J-43

ЯФ, 1969, т.9, в.2, с.458-461

ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2 - 3470

J.Jersák

ON WAVE FUNCTIONS OF UNSTABLE PARTICLE

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1967.

E2 - 3470

J.Jersák

ON WAVE FUNCTIONS OF UNSTABLE PARTICLE

To be submitted to JETP

1. The description of properties of unstable particles in quantum theory meets with the difficulty that these properties are not defined exactly. This fact makes certain arbitrariness possible in choosing the wave functions describing the unstable particle, though their basic characteristics are well known[1,2]. The requirements such as an approximate localization or an approximate exponentiality of the decay law are fulfilled by many functions and in this situation some additional restrictions are needed. We show that a simple condition (as far as we know nowhere used explicitely) that in the course of decay of some unstable particle only either unstable particle itself or outgoing decay products may be observed leads in the case of the non-exponential decay law to some relation between the number of wave functions describing unstable particle and its decay laws.

As has been shown in[2,3] a strong non-exponentiality of the decay law is conceivable for unstable states corresponding to higher order poles in the S-matrix. Another reason of the non-exponentiality is the lower energy bound: if we express a state $|U>$ of an unstable particle U in the total Hamiltonian representation[4]

$$| U > = \int_{E_0}^{\infty} dE \quad c(E) | \quad E > , \tag{1}$$

then in consequence of the lower energy bound E_0 the decay law $L(t) = |a(t)|^2$,

$$a(t) = < U | e^{-iHt} | U > = \int_{E_0}^{\infty} dE \; |c(E)|^2 \; e^{-iEt} \tag{2}$$

necessarily must differ from an exponential in order that the inequality

$$\int_{-\infty}^{+\infty} \frac{|\ln L(t)|}{1+t^2}\, dt < \infty \tag{3}$$

may be satisfied/5-7/. Since these departures from an exponential are small for long-living states, they are usually neglected (by taking $E_0 = -\infty$ in (2)).

In Sec. 4 we show that unstable particles connected with higher-order poles must be described by number $N > 1$ of wave functions and that the number N determines polynomial form of its decay laws. In Sec. 5 we briefly discuss the case of the non-exponentiality caused by lower bound E_0. We show that in this case $N = \infty$.

2. Let us assume that there exists a state $|U>$ corresponding to some unstable particle U in the moment of its production or detection. The decay of the state $|U>$ may be written in the form

$$e^{-iHt}|U> = a(t)|U> + |\Psi_t>, \tag{4}$$

where a(t) is a non-decay amplitude and

$$|\Psi_t> = \sum_n \sigma_n(t)|n>,$$

$$<U|\Psi_t> = 0, \quad |\Psi_0> = 0.$$

If the decay of the state $|U>$ is non-exponential, Eq./4/ implies the following inequality

$$<U|e^{-iHt'}|\Psi_t> = a(t+t') - a(t)a(t') \neq 0. \tag{5}$$

The nonzero matrix element (5) means that some states $|n>$ "regenerate" partially the state $|U>$ in the course of the decay. Just this "regeneration" makes the non-exponentiality possible, for its contribution to the probability $L(t) = |a(t)|^2$ violates the phenomenological idea that the decay probability of an unstable particle is independent of its age.

Among the states $|n>$ there must be states corresponding to free products of the decay. They are physical states described by outgoing waves, and thus cannot contribute to the "regeneration" of the localized state $|U>$ as expressed by (5). Assuming that only particle U or its decay products can be observed we must admit, among states $|n>$, the occurence of other states corresponding to the macroscopically observed unstable particle U. Hence, if particle U decays non-exponentially then there must exist the whole subspace $\mathcal{H}_U$ of dimension $N > 1$ of states describing the particle U.

Choosing some orthonormal system $|U^a>$, complete in $\mathcal{H}_U$, we then may write all decay laws of U in the general form

$$e^{-iHt}|U^a> = \sum_{\beta=1}^{N} a_{a\beta}(t)|U^\beta> + |\phi_t^a>, \quad a = 1,\dots,N$$

$$|\phi_t^u> = \sum_n{}' \sigma_n^u(t)|n>, \quad a_{a\beta}(0) = \delta_{a\beta}, \tag{6}$$

here Σ_n' means summation over states of the decay products, only. This, however, implies that

$$<U^\gamma|e^{-iHt'}|\phi_t^a> = 0 \tag{7}$$

must hold for arbitrary $t, t' > 0$. These relationships represent conditions the states in $\mathcal{H}_U$ must satisfy. With the use of Eq. (6) we may rewrite them in the form

$$a_{a\gamma}(t+t') - \sum_{\beta=1}^{N} a_{a\beta}(t)\, a_{\beta\gamma}(t') = 0. \tag{8}$$

