (2.43).

We perform a pseudounitary transformation on the wave function

$$\tilde{\Psi}_0 = U \cdot \Phi_0 ; \qquad (2.45)$$

which conserves the norm (2.39) i.e.

$$U^{+}\gamma_0^{(1,2)}U = \gamma_0^{(1,2)}. \qquad (2.46)$$

Eqs. (2.38) take the form

$$\left[\gamma_0^{(1,2)}E - M'\right]\Phi_0 = 0 ; \qquad (2.47)$$

where

$$M' = U^{+} \cdot M \cdot U .$$

If we require that in the new representation the operator M' be diagonal, i.e.

$$\gamma_0^{(1,2)} \cdot M' = M' \gamma_0^{(1,2)}, \tag{2.48}$$

then the wave function $\Phi_0(\vec{q})$ being the solution of eqs. (2.47) is the eigenfunction of the matrices $\gamma_0^{(1,2)}$:

$$\gamma_0^{(1)} \Phi_0 = \gamma_0^{(2)} \Phi_0 = \pm \Phi_0 . \tag{2.49}$$

Let us call such a representation the "standard" one. Note that the transformation $\mathcal{U}$ which diagonalizes the mass operator (2.48) may not, generally speaking, exist. However, we obtain approximate equations describing the system of two interacting spin particles in the nonrelativistic limit, when $|\vec{q}| \ll m$, the transformation $\mathcal{U}$, which diagonalizes the mass operator, can be constructed with any degree of accuracy by means of expansion in powers of $\frac{|\vec{q}|}{m}$. The wave function in the "standard" representation $\Phi_0(\vec{q})$, corresponding to the solutions when $\gamma_0^{(1)} = \gamma_0^{(2)} = 1$, is normalized in the following manner

$$\int \Phi_0^+(\vec{q}, E') \Phi_0(\vec{q}, E) d\vec{q} = \delta_{E',E} . \tag{2.50}$$

The relativistically covariant generalization of the quasipotential equations in the "standard" representation (2.47) has the form

$$\left[\gamma^{(1,2)} p - M\right] \Phi_p(q) = 0 ; \qquad p \cdot q = 0 . \tag{2.51}$$

The normalization condition (2.50) takes on the relativistically invariant form:

$$\int \bar{\bar{\Phi}}_{p'}(q) \cdot \Phi_p(q) \, \delta(n \cdot q) \, dq = \delta_{M',M} \tag{2.52}$$

3. Instantaneous Local Interaction of Two Particles

Let us consider a simple example when the interaction of two particles with spin 1/2 in the c.m.s. may be considered as local and nonretarded.

In this case the equation for the B.S. amplitude is of the form:

$$(i\gamma^{(1)} \partial_{x_1} - m)(i\gamma^{(2)} \partial_{x_2} - m) \chi_{\vec{p}=0}(x_1, x_2) = i \delta(x_{10} - x_{20}) V(\vec{x}_1 - \vec{x}_2) \chi_{\vec{p}=0}(x_1, x_2) , \tag{3.1}$$

or, passing to the momentum representation

$$\left[\gamma_0^{(1)}\left(\frac{E}{2}+q_0\right)-\vec{\gamma}^{(1)}\vec{q}-m\right]\left[\gamma_0^{(2)}\left(\frac{E}{2}-q_0\right)+\vec{\gamma}^{(2)}\vec{q}-m\right]\chi_{\vec{p}=0}(\vec{q},q_0)=-\frac{1}{2\pi i}\int V(\vec{q}-\vec{q}\,')\,\chi_{\vec{p}=0}(q')\,dq', \tag{3.2}$$

where

$$V(\vec{q})=\frac{1}{(2\pi)^3}\int V(\vec{x})\,e^{-i\vec{q}\vec{x}}\,d\vec{x}\,. \tag{3.3}$$

