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A.V.Efremov

DOUBLE SPECTRAL REPRESENTATION, DYNAMICS AND SUM RULES

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The so-called sum rules have recently attracted a great attention of many physicist.

Historically, they were written long before the current algebra had appeared /1/ but at that time they were not considered so important. It was the current algebra /2/ that gave a strong push to this approach and originated a lot of different sum rules. But current algebra is not the only source of these sum rules. Many of them can be obtained on the basis of the dispersion relation (d.r.) superconvergency assumption. Let us try to make the comparison between these two sources of sum rules. The first one is based upon the vanishing of two current commutators under space-like separation of their arguments and upon the definite form of singularity when their arguments coincide. Besides that an additional assumption is necessary about the validity of unsubtracted d.r. for the Fourier transforms of the retarded commutator of these currents. The second method is characterized by more severe condition on the asymptotic behaviour of the scattering amplitudes of the type

$$E f(E) \rightarrow \begin{cases} 0 \\ C \end{cases} \qquad \text{when} \quad E \rightarrow \infty$$

which lead to the relation

$$\int \mathrm{Im}\, f(E)dE = \left\{ \begin{matrix} 0 \\ C \end{matrix} \right. .$$

But, inspite of its generality , the second approach has two main defects. The first is the lack of common and reliable criteria, which of the asymptotical behaviour may be imposed on different invariant amplitudes of the process in question. (In the frame of the first method all of them are considered as vanishing). The other defect is that the second method says nothing to us about the character of the constant C. Some additional assumptions are necessary here, while the current algebra connects it with the form factors. Of course, all these advantages are the consequences of more wider information underlying current algebra. In the present work we try to overcome, to some extent, the above-mentioned defects of the dispersion relation approach using validity of a double spectral representation for any of the invariant amplitudes of a process with a virtual photon (or with a lepton pair described by the conserved vector current), gauge invariance and some assumption about the "character of potential", the sence of which will be clear what it follows /6/. We are going to outline here only the main idea of the method by the example of photoproduction of a hypothetical scalar meson and a W-boson (lepton pair) on a scalar meson. We hope to consider more realistic cases elsewhere.

2. Let us consider the process shown in Fig.1.

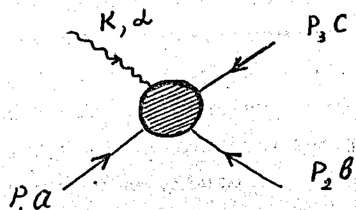

Fig.1

The amplitude T_μ of the process has the following decomposition into the invariant functions of the variable $s = (p_1 + k)^2$ $u = (p_2 + k)^2$ and $t = (p_3 + k)^2$

$$T_\mu = P_\mu A_1 + k_\mu A_2 + \Delta_\mu A_3 \quad ,$$

where $P_\mu = (p_1 - p_2)_\mu$ and $\Delta_\mu = (p_1 + p_2)_\mu$ (Here and further we shall omit the isotopic indices when they are not necessary). Then we suppose the validity of the double spectral representation for the function A_1

$$A_1(s,t,u) = \frac{1}{\pi^2}\iint d\sigma\, d\sigma' \left(\frac{\rho_1^{(12)}(\sigma\sigma')}{(\sigma - s)(\sigma' - u)} + \frac{\rho_1^{(13)}(\sigma,\sigma')}{(\sigma - s)(\sigma - t)} + \frac{\rho_1^{(23)}(\sigma,\sigma')}{(\sigma - u)(\sigma' - t)}\right) +$$

$$+ \frac{1}{\pi}\int d\sigma \left(\frac{r_1^{(1)}(\sigma)}{\sigma - s} + \frac{r_1^{(2)}(\sigma)}{\sigma - u} + \frac{r_1^{(3)}(\sigma)}{\sigma - t}\right). \qquad (I)$$

