ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2 - 3347

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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1967.

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Submitted to "Communications in Mathematical Physics"

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1. Introduction

In the Wightman axiomatical approach a quantum field is defined by a continuous representation of a $*$-algebra R of test-functions in an algebra of (unbounded) operators in a Hilbert space with the same invariant domain and a cyclic vector (vacuum). By the Gelfand-Segal-Theorem a continuous cyclic representation of a $*$-algebra is given by a continuous linear positive functional $W(a)$ on R i.e. a continuous linear functional for which $W(a^*a) \geq 0$, $a \in R$, holds. For a quantum field the positive functional W satisfies certain further conditions, i.e. Lorentz invariance, spectrality and locality. Such a functional is called Wightman-functional /1,2/.

The mathematical structure of such a $*$-algebra is described in the following section.

In this paper it is proved that there exists a set F of positive functionals on the real algebra R which all are bounded by one continuios norm on (the linear space) R so that for every $b \in R$, $b \neq 0$, there exists a $W \in F$ with $W(b) \neq 0$ and consequently, the algebra R is reduced /3/.

It is not proved that there exist "sufficient many" Wightman-functionals among which are some with non trivial quantum field, i.e. the S-matrix is not the identity, but one may hope that the proved result is a step to the solution of this problem.

II. The *-algebra R

Let M be a topological space (M is the Minkowski space or the mass shell, for example) and $M^{(n)} = M \times \ldots \times M$ the Cartesian product of n exemplares of M. $C(M^{(n)})$ is the normed linear space of the continuous complex-valued bounded functions $a_n(x_1, \ldots, x_n)$ $x_i \in M$, on $M^{(n)}$ with the norm $\| a_n \|_0 = \sup_{x_1 \ldots x_n \in M} | a_n(x_1, \ldots, x_n)|$. Let R_0 be the complex field C and for $n = 1, 2, \ldots$ R_n a locally real convex linear topological space (over the real field) of continuous complex-valued bounded functions on $M^{(n)}$ with a stronger topology than is determined by the norm $\| \ \|_0$, i.e. $\| a_n \|_0$, $a_n \in R_{n1}$ is a continuous function on the topological space R_n.

Furthermore we assume that for $a_n(x_1, \ldots, x_n) \in R_n$, $b_m(x_1, \ldots, x_m) \in R_m$ $\bar{a}_n(x_n, \ldots, x_1)$ is an element of R_n and $c_{n+m}(x_1, \ldots, x_{n+m}) = a_n(x_1, \ldots, x_n) b_m(x_{n+1}, \ldots, x_{n+m})$ is an element of R_{n+m} and that the so defined mappings from R_n into R_n resp. from $R_n \times R_m$ into R_{n+m} are continuous.

The algebra R if the linear space

$$R = \bigoplus_{n=0}^{\infty} R_n \qquad \text{(topological direct sum}^{/4/}) \qquad (1)$$

Consequently, every element $a \in R$ has the form $a = \sum_{n \geq 0} a_n, a_n = a_n(x_1, \ldots, x_n) \in R_n$ and only for a finite number of indices is a_n different from zero. a_n is called the homogeneous component of the degree n of a. The multiplication for two elements $a, b \in R$ is defined by

$$(a b)_n (x_1, \ldots x_n) = \sum_{\substack{k+\ell=n \\ k,\ell \geq 0}} a_k(x_1, \ldots, x_k) b_\ell (x_{k+1}, \ldots, x_n) \qquad (2)$$

(the product on the right-hand side is the usual product of functions) and the *-operation is defined by

$$(a^*)_n (x_1, \ldots, x_n) = \bar{a}_n(x_n, \ldots, x_1) \qquad (3)$$

(the bar on the right-hand side labels the complex conjugate function).

