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UNITARY REPRESENTATIONS OF THE LORENTZ GROUP AND FUNCTIONS WITH SPIN

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1. Introduction

In the paper [1] the functions were considered, which form the representation basis of the group of motions on the hyperboloid or, what is the same, the basis for representations of the class I (the most degenerate representations). These functions can serve as a basis for integral representations of the scattering amplitudes in the form, indicated in the paper, mentioned above. At the same time we can consider them as solutions of the wave equations for free spinless particles and having a complete set of these functions use it for the integral representation of the wave function of the spinless particles in the external field. It is natural now to generalize the method developed in [1] to obtain other representations of the Lorentz group.

The construction of the representation begins with the choice of a space, in which this representation will be realized. In the mathematical statement of a problem the group properties themselves from which the corresponding Lie algebra follows play the main role and the problems of the specific realization take the subordinate (often only illustrative) part. In physics the problem of realizing the representations (choosing the functions) is connected with the solution of the equations and usually is a principal goal of the research. Moreover different realizations of the representations of the same group often meet various problems. For instance, the group SU(3) meets the problem of oscillator (Hermite functions) and the symmetry of the elementary particles (Legendre functions).

In physical problems the space is usually defined by the conditions of the problem. The displacements (differentiation) in this space (together with symmetry conditions) define the Lie algebra according to which the group is now being built. It is clear that with such a costruction not all the representations of the group turn out to be used, but only some part of them(as for instance in the Coulomb problem or in the oscillator problem). Formally the restriction of the representations turns out to be the consequence of the additional symmetry, inherent in the basis, what leads sometimes to the appearance of the additional quantum numbers.

The well-known example of such a kind is the spherical functions with spin introduced by Berestetsky /2/. These functions realize the basis of all the representations of the group O(3). At the same time they are constructed as the linear combinations of functions on which one may build the wider group, namely the group O(3) x O(3). Therefore the Berestetsky's functions have the additional property - they are tensors of definite rank (spinors), while in the group O(3) there is no, generally speaking, separate spin quantum number, but there is only the quantum number of the total angular momentum.

The reference to the Berestetsky's functions makes clear further choice of the realization of representation of the Lorentz group.

In the homogeneous Lorentz group there is obviously no operator to which one would ascribe spin. But one may introduce the additional spin quantum number, if one chooses such a realization of the representation of the group $\mathfrak{L}$, which would admit the extension to the Poincare group $\mathfrak{P}$. This can be achieved, for instance, if one defines on the cone not the scalar spherical function, but the Berestetsky's one and makes the horisphere transformation of Gelfand-Graev /3/. However, this way leads to cumbersome calculations, which for the present we have not been able yet to reduce to more simple form.

Therefore we have chosen another-more straight forward way, having realized the representation on the space of functions, defined on the direct product of two spaces the hyperboloid (with the infinitedimensional representation) and the sphere (with the finite-dimensional spinor representation) or the cone. Thus the representation will be built as the direct product of the irreducible representations: infinite-dimensional nonunitary and finite-dimensional nonunitary. For the basis of the infinite-dimensional representation the functions from /1/ are chosen, for the basis of the finite-dimensional representation the representation of the special kind (s,o) is chosen, which corresponds to the function with one spin value. The general case and the application of the results to the equations for particles with higher spins will be presented elsewhere.

2. The algebra of the Lorentz group

Only the most degenerate representations of the type $(i p_0, i p_0)$ can be realized on the upper sheet of the hyperboloid which corresponds to scalar particles. If one wishes to obtain representations, corresponding to the states of particles with arbitrary spin s, it is necessary, as it was mentioned in Introduction, to extend the hyperboloid to the space, which is the direct product of a hyperboloid and a finite-dimensional spin space.

In such a space one may realize the representations (m, p), of the principal series of the Lorentz group with $-s \le \frac{m}{2} \le s$.