It is convenient for us to make such a picking out of the parts having the discontinuity in one variable only because the expression (I) being rewritten in the form of usual d.r. with a fixed t looks as

$$A_1(s,t\,u) = \frac{1}{\pi}\int d\sigma \frac{a_1^{(I)}(\sigma,t)}{\sigma - s} + \frac{1}{\pi}\int d\sigma \frac{a^{(II)}(\sigma,t)}{\sigma - u} + \frac{1}{\pi}\int \frac{r_1^{(3)}(\sigma)}{\sigma - t} d\sigma .$$

Usually, this d.r. is written in terms of $\nu = kP = \frac{s-u}{2}$

$$A_1(\nu,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a_1(\nu',t)}{\nu' - \nu} d\nu' + \phi_1(t) \qquad (2)$$

So, instead of the usual nonsubtracted d.r. we obtain d.r. with a constant contribution. Notice, that in the spectral representation (2) the boundaries of spectral regions are nonessential for us if only they give a correct physical thresholds in d.r. (2).

Gauge invariance, which is the result of current conservation, implies on T_μ the requirement:

$$k_\mu T_\mu = \nu A_1(\nu,t) + k^2 A_2(\nu,t) + (k\Delta) A_3(\nu,t) = 0. \qquad (3)$$

Besides that, ϕ_1 having only t-discontinuity the imaginary parts a_1 must satisfy the relation

$$\nu a_1(\nu,t) + k^2 a_2(\nu,t) + (k\Delta) a_3(\nu,t) = 0. \qquad (4)$$

The substitution (2) into (3) and the use of the relation (4) gives us immediately the sum rule

$$\frac{1}{\pi}\int_{-\infty}^{\infty} d\nu' \, a_1(\nu',t) = k_\mu \phi_\mu(t), \tag{5}$$

where

$$\phi_\mu = P_\mu \phi_1(t) + \Delta_\mu \phi_3(t) + k_\mu \phi_2(t) \quad .$$

Now we give some comments about the convergency of integral in (5). The validity of the double spectral representation (1) results in the convergency of $\int \frac{a_1(\nu',t)}{\nu'-\nu} d\nu'$. This means that each of a_1 goes to zero when $\nu \to \infty$ faster than $1/\ln \nu$. But gauge invariance condition for the imaginary parts (4) shows that a_1 is to decrease, in fact, faster than $1/\ln \nu$. So, the integral in the l.h.s. of (5) is meaningful.

Let us go now to the interpretation of ϕ_μ and comparison of (5) with the sum rule from commutator

$$[j_0^a(x), j^c(y)]\,\delta(x_0 - y_0) = i(F^a)_{cd}\, j^d(x)\,\delta^{(4)}(x-y), \tag{6}$$

where $(F^a)_{cd}$ are the isotopic group generators in the representation to which belongs our scalar particles, and their sources $j^d(x)$. Topologically ϕ_μ is a weakly connected diagrams of the type of Fig.2, that is

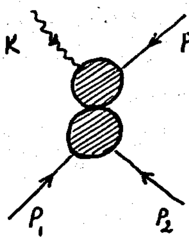

Fig.2

diagrams which consist of two threepoint blocks of different tensor nature depending on the character of the vertex which connects them. But the comparison with the sum rule following from (6) shows that this vertex is to be a scalar one, so that

$$k_\mu \phi_\mu(t) = (t - m^2)\,\tilde{\tilde{\Gamma}}(t)\,\tilde{\tilde{G}}(t), \tag{7}$$

where $\tilde{\tilde{\Gamma}}$ and $\tilde{\tilde{G}}$ are the invariant functions of the threepoint vertices with one vector and two scalar tails and three scalar tails correspondingly. In order that r.h.s. of (7) should not vanish at $t = m^2$ the product must have a pole at this point. This means topologically that ϕ_μ is to be the sum of diagrams of Fig.3, which leads

