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FREE LOCAL FIELDS
WITH INFINITE MULTIPLETS

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1. Introduction

For a relativistic generalization of $SU(6)$ Budini and Fronsdal [1] and Michel [2] proposed the symmetry group $G = PS$ being the semidirect product of the Poincaré group P and the internal symmetry group S which contains the subgroup $SL(2,C)$. In such a theory with infinite multiplets the symmetry properties are consistent with the unitarity of the S-matrix [3-5].

In the papers of Feldman and Matthews [6], Nguyen van Hieu [5] and Fronsdal [7] different attempts have been made to introduce quantized fields describing these infinite multiplets. For the description of each (infinite-dimensional) multiplet the authors of the paper [6] proposed to use a field operator transforming according to a unitary irreducible representation of an auxialiary group isomorphic to S. Such an operator has indices running over infinite values and a particle corresponds to a component of this operator. For such a field irrespectively of the particle spin only one wave equation can be written - the Klein-Gordon equation. Thus all particles independently of the values of spin must satisfy the Bosé-Einstein statistics. A different method was proposed in the papers [5,7]. According to that method an infinite number of spinor fields transforming according to non-unitary finite-dimensional representations of the auxiliary group and satisfying the Bargmann-Wigner equation (component or projective physical fields) are to be introduced for the description of each infinite multlplet. In the framework of this scheme the conventional relation between spin and statistics holds. Further it was shown in [5] (using general considerations) that the amplitudes of crossing processes are related among themselves by the Low substitution rule. In other words, the Low substitution rule and symmetry properties are compatible[x]. This statement have been proved for special cases by Dao vong Duc and Nguyen van Hieu[8] and Fronsdal[9][xx]. It has been shown also that the Bargmann-Wigner equation is invariant with respect to the group G.

One may naturally ask: is it possible to introduce a unique field (big field) in the framework of the second method? Will then this field and its component physical fields be local simultaneously? Further will the field energy derived from the Lagrangian be positive definite and invariant with respect to the internal symmetry group? In this paper we shall give positive answers to all these questions. We follow the method proposed by one of us in the paper[5]. For the sake of simplicity we consider the symmetry group $SL(2,C)$. We note that the big field has also been considered in the paper[7]. However, for particles with half-integer spin, the big field introduced in that paper and its component field cannot be local simultaneously without symmetry breaking.

2. Basis for the Reduction $SL(2,C) \supset SU(2)_p$

As it is well known for the classification of elementary particles the basis of representations corresponding to the reduction $SL(2,C) \supset SU(2)$ must be constructed. For particles in motion we have to use the basis corresponding to the reduction $SL(2,C) \supset SU(2)_p$. The explicit form of the bases for both reductions was given in the papers[8,11]. The first one can be constructed from generalized spinors of $SU(2)$: $\Phi^{\tilde{b}_1 \dots \tilde{b}_{j-\nu}}_{\tilde{a}_1 \dots \tilde{a}_{j+\nu}}(z)$ (the sign $\sim$ means that corresponding indices

[x] In the paper[5] the term "crossing symmetry" is used. It turned out, however,[7,8] that the scattering amplitude has not the usual analyticity properties. Therefore, as it has been pointed out in the papers[8,9] the Low substitution rule should be used instead of the term "crossing symmetry".

[xx] We have not seen the paper[9], we know about its content due to another paper[10].

are connected with the frame, in which particles are at rest), the second is formed from spinors of the form $\Phi^{\dot b_1 \dots \dot b_{j-\nu}}_{a_1 \dots a_{j+\nu}}(p;z)$ transforming according to corresponding spinor representations of the homogeneous Lorentz group. With the help of momenta $\left(-\frac{ip}{m}\right)^{c}_{\dot b}$ the dotted indices of the spinor $\Phi^{\dot b_1 \dots \dot b_{j-\nu}}_{a_1 \dots a_{j+\nu}}(p;z)$ can be transformed into undotted indices. Further all upper indices can be lowered using the antisymmetric spinor ε_{ab}. We obtain thus symmetric spinors, denoted for the sake of symplicity by

$$\Phi_{\tilde a_{2j}}(z) \equiv \Phi_{\tilde a_1 \dots \tilde a_{2j}}(z) \ , \ \Phi_{a_{2j}}(p;z) \equiv \Phi_{a_1 \dots a_{2j}}(p;z) \ .$$