Such a consideration is a direct generalization of the paper /1/. We shall describe the coordinate system in the velocity space by expressing the four homogeneous cartesian coordinates of the velocity $u_0, \vec{u}$ through angles; for this we introduce the spherical coordinate system on the hyperboloid:

$$u_0 = \mathrm{ch}\, a \qquad u_2 = \mathrm{sh}\, a \sin\theta \sin\phi$$
$$u_3 = -\mathrm{sh}\, a \cos\theta \qquad u_1 = \mathrm{sh}\, a \sin\theta \cos\phi \tag{1}$$

The spherical system of this special type is chosen in order to preserge the standard form of spin matrices. In this coordinate system the generators of the Lorentz group are written in the form

$$L_1 = -i\left(\sin\phi \frac{\partial}{\partial\theta} + \mathrm{ctg}\,\theta \cos\phi \frac{\partial}{\partial\phi}\right) I + L_1^s$$

$$L_2 = i\left(\cos\phi\frac{\partial}{\partial\theta} - \operatorname{ctg}\theta\sin\phi\frac{\partial}{\partial\phi}\right)I + L_2^s$$

$$L_3 = -i\frac{\partial}{\partial\phi}I + L_3^s$$

$$K_1 = -i\left(\sin\theta\cos\phi\frac{\partial}{\partial a} + \operatorname{cth} a\cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{1}{\sin\theta}\operatorname{cth} a\sin\phi\frac{\partial}{\partial\phi}\right)I + K_1^s$$

$$K_2 = -i\left(\sin\theta\sin\phi\frac{\partial}{\partial a} + \operatorname{cth} a\cos\theta\sin\phi\frac{\partial}{\partial a} + \frac{1}{\sin\theta}\operatorname{cth} a\cos\phi\frac{\partial}{\partial\phi}\right)I + K_2^s$$

$$K_3 = i\left(\cos\theta\frac{\partial}{\partial a} - \operatorname{cth} a\sin\theta\frac{\partial}{\partial\theta}\right)I + K_3^s \quad ,$$

where I is the unit matrix $(2s+1)\times(2s+1)$ and satisfy the usual commutation relations /4/ :

$$[L_i, L_j] = i\epsilon_{ijk}L_k \qquad [L_i, K_j] = i\epsilon_{ijk}K_k \tag{3}$$

$$[K_i, K_j] = -i\epsilon_{ijk}L_k$$

or using the operators $H_\pm = L_1 \pm iL_2$, $F_\pm = K_1 \pm iK_2$, $H_3 = L_3$, $F_3 = K_3$, we have /4/

$$[H_+, H_3] = -H_+ \; , \; [H_-, H_3] = H_- \; , \; [H_+, H_-] = 2H_3 \; .$$

$$[H_+, F_+] = [H_-, F_-] = [H_3, F_3] = 0$$

$$[F_+, F_3] = H_+ \; , [F_-, F_3] = -H_- \; , [F_+, F_-] = -2H_3$$

$$[H_+, F_3] = -F_+ \; , \quad [H_-, F_3] = F_-$$

$$[H_+, F_-] = -[H_-, F_+] = 2F_3$$

$$[F_+, H_3] = -F_+ \; , \quad [F_-, H_3] = F_- \; .$$

Since the representations are realized in the space, which is a direct product of the hyperboloid space and the finite-dimensional spin space, we have $L_i = L_i^0 + L_i^s$, $K_i = K_i^0 + K_i^s$, where L_i^0, K_i^0 are the "orbital" generators on the hyperboloid, and the representation we are searching for is a product of the representations of the type (ip_0, ip_0) on the hyperboloid and (s, o) in the finite-dimensional spin space/5/x/. Here L_i^s and K_i^s are the matrices of the dimension $(2s+1)x(2s+1)$, obeying the commutation relations (3). Due to such a choice of representations we are working with a minimum number of spinor components.