In [1,2] is $R_n = D(M^{(n)})$ resp. $S(M^{(n)})$ the well-known Schwartz' spaces of test-functions, but other spaces are regarded in the quantum field theory, too [5]. Here M is the Minkowski space. Let K_0 be the algebraical convex cover of the set of elements $a^* a$, $a \in R$. Each element $k \in K_0$ has the form

$$k = \sum_{i=1}^{N} a^{(i)} * a^{(i)}, \quad a^{(i)} \in R$$

$$a^{(i)} = \sum_{n \geq 0} a_n^{(i)}(x_1, \ldots, x_n) \tag{4}$$

K_0 is a cone, i.e.

a) for $k, k' \in K_0$ and two arbitrary positive numbers s, t is $sk + tk' \in K_0$

and

b) if $k \in K_0$, $k \neq 0$, then $-k \notin K_0$.

The statement a) follows direct from the definition of K_0 and the statement b) holds, because for each $g \in K_0$, $g \neq 0$;

i) the homogeneous component g_r of g with the smallest degree, which does not vanish identically, has an even degree, i.e., $g_r = g_{2s}$

ii) $g_r = g_{2s}(x_1, \ldots, x_{2s})$ is nonnegative on the set
$\Gamma_{2s} = \{ x = (x_1, \ldots, x_{2s});\ x_1 = x_{2s}, x_2 = x_{2s-1}, \ldots, x_n = x_{n+1} \}$ (5)

iii) for at least one $\overset{0}{x} \in \Gamma_{2s}$ we have $g_{2s}(\overset{0}{x}_1, \ldots, \overset{0}{x}_{2s}) > 0$.

Now we define for $k \in K_0$

$$\rho_n = \sup_{x_1, \ldots, x_n \in M} \left(|a_n^{(1)}|^2 + \ldots + |a_n^{(N)}|^2 \right)^{1/2}, \quad n = 0, 1, 2, \ldots \tag{6}$$ x)

Lemma 1

For an arbitrary $k \in K_0$ the following relations hold (k_n is the homogeneous component of the degree n of k):

x/ The application of this expression has been proposed by T.Görnitz, Karl-Marx-Universität, Leipzig.

$$\| \sum_i a_p^{(1)*} a_q^{(1)} \|_0 \leq \ell_p \ell_q \tag{7}$$

$$\| \sum_i a_n^{(1)*} a_n^{(1)} \|_0 = \ell_n^2 \tag{8}$$

$$\| k_n \|_0 \leq \sum_{\nu=0}^{n} \ell_{n-\nu} \ell_\nu \tag{9}$$

$$\ell_n^2 - 2 \sum_{\nu=1}^{n} \ell_{n+\nu} \ell_{n-\nu} \leq \| k_{2n} \|_0 , \; n = 0,1,2,.. \tag{10}$$

$$(\ell_{-1} = 0) .$$

<u>P r o o f:</u>

(7) follows immediately from the Cauchy-Schwarz inequality by the definitions (2) and (6). Further we have

$$\| \sum_i a_n^{(1)*} a_n^{(1)} \|_0 = \sup_{x_1,..,x_{2n}} | \sum \bar{a}_n^{(1)} (x_n, .., x_1) a_n^{(1)} (x_{n+1}, ..., x_{2n}) | \geq$$

$$\geq \sup_{x_1,...,x_n} | \sum_i \bar{a}_n^{(1)} (x_1, ..., x_n) a_n^{(1)} (x_1, ..., x_n) | = \ell_n^2$$

and from this, together with (7) for $n = m$, follows (8). (9) follows from (7) by summing over all p, q, $p + q = n$. From the definition of k_{2n} we obtain

$$\| \sum_i a_n^{(1)*} a_n^{(1)} \|_0 - \| \sum_i \sum_{\substack{\nu=0 \\ \nu \neq n}}^{2n} a_{2n-\nu}^{(1)*} a_\nu^{(1)} \|_0 \leq \| k_{2n} \|_0$$

and from this follows (10) by (7) and (8).