Besides the spinor $\Phi_{a_{2j}}(p;z)$ by multiplying the spinor $\Phi_{a_{2j}}(p;z)$ by a necessary number of momenta $\left(-\frac{ip}{m}\right)^{a}_{\dot c}$ it is possible to introduce another set of $2^{2j}-1$ spinors with all possible number of lower dotted indices. Combining all these spinors we get spinor $\Phi_{\alpha_{2j}}(p;z)$, α_i running now over four values $1,2,\dot 1,\dot 2$. Instead of the basis $\Phi_{\tilde a_{2j}}(z)$ we can use the basis $\xi_{jj_3}(z)$ which is a linear combination of $\Phi_{\tilde a_{2j}}(z)$ (j having a fixed value). A simple relation can be established between the two bases $\xi_{jj_3}(z)$ and $\Phi_{\alpha_{2j}}(p;z)$

$$\Phi_{a_{2j}}(p;z) = \sum_{kk_3} \Delta_{\alpha_{2j}}(p;kk_3)\xi_{kk_3}(z) \quad .$$

The matrix $\Delta_{\alpha_{2j}}(p;kk_3)$, up to some transformations, is the finite Lorentz transformation matrix, transfering particles from the rest-state into the state with 4-momenta p /11/.

We introduce, besides the basis $\Phi_{\alpha_{2j}}(p;z)$, another one, namely $\Phi_{\alpha_{2j}}(-p;z)$, derived from $\Phi_{\alpha_{2j}}(p;z)$ by substituting $p \to -p$. This will ensure the "crossing symmetry" of the theory (by "crossing symmetry" the possibility of the application of the Low substitution rule is to be understood). In our paper we use only self-conjugate representations $\tau \sim (\nu, 0)$.

From the orthogonality condition of the basis vectors $\xi_{jj_3}(z)$ the relation:

$$\Delta_{\alpha_{2j}}(\pm p;kk_3) = \int \Phi_{\alpha_{2j}}(\pm p;z)\xi^{+}_{kk_3}(z)d\mu(z) \qquad (1)$$

can be easily derived, where $d\mu(z)$ is the invariant measure on the group [11]. We prove now the following important formula

$$\sum_j \Delta_{\alpha_{2j}}(\pm p; kk_3)\overline{\Delta}^{\alpha_{2j}}(\pm p; ii_3) = \delta_{ik}\delta_{i_3k_3} \quad , \tag{2}$$

where $\overline{\Delta}^{\alpha_{2j}}(\pm p; ii_3) = \left[\Delta_{\beta_1\cdots\beta_{2j}}(\pm p; ii_3)\right]^{+}(\gamma_4)^{\alpha_1}_{\beta_1}\cdots(\gamma_4)^{\alpha_{2j}}_{\beta_{2j}} \; ; \; \gamma_4 = \begin{pmatrix} 0 & \sigma_4 \\ \sigma_4 & 0 \end{pmatrix}$.

First we consider the case with the sign + before p. Using (1) we have

$$\sum_j \Delta_{\alpha_{2j}}(p; kk_3)\overline{\Delta}^{\alpha_{2j}}(p; ii_3) = \iint \sum_j \Phi_{\alpha_{2j}}(p; z)\overline{\Phi}^{\alpha_{2j}}(p; w)\xi^{+}_{kk_3}(z)\xi_{ii_3}(w)d\mu(z)d\mu(w). \tag{3}$$

As spinors $\Phi_{\alpha_{2j}}(p; z)$ form an orthonormalized system of vectors in the Hilbert space, we have the following covariant relation

$$\sum_j \Phi_{\alpha_{2j}}(p; z)\overline{\Phi}^{\alpha_{2j}}(p; w) = \delta_\mu(z-w) \quad , \tag{4}$$

where the generalized function $\delta_\mu(z-w)$ is defined by the following formula

$$\int \delta_\mu(z-w) f(w)\, d\mu(w) = f(z) \quad .$$

Inserting (4) into (3) we get (2) immediately.