3. The Contruction of Basis Functions

An irreducible representation of the Lorentz group is characterized by two Casimir operators:

$$\Delta = L_i^2 - K_i^2 , \qquad \Delta' = 2i L_i K_i .$$

The functions f, forming the basis of an irreducible representation of the Lorentz group, satisfy the relations

$$\Delta f = [(\frac{m}{2})^2 - p^2 - 1] f$$

$$\Delta' f = impf . \qquad (4)$$

For unitary irreducible representations from the principal series m is integral and p is real. Since the representations (m, p) and $(-m, -p)$ are equivalent, for uniqueness we choose the range of definition as $-\infty < m < \infty$ and $0 \leq p < \infty$. We construct the basis function in the form:

x/ In other words, we use spinor, which has all the indices of the same type.

$$< a,\theta,\phi \,|\, p\frac{m}{2} jM > = \sum_{i=1}^{2s+1} \sum_{\mu=-s}^{s} B_i^s(j,m,p)(\ell_i s j M | \ell_i s m_\ell \mu) f_{m_\ell}^{\ell_i}(a,\theta,\phi)\phi_\mu^s \quad , \qquad (5)$$

where $\ell_i = j+s-i+1$, the Clebsh-Gordon coefficients correspond to the addition $\vec{s} + \vec{\ell}_i = \vec{j}$, so that $|j-s| \le \ell_i < j+s$, $f_{m_\ell}^{\ell_i}(a,\theta,\phi)$ are the basis functions for the infinite-dimensional representation on the hyperboloid (ip_0, ip_0) and ϕ_μ^s are the basis functions for the finite-dimensional representation, corresponding to the particle with spin s. For such representations $K_i^s = -iL_i^s$. The diagonalization of L^2 defines the dependence of (5) from M. The functions $f_{m_\ell}^{\ell_i}(a,\theta,\phi)$ are the eigenfunctions of the operator $\Delta_- = \Delta - \Delta'$

$$\Delta_- f_{m_\ell}^{\ell_i}(a,\theta,\phi) = \left[\left(\frac{m}{2} - ip\right)^2 - 1\right] f_{m_\ell}^{\ell_i}(a,\theta,\phi). \qquad (6)$$

We note, that the operator Δ_- does not contain spin matrices. Solution of (6) which is regular at the origin have the form x/

$$f_{m_\ell}^{\ell_i}(a,\theta,\phi) = \frac{N(\ell_i)}{sh^{1/2} a} P_{-\frac{1}{2}+ip_0}^{-(\ell_i+\frac{1}{2})}(ch\, a) Y_{\ell_i m_\ell}(\theta,\phi), \qquad (7)$$

where $-ip_0 = \frac{m}{2} - ip$ and the spherical functions are normalized so that an integral of their modulus squared over the sphere is equal to 1, and have the symmetry property $Y_{\ell,m} = (-1)^m Y_{\ell,-m}^*$, fixing the phase.

The expression for $N(\ell_i)$ is obtained, if one demands, for the matrix elements K_i^0 to have the canonical form in the representation (ip_0, ip_0):

$$(iF_3^0) f_{m_\ell}^{\ell_i} = -\sqrt{(\ell_i - m_\ell)(\ell_i + m_\ell)} \sqrt{\frac{\ell_i^2 + p_0^2}{4\ell_i^2 - 1}} f_{m_\ell}^{\ell_i - 1} +$$
$$+ \sqrt{(\ell_i + m_\ell + 1)(\ell_i - m_\ell + 1)} \sqrt{\frac{[(\ell_i+1)^2 + p_0^2]}{4(\ell_i+1)^2 - 1}} f_{m_\ell}^{\ell_i+1} \qquad (8)$$

Hence, with the use of (2), we get:

$$(iF_3^0)f^{\ell_1}_{m_\ell}(a,\theta,\phi)=[-\cos\theta\frac{\partial}{\partial a}+\mathrm{cth}a\,\sin\theta\frac{\partial}{\partial\theta}]\frac{N(\ell_1)}{\mathrm{sh}^{1/2}a}P^{-(\ell_1+\frac{1}{2})}_{-\frac{1}{2}+ip_0}(\mathrm{ch}a)Y_{\ell_1 m_\ell}(\theta,\phi)=$$

$$=[-\cos\theta\frac{\partial}{\partial a}+\mathrm{cth}a\,\sin\theta\frac{\partial}{\partial\theta}]\frac{\bar N(\ell_1,m_\ell)}{\mathrm{sh}^{1/2}a}P^{-(\ell_1+\frac{1}{2})}_{-\frac{1}{2}+ip_0}(\mathrm{ch}a)P^{m_\ell}_{\ell_1}(\cos\theta)e^{im_\ell\phi} \tag{9}$$