We need further a relation for a special infinite hermitian matrix H, which is defined for a sequence $a_0, a_1, \ldots$ of positive numbers by

$$H = (h_{ij})_{i,j=0,1,\ldots}$$

$$h_{rr} = a_r\ , \qquad h_{ij} = \begin{cases} -a_r, & i \neq j \text{ and } i+j = 2r \\ 0, & i \neq j \text{ and } i+j \text{ an odd number} \end{cases} \tag{11}$$

Lemma 2

There exists such a sequence $a_0, a_1, a_2, \ldots$ of positive numbers that for an arbitrary infinite vector $\ell = (\ell_0, \ell_1, \ldots)$ for which only finite components are not zero the relation

$$\sum_{i,j \geq 0} h_{ij}\, \ell_i\, \ell_j \geq \sum_i \ell_i^2 \tag{12}$$

holds.

Proof:

We construct by induction a sequence of positive numbers $a_0, a_1, \ldots$ such that

$$\sum_{i,j=0}^{m} h_{i,j}\, \ell_i\, \ell_j \geq c_m \sum_{i=0}^{m} \ell_i^2\ , \quad m = 0, 1, \ldots \tag{13}$$

holds, with certain numbers $c_m > 1$. For $m = 0$ we can set $a_0 = 2$. Now we assume that (13) holds for $m = n-1$ and show that we can choose a_n such that (13) holds for n, too, with a certain $c_n > 1$. From the definition of H we obtain

$$\sum_{i,j=0}^{n} h_{ij}\, \ell_i\, \ell_j = \sum_{i,j=0}^{n-1} h_{ij}\, \ell_i\, \ell_j + a_n \ell_n^2 - 2 \sum_{\nu \geq 1} a_{n-\nu}\, \ell_n\, \ell_{n-2\nu}\ .$$

The sum on the right-hand side runs over all ν for which the other indices are nonnegative.

From this and the induction assumption we obtain with an arbitrary positive β

$$\sum_{i,j=0}^{n} h_{ij}\,\ell_i\,\ell_j \geq c_{n-1}\sum_{\nu\geq 1}(\ell_{n-2\nu+1})^2 + (a_n - \frac{1}{\beta^2})\,\ell_n^2 +$$

$$+ (\beta\sum_{\nu\geq 1} a_{n-\nu}\,\ell_{n-2\nu} - \frac{1}{\beta}\,\ell_n)^2$$

$$+ c_{n-1}\sum_{\nu\geq 1}(\ell_{n-2\nu})^2 - \beta^2(\sum_{\nu\geq 1} a_{n-\nu}\,\ell_{n-2\nu})^2 \geq$$

$$\geq c_{n-1}\sum_{\nu\geq 1}(\ell_{n-2\nu+1})^2 + (a_n - \frac{1}{\beta^2})\,\ell_n^2 +$$

$$+ (c_{n-1} - \beta^2\sum_{i=0}^{n-1} a_i^2)\sum(\ell_{n-2\nu})^2 .$$

How we choose β such that $c_{n-1} - \beta^2\sum_{i=0}^{n-1} a_i^2 > 1$ and then a_n such that $a_n - \frac{1}{\beta^2} > 1$

In this manner we have constructed an a_n such that the relation (13) holds for $m = n$, too.

Now let $a_0, a_1, \dots$ be a sequence of positive numbers for which the assertion of the preceding Lemma holds. Then we define for $a \in R$

$$\| a \|_\rho = \sum_{\nu\geq 0} a_\nu \, \| a_{2\nu} \|_0 \quad , \tag{14}$$

where $\| \ \|_0$ is the norm in $C(M^{(2\nu)})$ and $a_{2\nu}$ is the homogeneous component of the degree 2ν of a. $\| \ \|_\rho$ is a continuous semi-norm on R which gives a continuous norm on $R_\rho = \bigoplus_{\nu=0}^{\infty} R_{2\nu}$ (topological direct sum). Let $\| \ \|_u$ be another continuous semi-norm in R such that for each $a \in R$ and $r = 0.1, \dots$ $\| a_{2r+1} \|_0 < \mu_{2r+1} \| a \|_u$ holds, where

$a_{2\nu+1}$ is a homogeneous component of a and $\mu_{2\nu+1}$ are positive constants. Then $\| \; \|_u$ is a continuous norm in $R_u = \bigoplus_{\nu=0} R_{2\nu+1}$ and

$$\| a \| = \| a \|_\ell + \| a \|_u \qquad a \in R \tag{15}$$

is a continuous norm in R. Beside the basic-topology we regard in R a second topology which is determined by the norm $\| \; \|$. This topology is called the norm-topology or $\| \; \|$-topology. With this topology is R a (uncomplete) normed linear space, but not a normed algebra.