We go over in eq. (3.2) to the Foldy-Wouthuysen representation (1.12):

$$\chi_{\vec{p}=0}(q)=T_0(\vec{q})\Psi_0(q);$$
$$\left[\gamma_0^{(1)}\left(\frac{E}{2}+q_0\right)-W\right]\left[\gamma_0^{(2)}\left(\frac{E}{2}-q_0\right)-W\right]\Psi_0(q)=-\frac{1}{2\pi i}\int K(\vec{q},\vec{q}\,')\,\Psi_0(q')\,dq', \tag{3.4}$$

where

$$K(\vec{q},\vec{q}\,')=T_0(\vec{q})\,V(\vec{q}-\vec{q}\,')\,T_0(\vec{q}\,'). \tag{3.5}$$

Use now the fact that the right-hand side of eq. (3.4) is independent of the relative energy q_0 and obtain the equation for the function $\tilde{\Psi}_0(\vec{q})$

$$\tilde{\Psi}_0(\vec{q})=\int_{-\infty}^{+\infty}dq_0\,\tilde{\Psi}_0(\vec{q},q_0), \tag{3.6}$$

connected with the one-time wave function of two particles (1.10). Taking into account eqs. (2.21) and (2.29) we get

$$\tilde{\Psi}_0(\vec{q})=\frac{1}{\gamma_0^{(1,2)}E-2W+i0}\cdot\left(\frac{1+\gamma_0^{(1)}\gamma_0^{(2)}}{2}\right)\int K(\vec{q},\vec{q}\,')\,\tilde{\Psi}_0(\vec{q}\,')\,d\vec{q}\,'. \tag{3.7}$$

Thus, if the interaction of two particles is instantaneous (or nonretarding) the wave function $\tilde{\Psi}_0(\vec{q})$ belongs to the subspace $\Lambda^{(+)}\varphi$ of the 16-component spinors and obeys the equation

$$\left[\gamma_0^{(1,2)}E-2W\right]\tilde{\Psi}_0(\vec{q})=\int K^{++}(\vec{q},\vec{q}\,')\,\tilde{\Psi}_0(\vec{q}\,')\,d\vec{q}\,'; \tag{3.8}$$

where

$$K^{++}=\left(\frac{1+\gamma_0^{(1)}\gamma_0^{(2)}}{2}\right)K(\vec{q},\vec{q}\,')\left(\frac{1+\gamma_0^{(1)}\gamma_0^{(2)}}{2}\right). \tag{3.9}$$

The mass operator corresponding to eq. (3.8) is nondiagonal even if the original local interaction in the B.S. equation (3.1) was described by the scalar potential $V(\vec{x})$ not containing the Dirac $\gamma-$ matrix.

To demonstrate it we expand the mass operator of eq. (3.8) in inverse powers of m

with the accuracy not lower than the second power of $\frac{1}{m}$. Using the formula

$$T_0(\vec{q}) = 1 - \frac{1}{2m}(\vec{\gamma}^{(1)} - \vec{\gamma}^{(2)})\cdot\vec{q} - \frac{\vec{q}^2}{4m^2} - \frac{\vec{\gamma}^{(1)}\vec{q}\cdot\vec{\gamma}^{(2)}\vec{q}}{4m^2} + O\left(\frac{1}{m^3}\right), \tag{3.10}$$

and passing to the x-space we get the following approximate quasipotential equation

$$\left[\gamma_0^{(1,2)} E - 2m - \frac{\vec{q}^2}{m} - V(x) + \frac{1}{4m^2}\left\{\vec{q}^2, V(x)\right\}\right]\tilde{\Psi}_0(x) = \tag{3.11}$$

$$= \left[\frac{1}{4m^2}\vec{\gamma}^{(1)}\vec{q}\cdot V(x)\,\vec{\gamma}^{(1)}\vec{q} + \frac{1}{4m^2}\vec{\gamma}^{(2)}\vec{q}\cdot V(x)\,\vec{\gamma}^{(2)}\vec{q} - \frac{1}{4m^2}\left\{\vec{\gamma}^{(1)}\vec{q}, \left\{\vec{\gamma}^{(2)}\vec{q}, V(x)\right\}\right\}\right]\tilde{\Psi}_0(x) ,$$

where

$$\vec{q} = -i\vec{\nabla}_x .$$

It is easily seen that the last term in the right-hand side of eq. (3.11) does not commute with the matrices $\gamma_0^{(1,2)}$ and mix the upper and lower components in the wave function

$$\tilde{\Psi}_0 = \begin{pmatrix} \varphi^{(++)} \\ \varphi^{(--)} \end{pmatrix} .$$