If one uses the recurrence relations for the Legendre functions, then the expression (9) can be taken in the form:

$$(iF_3^0)f^{\ell_1}_{m_\ell}(a,\theta,\phi)=\frac{\bar N(\ell_1,m_\ell)}{\mathrm{sh}^{1/2}a}\{\frac{(\ell_1-m_\ell+1)}{(2\ell_1+1)}[(\ell_1+1)^2+p_0^2]\times$$

$$\times P^{-(\ell_1+\frac{3}{2})}_{-\frac{1}{2}+ip_0}(\mathrm{ch}a)P^{m_\ell}_{\ell_1+1}(\cos\theta)e^{im_\ell\phi}-$$

$$-\frac{(\ell_1+m_\ell)}{(2\ell_1+1)}P^{-(\ell_1-\frac{1}{2})}_{-\frac{1}{2}+ip_0}(\mathrm{ch}a)P^{m_\ell}_{\ell_1-1}(\cos\theta)e^{im_\ell\phi}\}. \tag{10}$$

Comparing (10) with (8), we get the formula for the increase of the value of ℓ_1 by unit:

$$\bar N(\ell_1,m_\ell)=\sqrt{\frac{(\ell_1^2+p_0^2)(\ell_1-m_\ell)(2\ell_1+1)}{(\ell_1+m_\ell)(2\ell_1-1)}}\,\bar N(\ell_1-1,m_\ell) \tag{11}$$

Taking into account, that

$$\bar N(\ell_1,m_\ell)=N(\ell_1)\frac{(-1)^{m_\ell}}{\sqrt{2\pi}}\sqrt{\frac{(2\ell_1+1)(\ell_1-m_\ell)!}{2(\ell_1+m_\ell)!}}$$

we find

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x/ A choice of the regular at the origin solution of the equation (6) allows to define the nonrelativistic limit.

$$N(\ell_1) = \sqrt{\ell_1^2 + p_0^2}\; N(\ell_1 - 1). \tag{12}$$

The solution of (12), taking into account (7) have the form

$$N(\ell_1) = \sqrt{\frac{\Gamma[\ell_1 + 1 + (\frac{m}{2} - ip)]\,\Gamma[\ell_1 + 1 - (\frac{m}{2} - ip)]}{\Gamma[\ell_{1\,min} + (\frac{m}{2} - ip)]\,\Gamma[\ell_{1\,min} - (\frac{m}{2} - ip)]}}, \tag{13}$$

where

$$\ell_{1\,min} = \frac{|m|}{2} + s - i + 2$$

The equation

$$\Delta_+ f^{\ell_1}_{m\ell}(a, \theta, \phi) = [(\frac{m}{2} + ip)^2 - 1]\, f^{\ell_1}_{m\ell}(a, \theta, \phi),$$

where

$$\Delta_+ = \Delta + \Delta' .$$

leads to the restriction on m

$$\frac{m}{2} = -s, -s+1, \dots, s-1, s. \tag{14}$$

For the basis functions ϕ^s_μ we choose

$$\phi^s_\mu(a) = \delta_{a,\, s-\mu+1} \qquad (-s \leq \mu \leq s).$$

In the scalar case $m = 0$, $p_0 = p$, the functions $f^{\ell_1}_{m\ell}(a, \theta, \phi)$ turn into the functions, obtained in the paper /1/, and realize unitary representations on the hyperboloid.

In the general case of non-zero s, p_0 is complex with $m \neq 0$ and the representation, which is realized on these functions, is infinite-dimensional and nonunitary. Therefore the problem is to take a product of two nonunitary representations: infinite-dimensional on the hyperboloid and finite-dimensional in the spin space and to extract the unitary part from the product. This is achieved in the following way. We act by the generators of the group on the function (5), orbital and spin parts acting on $f^{\ell_1}_{m_\ell}$ and ϕ^s_μ independently in the following way[5]:

$$H^0_+ f^{\ell_1}_{m_\ell} = \sqrt{(\ell_1+m_\ell+1)(\ell_1-m_\ell)}\, f^{\ell_1}_{m_\ell+1}\,,\quad H^0_- f^{\ell_1}_{m_\ell} = \sqrt{(\ell_1+m_\ell)(\ell_1-m_\ell+1)}\, f^{\ell_1}_{m_\ell-1}$$