On differentiating the above conditions with respect to t or t' and assuming, for the present time, $N < \infty$ we obtain in the matrix form:

$$\frac{d}{dt} A(t) = -iM^+ A(t), \quad t > 0, \tag{9}$$

where

$$m^+_{\alpha\beta} = i \lim_{t\to 0+} \frac{d}{dt} a_{\alpha\beta}(t) .$$

There exists at least one single-row matrix D^+ obeying the following equation

$$D^+ M^+ = m^+ D^+ . \tag{10}$$

Multiplication by the matrix D^+ of eq. (9) yields

$$\frac{d}{dt} \sum_{\alpha=1}^{N} d^+_\alpha a_{\alpha\beta}(t) = - i m^+ \sum_{\alpha=1}^{N} d^+_\alpha a_{\alpha\beta}(t) \tag{11}$$

that is

$$\sum_\alpha d^+_\alpha a_{\alpha\beta}(t) = d^+_\beta e^{-im^+t}, \quad t > 0 .$$

Therefrom it follows that the decay of a state

$$| U^+_{ex} > = \sum_{\alpha=1}^{N} d^+_\alpha | U^\alpha >$$

is described by the equation

$$e^{-iHt} | U^+_{ex} > = e^{-im^+t} | U^+_{ex} > + | \phi_t > , \; t > 0 \tag{12}$$

$$| \phi_t > \perp \mathcal{H}_U .$$

This equation implies that in the space $\mathcal{H}_U$ there exists at least one state which gives an exponential decay of particle U and conserves a constant direction in $\mathcal{H}_U$ during the decay.

3. If we apply the above considerations to the non-exponentiality due to the higher order pole (neglecting the effect of the lower energy bound), we find that to a pole of the order $r > 1$ in the S-matrix there corresponds a space $\mathcal{H}_U$ of unstable states with the dimension $N > 1$.

(Kerler and Petzold drawn an analogous conclusion in Ref.[8]). We may expect that $N \geq r$. Namely, if we write some orthonormal system $|U^\alpha>$ in the space $\mathcal{H}_U$ in the form (1), where

$$c_\alpha(E) = \frac{1}{(E - E_p)^r} \sum_{k=0}^{\infty} \rho^\alpha_k (E - E_p)^k, \quad \alpha = 1, \ldots, N$$

then the occurence of a state of the type (12) in $\mathcal{H}_U$ requires the equation

$$\sum_{\alpha=1}^{N} d^+_\alpha \rho^\alpha_k = 0 \qquad \text{for} \quad k = 0, 1, \ldots, r-2, \tag{13}$$

to be satisfied in order that the state $|U^+_{ex}> = \sum_\alpha d^+_\alpha |U^\alpha>$ has a pole of the first order only. In general, the coefficients d^+_α satisfying Eq.(13) may be found only for $N \geq r$. (States $|U^\alpha>$ are restricted by (8) which is stronger condition than (12)).

If $N > 1$, then there exists a continuous variety of decay laws depending on the way of measuring and on the initial conditions under which the unstable particle has been generated. These decay laws are determined by the matrix $A(t)$. We shall examine its form for the case $N = r = 2$, which has been investigated on models in Refs.[9,10]:

Existence of the state

$$|U^+_{ex}> = \frac{1}{\sqrt{1 + |d^+|^2}} [|U^1> + d^+ |U^2>]$$

obeying Eq. (12) leads to relations

$$\begin{aligned} a_{11}(t) + d^+ a_{21}(t) &= e^{-im^+t} \\ a_{12}(t) + d^+ a_{22}(t) &= d^+ e^{-im^+t} \end{aligned} \qquad t > 0 . \tag{14}$$

However, Eq. (6) holds formally for any $t \in (-\infty, +\infty)$, and then

$$a_{\alpha\beta}(t) = a^*_{\beta\alpha}(-t) \tag{15}$$

so that for $t < 0$ we obtain the equation analogous to (9)

$$\frac{d}{dt} A(t) = - i M^{-} A(t), \quad t < 0 \tag{16}$$

$$m^{-}_{\alpha\beta} = (m^{+}_{\beta\alpha})^{*} \tag{17}$$

Eq. (16) implies that such a state

$$| U^{-}_{ex} > = \frac{1}{\sqrt{1 + |d^{-}|^2}} [| U^1 > + d^{-} | U^2 >]$$

must exist that

$$e^{-iHt} | U^{-}_{ex} > = e^{-im^{-}t} | U^{-}_{ex} > + | \phi_t >, \quad t < 0 \tag{18}$$