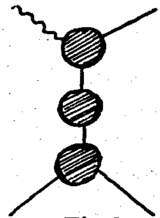

Fig.3

after the renormalization to the expression of the form

$$k_\mu \phi_\mu = k_\mu \Gamma_\mu (k, p_3) \Delta_R(t) G(t),$$

where $\Delta_R(t)$ is the propagator of the scalar meson. But due to the Ward-Takahashi identity for $k^2 = 0$

$$k_\mu \Gamma_\mu (k, p_3) \Delta_R(t) = e .$$

Thus, we obtain in this case exactly the current algebra sum rule. If $k^2 \neq 0$ there is no full compensation and we have the expression

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\nu' a_1(\nu', t) = (e + k^2 \gamma(t)) G(t). \tag{8}$$

This means that when $k^2 \neq 0$ the unsubtracted d.r. for the Fourier--transform of the retarded commutator is not valid. This conclusion coincides with the one of the work /5/. By the way, if the particle with momentum p_3 took part only in weak or electromagnetic interactions, the neglect of higher orders in those small constants would give us only one pole diagram of the type of Fig.3 with a bare propagator and a bare vector vertex. As a result we would obtain the current algebra sum rule even for the case $k^2 \neq 0$.

3. Now let us go over to the process of the type of the W-boson photoproduction on a scalar meson, which has two vector tails. This

amplitude is decomposed as follows:

$$T_{\mu\rho} = P_\mu P_\rho A + P_\mu k_\rho B_1 + P_\mu \Delta_\rho B_2 + P_\rho k_\mu B_3 + P_\rho \Delta_\mu B_4 +$$

$$+ k_\mu k_\rho C_1 + k_\mu \Delta_\rho C_2 + k_\rho \Delta_\mu C_3 + \Delta_\mu \Delta_\rho C_4 + \delta_{\mu\rho} C_5 .$$

As in the previous section we shall assumed for each of the invariant functions a double spectral representation of the form (1) which gives us after reformulation the usual d.r. of the type

$$T_{\mu\rho} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{t_{\mu\rho}(\nu',t)}{\nu'-\nu} d\nu' + \phi_{\mu\rho}(t) .$$

Gauge invariance with respect to the photon tails, results

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\nu' \frac{\nu a_1(\nu',t) + (k\Delta) b_4(\nu',t) + k^2 b_3(\nu',t)}{\nu'-\nu} = -\xi_1(t)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\nu' \frac{\nu b_1(\nu',t) + (k\Delta) C_3(\nu't) + k^2 C_1(\nu't) + C_5(\nu',t)}{\nu'-\nu} = -\xi_2(t) \qquad (9)$$

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\nu' \frac{\nu b_2(\nu',t) + (k\Delta) C_4(\nu',t) + k^2 C_2(\nu',t)}{\nu'-\nu} = -\xi_3(t) ,$$

where the functions ξ_1 are determined by the equality

$$k_\mu \phi_{\mu\rho} = \xi_1 \cdot P_\rho + \xi_2 k_\rho + \xi_3 \Delta_\rho . \qquad (10)$$

Now because of the fact that the r.h.s. of (9) is independent of ν the following equalities for the imaginary parts are valid

$$\begin{aligned} &\nu a(\nu,t) + k^2 b_3(\nu,t) + (k\Delta) b_4(\nu,t) = 0 \\ &\nu b_1(\nu,t) + k^2 C_1(\nu,t) + (k\Delta) C_3(\nu,t) + C_5(\nu,t) = 0 \\ &\nu b_2(\nu,t) + k^2 C_2(\nu,t) + (k\Delta) C_4(\nu,t) = 0 . \end{aligned} \qquad (11)$$

that gives us, together with (9), the sum rules

$$\frac{1}{\pi}\int_{-\infty}^{\infty} d\nu'\, a(\nu',t) = \xi_1(t)$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} d\nu'\, b_1(\nu,t) = \xi_2(t) \qquad (12)$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} d\nu'\, b_2(\nu,t) = \xi_3(t) \quad .$$