$$H^0_3 f^{\ell_1}_{m_\ell} = m_\ell f^{\ell_1}_{m_\ell}$$

$$F^0_+ f^{\ell_1}_{m_\ell} = \sqrt{(\ell_1-m_\ell)(\ell_1-m_\ell+1)}\, i\sqrt{\frac{\ell_1^2+p_0^2}{4\ell_1^2-1}}\, f^{\ell_1-1}_{m_\ell+1} + \sqrt{(\ell_1+m_\ell+2)(\ell_1+m_\ell+1)}\times \tag{15}$$

$$\times\; i\sqrt{\frac{(\ell_1+1)^2+p_0^2}{4(\ell_1+1)^2-1}}\, f^{\ell_1+1}_{m_\ell+1}$$

$$F^0_- f^{\ell_1}_{m_\ell} = -\sqrt{(\ell_1+m_\ell)(\ell_1+m_\ell-1)}\, i\sqrt{\frac{\ell_1^2+p_0^2}{4\ell_1^2-1}}\, f^{\ell_1-1}_{m_\ell-1} - \sqrt{(\ell_1-m_\ell+1)(\ell_1-m_\ell+2)}\, i\sqrt{\frac{(\ell_1+1)^2+p_0^2}{4(\ell_1+1)^2-1}}\, f^{\ell_1+1}_{m_\ell-1}$$

$$F^0_3 f^{\ell_1}_{m_\ell} = \sqrt{(\ell_1-m_\ell)(\ell_1+m_\ell)}\, i\sqrt{\frac{\ell_1^2+p_0^2}{4\ell_1^2-1}}\, f^{\ell_1-1}_{m_\ell} - \sqrt{(\ell_1+m_\ell+1)(\ell_1-m_\ell+1)}\, i\sqrt{\frac{(\ell_1+1)^2+p_0^2}{4(\ell_1+1)^2-1}}\, f^{\ell_1+1}_{m_\ell}$$

$$H^s_+ \phi^s_\mu = \sqrt{(s+\mu+1)(s-\mu)}\, \phi^s_{\mu+1} \qquad H^s_- \phi^s_\mu = \sqrt{(s+\mu)(s-\mu+1)}\, \phi^s_{\mu-1}$$

$$H^s_3 \phi^s_\mu = \mu\, \phi^s_\mu \tag{16}$$

$$F^s_+ \phi^s_\mu = (-i)\sqrt{(s+\mu+1)(s-\mu)}\, \phi^s_{\mu+1}\,,\quad F^s_- \phi^s_\mu = (-i)\sqrt{(s+\mu)(s-\mu+1)}\, \phi^s_{\mu-1}$$

$$F^s_3 \phi^s_\mu = -i\mu\phi^s_\mu$$

The action of generators of the group on the functions $\langle a,\theta,\phi|p\,\frac{m}{2}\,j\,M\rangle \equiv \Phi^j_M$ for unitary representations from the principal series is defined in the following manner:

$$H_+ \Phi^j_M = \sqrt{(j+M+1)(j-M)}\, \Phi^j_{M+1}\,,\quad H_- \Phi^j_M = \sqrt{(j+M)(j-M+1)}\, \Phi^j_{M-1}$$

$$H_3 \Phi^j_M = M\Phi^j_M$$

$$F_+ \Phi^j_M = \sqrt{(j-M)(j-M-1)}\, \frac{i}{j}\sqrt{\frac{(j^2-\frac{m^2}{4})(j^2+p^2)}{4j^2-1}}\, \Phi^{j-1}_{M+1} -$$

$$-\sqrt{(j-M)(j+M+1)}\,\frac{mp}{2j(j+1)}\,\Phi^{j}_{M+1} + \sqrt{(j+M+1)(j+M+2)}\,\frac{i}{(j+1)}\sqrt{\frac{[(j+1)^2-\frac{m^2}{4}][(j+1)^2+p^2]}{4(j+1)^2-1}}\;\Phi^{j+1}_{M+1}$$