b) Let us take now the case with the sign - before p. We introduce the transformation

$$T \; : \; z_a \to \varepsilon_{ab} z^{*b}, \; z^{*a} \to z_b \varepsilon^{ba} \quad ; \quad \varepsilon_{ab} = \varepsilon^{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

changing the sign of the time component and leaving unchanged the signs of the spatial components of the vector $x_\mu = z_r(\sigma_\mu)^r_{\dot s} z^{*\dot s}$. With the help of the following relation, which can be easily verified

$$\Phi_{a_1\dots a_{2j}}(-p;z) = \varepsilon_{a_1 a_1'}\cdots\varepsilon_{a_{2j}a_{2j}'}\left[(-)^{2j}\Phi_{a_1'\dots a_{2j}'}(p_0,-\vec{p};Tz)\right]^{+}$$

we get

$$\sum \Delta_{\alpha_{2j}}(-p;kk_3)\overline{\Delta}^{\alpha_{2j}}(-p;ii_3) = \iint \sum_j \Phi_{\alpha_{2j}}(p_0,-\vec{p};w)\overline{\Phi}^{\alpha_{2j}}(p_0,-\vec{p};z)\xi^{+}_{ii_3}(w)\xi_{kk_3}(z)d\mu(Tz)d\mu(Tw)\ .$$

As the measure $d\mu(z)$ is also invariant with respect to the T-transformation we can repeat the proof of the preceding case and formula (2) can be proved for both cases. The following relations (their proof is omitted here) are also necessary in what follows

$$\Delta_{\alpha_1\dots\alpha_i\dots\alpha_{2j}}(\pm p;kk_3)\frac{1}{2}\left(1\mp\frac{i\hat{p}}{m}\right)^{\alpha_i}_{\beta_i} = \Delta_{\alpha_1\dots\beta_i\dots\alpha_{2j}}(\pm p;kk_3) \tag{5}$$

$$\Delta_{\alpha_{2i}}(\pm p;kk_3)\overline{\Delta}^{\beta_{2j}}(\pm p;kk_3) = \delta_{ij}\left(1\mp\frac{i\hat{p}}{m}\right)^{\beta_1}_{\alpha_1}\cdots\left(1\mp\frac{i\hat{p}}{m}\right)^{\beta_{2j}}_{\alpha_{2j}}\ . \tag{6}$$

3. Locality of the Big Field

Let now Ψ be an element of the Hilbert space realizing the self-conjugate unitary representation $\tau\sim(\nu,0)$. We consider Ψ as a unique field containing an infinite multiplet. The big field $\Psi = \Psi_+ + \Psi_-$ can be decomposed into components with the help of bases $\Phi_{\alpha_{2j}}(p;z)$ and $\Phi_{\alpha_{2j}}(-p;z)$ in the following way

$$\Psi(p;z) = \sum_{jkk_3}\Psi^{\alpha_{2j}}(p)\Delta_{\alpha_{2j}}(p;kk_3)\xi_{kk_3}(z) + \Psi^{\alpha_{2j}}(-p)\Delta_{\alpha_{2j}}(-p;kk_3)\xi_{kk_3}(z)\ . \tag{7}$$

The spinors $\Psi^{\alpha_{2j}}(p), \Psi^{\alpha_{2j}}(-p)$ satisfying the Bargmann-Wigner equations

$$\left(1+\frac{i\hat{p}}{m}\right)^{\beta_i}_{\alpha_i}\Psi^{\alpha_1\ldots\alpha_i\ldots\alpha_{2j}}(p)=0 \quad , \quad \left(1-\frac{i\hat{p}}{m}\right)^{\beta_i}_{\alpha_i}\Psi^{\alpha_1\ldots\alpha_i\ldots\alpha_{2j}}(-p)=0$$

can be in turn decomposed into spin states (summation over j_3 is understood)

$$\Psi^{\alpha_{2j}}(p)=u^{\alpha_{2j}}(p;j_3)\,a(\vec{p};jj_3)\quad , \quad \Psi^{\alpha_{2j}}(-p)=v^{\alpha_{2j}}(-p;j_3)\,b^+(\vec{p};jj_3)\;,$$

where a and b^+ are particle annihilation and antiparticle creation operators, respectively.

We write now the big field (7) in x-representation

$$\begin{aligned}\Psi^{\Gamma}(x;z) = \int\sum_{jkk_3}\Big\{ & u^{\alpha_{2j}}(p;j_3)\,a(\vec{p};jj_3)\,\Delta_{\alpha_{2j}}(p;kk_3)\,\xi_{kk_3}(z)\,e^{-ipx}+ \\ & + v^{\alpha_{2j}}(-p;j_3)\,b^+(\vec{p};jj_3)\,\Delta_{\alpha_{2j}}(-p;kk_3)\,\xi_{kk_3}(z)\,e^{ipx}\Big\}\,d\mu(\vec{p})\;,\end{aligned} \tag{8}$$