To reproduce the results of current algebra /2/ let us turn again to the topological interpretation of $\phi_{\mu\rho}$. In the same way as in the previous section we can easily conclude that $\phi_{\mu\rho}$ are the contributions of the weakly connected diagrams of the type of Fig.2, but in this case the particle of momentum p_3 is a vector one. Following the same way we conclude that these diagrams are to be of a pole type. In the lowest order in weak or electromagnetic interaction (the only interactions which involve the W-boson) there remains only the diagram of Fig.4.

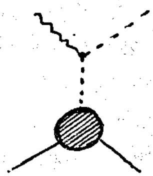

Fig.4

which goes for $\phi_{\mu\nu}$ in the following expression

$$\phi_{\mu\rho} = \frac{P_\rho (2p_3+k)_\mu}{t-m_W^2} F(t). \qquad (13)$$

The contraction with k_μ leads us immediately to:

$$\xi_1(t) = F(t) \qquad \xi_2 = \xi_3 = 0$$

that together with (12) reproduces the current algebra sum rule without Schwinger terms.

Thus, we conclude that in some cases the results of the current algebra are the consequences of gauge invariance, double spectral representation and assumption about the dynamics of interactions: from all the possible graphs having only t -dependence we pick out the pole graph of the type of Fig.4. By the way, these graphs correspond to the Fourier transform of the potential in quantum mechanical d.r. Unifortunately, we have not succeeded in showing such an equivalence for the commutator $[j_0^+(x), j_0^-(y)]\delta(x_0 - y_0) = 2j_0(x)\delta^4(x-y)$ and it is not clear for us what is the way to do this. The matter is that it is difficult to connect this commutator with any physical process because the l.h.s. of it can be considered only as "elastic scattering" of a lepton pair while the r.h.s. can not be described in terms of this process, due to the lack of weak neutral currents. As to the remaining commutators there is no wonder that they give the same sum rules as gauge invariance because one of the consequences of it is the necessity for any charged particle to interact with an electromagnetic field. This results /4/ in the graphs of Fig.4.

We have considered in this paper only the currents, but almost all the arguments are easily extended to the case of axial currents. The PCAC-hypothesis will play the role of gauge invariance. But in this case there remains the difficulty connected with the commutator $[A_0^+(x), A_0^-(y)]\delta(x_0 - y_0)$ and, consequently, with Adler-Weisberger sum rule.

In conclusion we want to make a remark about the Schwinger term. We have the impression that the absence or the presence of it is completely defined by the interaction mechanism. So, if we assumed a somewhat different interaction than that which have led to expression (13) then we would have $\xi_2 \neq 0$, $\xi_3 \neq 0$ that would be equivalent to the presence of the Schwinger terms in the current commutator.

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R e f e r e n c e s

1. See for instance

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2. For the most modern method of the sum rule deduction see S.Fubini. Nuovo Cim., A43, 475 (1966) and S.Fubini, G.Segre. Nuovo Cim., A45, 641 (1966).
3. See for instance Л.Д.Соловьев. ЯФ, 3, 188 (1966); V.A.Matveev, B.V.Struminsky, A.N.Tavkhelidze, Phys.Lett., 23, 146 (1966). R.V. Faustov, V.P.Pisarenko, R.E.Kallosh. Preprint E-2865, Dubna, 1966. И.Г.Азнаурян, Л.Д.Соловьев. ЯФ, 4, 615, (1966); С.Б.Герасимов. Препринт ОИЯИ Р2-2974, 1966.
4. See appendix of the work /5/ where the corrected deduction of Ward-Takahashi identity is given.
5. K.Bardakci, M.B.Halpern, G.Segré. Preprint "Ward Identities, Unsubtracted Dispersion Relations and Feynman Graphs", 1966.
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