$$F_{-}\Phi^{j}_{M} = -\sqrt{(j+M)(j+M-1)}\,\frac{i}{j}\sqrt{\frac{(j^2-\frac{m^2}{4})(j^2+p^2)}{4j^2-1}}\;\Phi^{j-1}_{M-1} + \sqrt{(j+M)(j-M+1)}\,\frac{mp}{2j(j+1)}\,\Phi^{j}_{M-1} -$$

$$-\sqrt{(j-M+1)(j-M+2)}\,\frac{i}{(j+1)}\sqrt{\frac{[(j+1)^2-\frac{m^2}{4}][(j+1)^2+p^2]}{4(j+1)^2-1}}\;\Phi^{j+1}_{M-1}$$

$$F_3\,\Phi^{j}_{M} = \sqrt{(j-M)(j+M)}\,\frac{i}{j}\sqrt{\frac{(j^2-\frac{m^2}{4})(j^2+p^2)}{4j^2-1}}\;\Phi^{j-1}_{M} + M\,\frac{mp}{2j(j+1)}\,\Phi^{j}_{M} -$$

$$-\sqrt{(j+M+1)(j-M+1)}\,\frac{i}{(j+1)}\sqrt{\frac{[(j+1)^2-\frac{m^2}{4}][(j+1)^2+p^2]}{4(j+1)^2-1}}\;\Phi^{j+1}_{M}$$

For our purpose it is convenient to use the relation:

$$(iF_3)\Phi^{j}_{j} = \frac{imp}{2(j+1)}\Phi^{j}_{j} + \frac{\sqrt{2j+1}}{(j+1)}\sqrt{\frac{[(j+1)^2-\frac{m^2}{4}][(j+1)^2+p^2]}{4(j+1)^2-1}}\;\Phi^{j+1}_{j} \quad (18)$$

Applying (15)(16)(18) to (5), we get

$$\frac{imp}{2(j+1)}\sum_{i=1}^{2s+1}\sum_{\mu=-s}^{s} B^{s}_{i}(j,m,p)(\ell_i\, s j j\,|\,\ell_i\, s j-\mu\mu)\, f^{\ell_i}_{j-\mu}\,\phi^{s}_{\mu} +$$

$$+\frac{\sqrt{2j+1}}{j+1}\sqrt{\frac{[(j+1)^2-\frac{m^2}{4}][(j+1)^2+p^2]}{4(j+1)^2-1}}\sum_{i=1}^{2s+1}\sum_{\mu=-s}^{s} B^{s}_{i}(j+1,m\,p)\times$$

$$\times(\ell_i+1\, s\, j+1\, j\,|\,\ell_i+1\, s\, j-\mu\,\mu)\, f^{\ell_i+1}_{j-\mu}\,\phi^{s}_{\mu} =$$

$$=\sum_{i=1}^{2s+1}\sum_{\mu=-s}^{s} B^{s}_{i}(j,m,p)(\ell_i\, s j j\,|\,\ell_i\, s j-\mu\mu)\Big[-\sqrt{(\ell_i-j+\mu)(\ell_i+j-\mu)}\sqrt{\frac{\ell_i^2+p_0^2}{4\ell_i^2-1}}\, f^{\ell_i-1}_{j-\mu} +$$

$$+\sqrt{(\ell_i+j-\mu+1)(\ell_i-j+\mu+1)}\sqrt{\frac{[(\ell_i+1)^2+p_0^2]}{4(\ell_i+1)^2-1}}\, f^{\ell_i+1}_{j-\mu} + \mu\, f^{\ell_i}_{j-\mu}\Big]\,\phi^{s}_{\mu}$$