where $d\mu(\vec{p})$ is the usual invariant measure on the mass shell. The Hermitian conjugate expression of the big field is

$$\begin{aligned}\Psi^{+}(y;w) = \int\sum_{jkk_3}\Big\{ & \bar{u}_{\alpha_{2j}}(p;j_3)\,a^+(\vec{p};jj_3)\,\bar{\Delta}^{\alpha_{2j}}(p;kk_3)\,\xi^+_{kk_3}(w)\,e^{ipy}+ \\ & + \bar{v}_{\alpha_{2j}}(-p;j_3)\,b(\vec{p};jj_3)\,\bar{\Delta}^{\alpha_{2j}}(-p;kk_3)\,\xi^+_{kk_3}(w)\,e^{-ipy}\Big\}\,d\mu(\vec{p})\,.\end{aligned} \tag{9}$$

Assuming that operators a and b satisfy the usual commutation relations

$$\left[a(\vec{p};jj_3), a^+(\vec{p}';j'j_3')\right]_\pm = \delta_{jj'}\delta_{j_3j_3'}\delta(\vec{p}-\vec{p}') \quad ,$$

$$\left[b(\vec{p};jj_3), b^+(\vec{p}';j'j_3')\right]_\pm = \delta_{jj'}\delta_{j_3j_3'}\delta(\vec{p}-\vec{p}') \quad , \qquad (10)$$

and using (5), (12) and the following generalized formulae for summation over the spin index

$$u^{\alpha_{2j}}(p;j_3)\overline{u}_{\beta_{2j}}(p;j_3) = \frac{1}{2^{2j}}\left(1-\frac{i\hat{p}}{m}\right)^{\alpha_1}_{\beta_1}\cdots\left(1-\frac{i\hat{p}}{m}\right)^{\alpha_{2j}}_{\beta_{2j}} \qquad (11)$$

$$v^{\alpha_{2j}}(p;j_3)\overline{v}_{\beta_{2j}}(p;j_3) = (-)^{2j}\frac{1}{2^{2j}}\left(1+\frac{i\hat{p}}{m}\right)^{\alpha_1}_{\beta_1}\cdots\left(1+\frac{i\hat{p}}{m}\right)^{\alpha_{2j}}_{\beta_{2j}} \qquad (12)$$

we get

$$\left[\Psi(x;z),\Psi^+(y;w)\right]_\pm = \sum_{kk_3}\xi_{kk_3}(z)\xi^+_{kk_3}(w)\left\{\int\left[e^{-ip(x-y)} \pm (-)^{2y} e^{ip(x-y)}\right]d\mu(\vec{p})\right\} . \qquad (13)$$

Thus the causality condition requires the big field to be quantized according to the Bosé statistics in the case of a boson infinite multiplet and according to Fermi-statistics in the case of a fermion infinite multiplet. Obviously for component physical field with definite values of spin, using (10), (11) and (12) we get the conventional relation between spin and statistics. Using (4) we can rewrite expression (13) in a more compact form

$$\left[\Psi(x;z),\Psi^+(y,w)\right]_\pm = \delta_\mu(z-w)\Delta(x-y) .$$

This commutation relation is invariant with respect to the symmetry group G.

Thus for the free particles both big field and component fields can be constructed to be local. The situation is quite different for interacting fields. If we suppose that the big interacting field $\Psi^{\Gamma}(x;z)$ is local, i.e. assume the relation

$$\left[\Psi(x;z),\Psi^{+}(y;w)\right]_{\pm} = 0 \qquad \text{for} \qquad (x-y)^2 > 0$$

which is consistent with symmetry requirements, then for the component interacting fields the locality will be violated:

$$\left[\Psi^{\Gamma\alpha_{2j}}(x),\overline{\Psi}_{\beta_{2j}}(y)\right]_{\pm} \neq 0 \qquad \text{for} \qquad (x-y)^2 > 0 \, .$$

A detailed consideration of this problem will be given in a subsequent paper.