In the usual cases, where M is the Minkowski space and $R_n = S(M^{(n)})$ or $D(M^{(n)})$, the Schwartz' spaces, the semi-norm $\| \; \|_\ell$ is Lorentz invariant and consequently, we can choose $\| \; \|_u$ such that the norm $\| \; \|$ is Lorentz invariant, too.

Now we state and prove the main relation for the proofs of the theorems:

L e m m a 3

For every $k \in K_0$ holds the relation

$$\sum_{n \geq 0} \ell_n^2 < \| k \|_\ell \quad , \tag{16}$$

where ℓ_n are the expressions (6).

P r o o f :

We obtain from (10)

$$\sum_{n \geq 0} a_n \ell_n - 2 \sum_{n \geq 0} \sum_{\nu \geq 1} a_n \ell_{n+\nu} \ell_{n-\nu} < \sum_{n \geq 0} a_n \| k_{2n} \|_0$$

and in consequence of the definition (11) of H this is equivalent to

$$\sum_{i,j \geq 0} h_{ij} \ell_i \ell_j \leq \sum_{n \geq 0} a_n \| k_{2n} \|_0 \quad .$$

From this the relation (16) follows, because $a_0, a_1 \dots$ is a sequence for which (12) holds.

III. Main Theorems

After the preparations in the preceding section we state and prove here the main theorems.

Theorem 1

The topological closure $K_{\|\ \|}$ of K_0 in R with the norm-topology, which is determined by the norm (15), is a cone. Consequently, the topological closure $\overline{K}_0$ of K_0 in R, with the basic-topology, is a cone, too, because $\overline{K}_0 \subset K_{\|\ \|}$.

Proof:

We prove that the relations (5) i) - iii) hold for a $g \in K_{\|\ \|}$, $g \neq 0$, too. Let $g \neq 0$ be an element of $K_{\|\ \|}$. Then we can write $g = \sum_{n=0}^{m} g_n$, where g_n is the homogeneous component of the degree n of g, $g_n = 0$ for $n > m$. There has to exist a sequence $k^\nu \in K_0$ with $\| k^\nu - g \| \leq 1$ and $\| k^\nu - g \| \to 0$ for $\nu \to \infty$ and consequently,

$$\| k_n^\nu - g_n \|_0 \to 0 \qquad \text{for} \quad \nu \to \infty \ , \quad n = 0,1,\ldots \ . \tag{17}$$

Each k^ν has the form $k^\nu = \sum_{l=1}^{N_\nu} a_\nu^{(l)*} a_\nu^{(l)}$, $a_\nu^{(l)} \in R$. Let ℓ_n^ν , $n = 0,1,2,\ldots$ be the numbers (6) of k^ν, then we obtain from (16)

$$\sum_{n \geq 0} (\ell_n^\nu)^2 < \| k^\nu \|_\rho < \| g \|_\rho + 1$$

and consequently, the sequence ℓ_n^ν $\nu = 1,2, \ldots,$ is bounded for every r. Because $g \neq 0$, there exists one $s \geq 0$ such that

$$\begin{aligned} &\lim_{\nu \to \infty} \ell_n^\nu = 0 \qquad \text{for} \quad 0 \leq n \leq s-1 \\ &\ell_s^\nu \ \text{does not tend to zero for} \ \nu \to \infty \ . \end{aligned} \tag{18}$$