After a number of simple transformations, we get the recurrence relation for B_i^s

$$B_i^s(j)(\ell_i\, s j j|\ell_i\, s j-\mu\mu)\left[\frac{imp}{2(j+1)}-\mu\right]+\frac{\sqrt{2j+1}}{(j+1)}\sqrt{\frac{\left[(j+1)^2-\frac{m^2}{4}\right]\left[(j+1)^2+p^2\right]}{4(j+1)^2-1}}\times$$

$$\times B_{i+1}^s(j+1)(\ell_i\, s j+1 j|\ell_i\, s j-\mu\mu)=-B_{i-1}^s(j)(\ell_i+1 s j j|\ell_i+1 s j-\mu\mu)\times \quad (19)$$

$$\times\sqrt{(\ell_i+1-j+\mu)(\ell_i+1+j-\mu)}\sqrt{\frac{[(\ell_i+1)^2+p_0^2]}{4(\ell_i+1)^2-1}}+B_{i+1}^s(j)(\ell_i-1 s j j|\ell_i-1 s j-\mu\mu)\times$$

$$\times\sqrt{(\ell_i+j-\mu)(\ell_i-j+\mu)}\sqrt{\frac{\ell_i^2+p_0^2}{4\ell_i^2-1}}$$

With the choice of $\mu = i-s-1$ in equation (19) we get for B_i^s :

$$B_i^s(j)(\ell_i\, s j j|\ell_i\, s\, \ell_i\, i-s-1)\left[\frac{imp}{2(j+1)}+(s-i+1)\right]+B_{i+1}^s(j+1)\times$$

$$\times(\ell_i\, s j+1 j|\ell_i\, s\, \ell_i\, i-s-1)\frac{\sqrt{2j+1}}{j+1}\sqrt{\frac{\left[(j+1)^2-\frac{m^2}{4}\right]\left[(j+1)^2+p^2\right]}{4(j+1)^2-1}}=$$

$$=-B_{i-1}^s(j)(\ell_i+1 s j j|\ell_i+1 s\, \ell_i\, i-s-1)\sqrt{2\ell_i+1}\sqrt{\frac{(\ell_i+1)^2+p_0^2}{4(\ell_i+1)^2-1}}$$

Putting the values of the Clebsh-Gordon coefficients in this expression, we get:

$$B_i^s(j)\left[\frac{imp}{2(j+1)}+(s+1-i)\right]+B_{i+1}^s(j+1)\frac{1}{(j+1)}\sqrt{\left[(j+1)^2-\frac{m^2}{4}\right]\left[(j+1)^2+p^2\right]}$$

$$\times\sqrt{\frac{i(2s-i+1)}{(2j-i+2)(2j+2s-i+3)}}=$$

$$=B_{i-1}^s(j)\sqrt{\left[(j+s-i+2)^2-\left(\frac{m}{2}-ip\right)^2\right]}\sqrt{\frac{(i-1)(2s-i+2)(2j+2s-2i+3)}{(2j+2s-i+3)(2j+2-i)(2j+2s-2i+5)}}$$

In the following it is convenient for us to pass from the coefficients $B_i^s(j)$ to the quantities $A_i^s(j)$, where

$$A_i^s(j) = \frac{B_i^s(j)N(\ell_i)}{B_1^s(j)N(\ell_1)}$$

Then with the help of the expression (5) we get a final form of the functions realizing a basis of the representation (p, m) of the Lorentz group:

$$\langle a, \theta, \phi | p\ \tfrac{m}{2}\ j M\rangle = G(j,s) \sum_{i=1}^{2s+1} \sum_{\mu=-s}^{s} A_i^s(j,m,p)(\ell_i\ s\ j M | \ell_i\ s\ m_\ell\ M) \times \tag{20}$$

$$\times \frac{1}{\operatorname{sh}^{1/2} a} P^{-(\ell_i + \frac{1}{2})}_{-\frac{1}{2}+ip_0}(\operatorname{ch} a) Y_{\ell_i m_\ell}(\theta, \phi)\ \phi^s_\mu ,$$

where

$$ip_0 = -\frac{m}{2} + ip$$

$$G(j,s,m,p) = B_1^s(j,m,p)N(\ell_1) \tag{21}$$

and $\frac{|m|}{2}$ defines the lower boundary of the possible values of s. The coe coefficients $A_i^s(j)$ are defined by the recurrence relations[x]

$$A_i^s(j)\left[\frac{imp}{2(j+1)} + (s+1-i)\right] + A_{i+1}^s(j+1)\left[(j+s+1)^2 - \left(\frac{m}{2} - ip\right)^2\right]$$

$$\times \sqrt{\frac{(2j+2s+2)(2j+1)i(2s-i+1)}{(2j+2)(2j+2s+1)(2j+2-i)(2j+2s-i+3)}} = A_{i-1}^s(j)\sqrt{\frac{(i-1)(2s-i+2)(2j+2s-2i+3)}{(2j+2s-i+3)(2j+2-i)(2j+2s-2i+5)}} \tag{22}$$