Whence we obtain

$$\begin{aligned} a_{11}(t) + d^{-} a_{21}(t) &= e^{-im^{-}t} \\ & \qquad t < 0 \\ a_{12}(t) + d^{-} a_{22}(t) &= d^{-} e^{-im^{-}t}, \end{aligned} \tag{19}$$

where in consequence of (17)

$$m^{+} = (m^{-})^{*} = \mu - i\gamma . \tag{20}$$

Equations (14) and (19) simplify discussion of the condition (7) because they allow, together with (15), to write the whole matrix $A(t)$ in terms of one amplitude $a_{\alpha\beta}(t)$ only. Writing

$$a_{11}(t) = e^{-i\mu t} \cdot e^{-\gamma |t|} a(t), \tag{21}$$

where

$$a^{*}(-t) = a(t), \quad a(0) = 1 \tag{22}$$

we obtain

$$t > 0 ; \quad A(t) = e^{-i\mu t} \cdot e^{-\gamma t} \begin{bmatrix} a(t) & \frac{1}{(d^{-})^{*}} [1 - a(t)] \\ \frac{1}{d^{+}} [1 - a(t)] & 1 - \frac{1}{d^{+}(d^{-})^{*}} [1 - a(t)] \end{bmatrix} \tag{23}$$

and the condition (7) leads to the equation

$$(1 + \alpha) a(t) a(t') - \alpha [a(t) + a(t') - 1] = a(t + t'); \quad \alpha = \frac{1}{d^{+}(d^{-})^{*}} . \tag{24}$$

On differentiating it with respect to t' we obtain

$$\frac{d}{dt} a(t) = a(t)(1 + \alpha)\rho - \alpha\rho \quad ; \quad \rho = \lim_{t \to 0+} \frac{d}{dt} a(t).$$

For $\alpha \neq -1$ we have

$$a(t) = \frac{1}{1 + \alpha} [e^{(1+\alpha)\rho t} + \alpha] . \tag{25}$$

In this case (as well in the $\rho = 0$ one) all $a_{\alpha\beta}$ are superpositions of pure exponentials, so this case corresponds to a mixing of two single-pole particles.

Only in the case $\alpha = -1$ we obtain

$$a(t) = \rho t + 1, \quad t \geq 0$$

which leads to the decay law to be expected in occurence of a double pole in the S-matrix /2/. So the condition (7) restricts possible decay laws in the case $N = r = 2$ to the form

$$A(t) = e^{-i\mu t} \cdot e^{-\gamma t} \begin{bmatrix} 1 + \rho t & d^{+} \rho t \\ - \frac{1}{d^{+}} \rho t & 1 - \rho t \end{bmatrix}, \quad t > 0 . \tag{26}$$

Choosing $\rho = \gamma = \frac{\Gamma}{2}$ and $d^{+} = -i$ we obtain results of dynamical models[9].

We see that for to describe a particle connected with the third-order pole when the decay law of the form

$$e^{-i\mu t} \cdot e^{-\gamma t}(1+\alpha t+\beta t^{2})$$

is to be expected[2,3] we must have $N > 2$.

4. However, taking into account the lower energy bound, we see that the condition (12) cannot be fulfilled, since for any arbitrary state in $\mathcal{H}_U$ Eq.(3) must be satisfied. By requiring Eq.(7) to be satisfied, the considerations of Sec. 2 leading to (12) cannot be justified. This fact implies, however, that the dimension N of the space $\mathcal{H}_U$ is infinite (which may, e.g., hinder interchanging of the operations of derivation and summation in Eq. (9)). Finally, because of the lower energy bound the exact space of states describing an unstable particle has infinite dimension with the decay laws obeying Eq. (8).

The opinion that unstable particles observed macroscopically are associated with more states was mentioned in a number of papers[1,11], where it is a natural consequence of the fact that no unique definition of properties of unstable particle is given. Such a situation would lead to the violation of the symmetrization principle by unstable particles[12]. Though the experimental situation in this region is not clear[13], one would nevertheless prefer to have such a description of unstable particles in which the symmetrization postulate is preserved. But because the non-exponentiality of the decay law requires an internal degree of freedom, such a description seems in the present formalism impossible.

We are indebeted to Prof. V.Votruba for his kind interest and to Prof. J.A.Smorodinsky and Dr. J.Stern for discussions.

R e f e r e n c e s

1. J.McEwan. Phys.Rev., 132, 2353 (1963) (and references contained in this paper).

2. M.L.Goldberger, K.M.Watson. Phys.Rev., 136, B, 1473 (1964).
3. J.Lukierski. Preprint, Wroclaw, March 1967.
4. N.S.Krylov, V.A.Fok. J.Exptl.Theoret.Phys. (USSR), 17, 93 (1947).
5. R.Paley, N.Wiener. Fourier Transforms in the Complex Domain, New York 1934. Theorem XII.
6. L.A.Chalfin. J.Exptl.Theoret.Phys. (USSR), 33, 1371 (1957).
7. M.Lévy. Nuovo Cimento, 14, 612 (1959).
8. W.Kerler, J.Petzold. Zs. Physik, 186, 168 (1965).
9. J.S.Bell, C.J.Goebel. Phys.Rev., 138, B, 1198 (1965).
10. K.E.Lassila, V.Ruuskanen. Phys.Rev.Let., 17, 490 (1966).
11. G.Hőhler. Zs.Physik, 152, 546 (1958).
12. L.A.Chalfin. DAN (USSR), 165, 541 (1965).
13. A.M.L.Messiah, O.W.Greenberg. Phys.Rev., 136, B, 148 (1964).

Received by Publishing Department
August 1, 1967.