Then follows from (9)

$$\lim_{\nu \to \infty} \| k_n^\nu \|_0 = \| g_n \|_0 = 0 . \qquad \text{for} \quad 0 \leq n \leq 2s-1 \ . \tag{19}$$

Furthermore one has

$$\lim_{\nu\to\infty} \| k^{\nu}_{2s} \|_0 = \| g_{2s} \|_0 \neq 0 . \qquad (20)$$

If this is not true, i.e. $\lim_{\nu\to\infty} \| k^{\nu}_{2s} \| = 0$, then we obtain $\lim_{\nu\to\infty} \ell^{\nu}_{s} = 0$ by (10) and the first assertion of (18), which is in contradiction with the second assertion of (18). (19) and (20) are the assertion (5) i). Finally it remains to prove the statements ii) and iii) of (5). From (17) and (18) we obtain

$$\| g_{2s} - \sum_{1} a^{(1)*}_{\nu s} a^{(1)}_{\nu s} \|_0 \leq \| g_{2s} - k^{\nu}_{2s} \|_0 + \| \sum_{1} \sum_{\substack{p=0 \\ p\neq s}}^{2s} a^{(1)*}_{\nu\, 2s-p} a^{(1)}_{\nu p} \|_0 \leq$$

$$< \| g_{2s} - k^{\nu}_{2s} \|_0 + 2 \sum_{p=1}^{s} \ell^{\nu}_{s-p} \ell^{\nu}_{s+p} \to 0$$

for $\nu \to \infty$, i.e.

$$g_{2s}(x_1, \dots, x_{2s}) = \lim_{\nu\to\infty} \sum_{1} \bar{a}^{(1)}_{\nu s}(x_s, \dots, x_1)\, a^{(1)}_{\nu s}(x_{s+1}, \dots, x_{2s}) \qquad (21)$$

(in the $\| \; \|_0$ - convergence) and consequently, g_{2s} is nonnegative on Γ_{2s}. Because ℓ^{ν}_{s} does not tend to zero, we obtain straightforward from (21) that $g_{2s}(x_1, \dots, x_{2s})$ is not identically zero on Γ_{2s}. Hence, assertion (5) iii) holds, too. $\bar{K}_0 \subset K_{\| \; \|}$ holds, because the norm $\| \; \|$ is continuous in the basic-topology of R.

<u>Theorem 2</u>

For each $b \in R$, $b \neq 0$, there exists a positive continuous linear functional $W_b(a)$ on R, with $W_b(b) \neq 0$, for which $| W_b(a) | \leq \| a \|$, $a \in R$, holds, and consequently, the topological *-algebra R is reduced. This theorem follows from the

<u>Lemma 4</u>

Let K be a closed cone in a normed (or locally convex) linear space R and $b \in R$, $b \neq 0$, $b \notin K$, then exists a linear continuous functional $f(a)$ on R, with $f(b) \neq 0$ and $f(a) \geq 0$ for $a \in K$. /6/.

If $b \neq 0$ is an element of R, then either b or $-b$ does not lie in $K_{\|\ \|}$. Hence, from the Lemma 4 it follows the existence of such a functional $W_b(a)$ on R that $W_b(b) \neq 0$ and $W(a) \geq 0$ for $a \in K_{\|\ \|}$ and which is continuous in the $\|\ \|$ -topology of R and consequently in the basic-topology, too. $W_b(a)$ is a positive functional on the algebra R in the usual sence, because $W_b(a^*a) \geq 0$, then $a^*a \in K_0 \subset \bar{K}_0 \subset K_{\|\|}$. Evidently, we can choose $W_b(a)$ so that $|W_b(a)| \leq \|a\|$ holds. From the last property it follows that the set $\{W_b\}$ is bounded in the weak topology of R' and consequently, by a well known theorem /4/, we obtain the

Corollary:

The set $\{W_b\}$ of all these positive linear functionals of Theorem 2 is a relatively compact set in the weak topology in R' (the dual space of R).

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Received by Publishing Department
on May 23 1967.