4. Lagrangian Formalism and "Crossing Symmetry"

The Lagrangian for the whole infinite multiplet is

$$\mathcal{L} = \sum_j \left\{ -\tfrac{1}{2}\overline{\Psi}_{\alpha_1\ldots\rho_i\ldots\alpha_{2j}}\left(\gamma_\mu\right)^{\rho_i}_{\sigma_i}\frac{\partial\Psi^{\alpha_1\ldots\sigma_i\ldots\alpha_{2j}}}{\partial x_\mu} + \right. \tag{15}$$

$$\left. + \tfrac{1}{2}\frac{\partial\overline{\Psi}_{\alpha_1\ldots\rho_i\ldots\alpha_{2j}}}{\partial x_\mu}\left(\gamma_\mu\right)^{\rho_i}_{\sigma_i}\Psi^{\alpha_1\ldots\sigma_i\ldots\alpha_{2j}} - m\,\overline{\Psi}_{\alpha_1\ldots\alpha_{2j}}\Psi^{\alpha_1\ldots\alpha_{2j}} \right\} .$$

When performing variation the component fields with different spins are considered to be independent. Carrying out standard calculations we get from (15) the following expression for the energy-momentum tensor

$$T_{\mu\nu} = \sum_j \left\{ \overline{\Psi}_{\alpha_1\ldots\rho_i\ldots\alpha_{2j}}\left(\gamma_\nu\right)^{\rho_i}_{\sigma_i}\frac{\partial\Psi^{\alpha_1\ldots\sigma_i\ldots\alpha_{2j}}}{\partial x_\mu} - \frac{\partial\overline{\Psi}_{\alpha_1\ldots\rho_i\ldots\alpha_{2j}}}{\partial x_\mu}\left(\gamma_\nu\right)^{\rho_i}_{\sigma_i}\Psi^{\alpha_1\ldots\sigma_i\ldots\alpha_{2j}} \right\} .$$

After the quantization we get for the energy of the infinite multiplet the formula

$$\mathcal{E} = \int :T_{00}: d\vec{x} = \int d\vec{p}\sqrt{\vec{p}^{\,2}+m^2}\sum_j \left[a^+(\vec{p};jj_3)a(\vec{p};jj_3)+b^+(\vec{p};jj_3)b(\vec{p};jj_3)\right]$$

$$= \int d\vec{p}\sqrt{\vec{p}^{\,2}+m^2}\int :\Psi_+^+(p;z)\Psi_+(p;z)+\Psi_-^+(p;z)\Psi_-(p;z): d\mu(z) \,.$$

The energy is thus positive definite and is invariant with respect to the internal symmetry group.

We consider now the interaction of an infinite multiplet with a singlet field. The operator matrix element is of the form

$$F(s,t,u)\int \Psi^+(p_2;z)\Psi(p_1;z)\,d\mu(z) \quad , \tag{16}$$

where $F(s,t,u)$ is a form-factor. In the expression (16) the field operators of the singlet field are omitted. Inserting (8),(9) into (16) we get the following matrix elements for the scattering and annihilation channels:

$$F(s,t,u)\sum_{jkk_3}\overline{u}_{\alpha_2 j}(p_2)\overline{\Delta}^{\alpha_2 j}(p_2;kk_3)\Delta_{\beta_2 j}(p_1;kk_3)u^{\beta_2 j}(p_1) \tag{17}$$

$$F(t,s,u)\sum_{jkk_3}\overline{v}_{\alpha_2 j}(-p_2)\overline{\Delta}^{\alpha_2 j}(-p_2;kk_3)\Delta_{\beta_2 j}(p_1;kk_3)u^{\beta_2 j}(p_1) \,. \tag{18}$$

Expressions (17) and (18) demonstrate in the most general manner that the Low substitution rule holds for this case. However, there are no yet theoretical arguments for analytical continuation of (17) to (18). For illustrating let us consider the annihilation process $\frac{1}{2}+\frac{1}{2} \longrightarrow 0+0$. First we calculate explicitly the matrix element. Using (18), (1) we get

$$
\begin{aligned}
&F(t,s,u)\bar{v}_\alpha(-p_2)u^\beta(p_1)\int \bar{\Phi}^\alpha(-p_2;z)\Phi_\beta(p_1;z)\,d\mu(z) \\
&= F(t,s,u)\bar{v}_\alpha(-p_2)u^\alpha(p_1)\frac{1}{2(1-a)\sqrt{a^2-1}}\left[\sqrt{-a+\sqrt{a^2-1}}-\sqrt{-a-\sqrt{a^2-1}}\,\right],
\end{aligned}
\tag{19}
$$

where $a = -\dfrac{p_1 p_2}{m^2}$.

The expression (19) can really be derived from the matrix element of the scattering process (see (28) in /8/) using the Low substitution rule $t \to s$.

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