To pass to the "standard" representation we perform the following transformation

$$\tilde{\Psi}_0 = \mathcal{U}\cdot\tilde{\Psi}_0' \;; \qquad \mathcal{U}^+\gamma_0^{(1,2)}\mathcal{U} = \gamma_0^{(1,2)}, \tag{3.12}$$

where

$$\mathcal{U} = \exp\left(\frac{1}{8m^3}\left\{\vec{\gamma}^{(1)}\vec{q}, \left\{\vec{\gamma}^{(2)}\vec{q}, V(x)\right\}\right\}\right)$$

the action of which with a given accuracy reduces to the elimination of the last term in the r.h.s. of eq. (3.11). If the non-relativistic consideration is invalid it is necessary to use an exact expression for the kernel of eq. (3.8):

$$K^{++}(\vec{q},\vec{q}') = \frac{(m+W)^2 - \vec{\gamma}^{(1)}\vec{q}\cdot\vec{\gamma}^{(2)}\vec{q}}{2W(W+m)}V(\vec{q}-\vec{q}')\frac{(m+W')^2 - \vec{\gamma}^{(1)}\vec{q}'\cdot\vec{\gamma}^{(2)}\vec{q}'}{2W'(W'+m)} + \tag{3.13}$$

$$\frac{\vec{\gamma}^{(1)} - \vec{\gamma}^{(2)}}{2W}\vec{q}\cdot V(\vec{q}-\vec{q}')\frac{\vec{\gamma}^{(1)} - \vec{\gamma}^{(2)}}{2W'}\cdot\vec{q}' ,$$

where

$$W = \sqrt{m^2 + \vec{q}^2} \;; \quad W' = \sqrt{m^2 + \vec{q}'^2} .$$

The relativistically covariant generalisation of eq. (3.8) will be of the form

$$[\gamma^{(1,2)} p - 2W]\, \psi_p(q) = \int K_p(q,q')\, \delta(n \cdot q')\, \psi_p(q')\, dq' \; ; \qquad (3.14)$$

$$n_\mu = \frac{p_\mu}{\sqrt{p^2}} \,,$$

under the additional condition $pq = 0$, the kernel of the equation has a relativistically invariant form

$$K_p(q,q') = \frac{(m+W)^2 - \gamma^{(1)} q \cdot \gamma^{(2)} q}{2W(m+W)} \cdot V(q-q') \frac{(m+W')^2 - \gamma^{(1)} q' \cdot \gamma^{(2)} q'}{2W'(m+W')} + \qquad (3.15)$$

$$+ \frac{\gamma^{(1)} q - \gamma^{(2)} q}{2W} \cdot V(q-q') \cdot \frac{\gamma^{(1)} q' - \gamma^{(2)} q'}{2W'} \,,$$

where

$$W = \sqrt{m^2 - q^2} \;; \quad W' = \sqrt{m^2 - q'^2} \,. \qquad (3.16)$$

The potential $V(q-q')$ in eq. (3.15) is a relativistic generalisation of the Fourier transform of the space potential (3.3). If the space potential $V(\vec{x})$ is spherically symmetric, i.e.

$$V(\vec{q} - \vec{q}\,') = V[(\vec{q} - \vec{q}\,')^2] \,: \qquad (3.17)$$

the relativistically invariant potential is of the form

$$V(q - q') = V[-(q-q')^2] \,. \qquad (3.18)$$

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