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[x] The condition for the existence of a nontrivial solution of the system of homogeneous equations (22) leads to the restriction on $\frac{m}{2}$: $\frac{m}{2} = -s, -s+1, \ldots, s-1, s$.

We shall list the values of the first three coefficients $A_1^s(j)$. By the definition $A_1^s(j) = 1$. Having put in (22) $i = 1$, we find for the following coefficient:

$$A_2^s(j) = \frac{\left[\frac{imp}{2j} + s\right]}{\left[\left(\frac{m}{2} - ip\right)^2 - (j + s)^2\right]} \sqrt{\frac{j}{s}(2j + 2s - 1)}$$

Similarly, having put $i = 2$ we get:

$$A_3^s(j) = \frac{(2j + 2s - 1)}{\left[(j + s)^2 - \left(\frac{m}{2} - ip\right)^2\right]} \sqrt{\frac{2j}{(2s - 1)(2j + 2s)}} \times$$

$$\times \left\{ \frac{\left[\frac{imp}{2j} + s - 1\right]\left[\frac{imp}{2(j-1)} + s\right]}{\left[(j + s - 1)^2 - \left(\frac{m}{2} - ip\right)^2\right]} \sqrt{\frac{(i-1)}{s}(2j + 2s - 3)} + \frac{1}{(2j + 2s - 1)} \sqrt{\frac{s(2j + 2s - 3)}{(j-1)}} \right\}$$

We still owe to define $B_1^s(j)$. For this purpose choosing in the formula(19):

$$\ell_1 = j + s + 1 \qquad \text{and} \qquad \mu = -s\,,$$

We get the following expression:

$$\frac{\sqrt{2j + 1}}{j + 1} \sqrt{\frac{\left[(j+1)^2 - \frac{m^2}{4}\right]\left[(j+1)^2 + p^2\right]}{4(j + 1)^2 - 1}} \,(j + s + 1 \quad sj + 1j \,|\, j + s + 1 \quad sj + s - s) B_1^s(j + 1) =$$

$$= B_1^s(j)(j + s \; sjj \,|\, j + s sj + s - s) \sqrt{2j + 2s + 1} \quad \sqrt{\frac{\left[(j + s + 1)^2 + p_0^2\right]}{4(j + s + 1)^2 - 1}}$$

Inserting the expression for the Clebsh-Gordon coefficients we get

$$B_1^s(j+1) = B_1^s(j)(j+1)\sqrt{\frac{[(j+s+1)^2+p_0^2](2j+2s+2)(2j+1)}{[(j+1)^2-\frac{m^2}{4}][(j+1)^2+p^2](2j+2)(2j+2s+1)}} \qquad (23)$$

The recurrence relation (23) gives the following expression for $B_1^s(j)$:

$$B_1^s(j) = N_1(m,p,s)\,\frac{\Gamma(j+1)}{\Gamma(\frac{|m|}{2}+1)}\left\{\frac{\Gamma[(j+s+1)+(\frac{m}{2}-ip)]\,\Gamma[(j+s+1)-(\frac{m}{2}-ip)]}{\Gamma[(\frac{|m|}{2}+s+1)+(\frac{m}{2}-ip)]\,\Gamma[(\frac{|m|}{2}+s+1)-(\frac{m}{2}-ip)]}\right.$$

$$\times\frac{\Gamma[(\frac{|m|}{2}+1)+\frac{m}{2}]\,\Gamma[(\frac{|m|}{2}+1)-\frac{m}{2}]}{\Gamma(j+\frac{m}{2}+1)\,\Gamma(j-\frac{m}{2}+1)}\,\frac{|\Gamma(\frac{|m|}{2}+1+ip)|^2}{|\Gamma(j+ip+1)|^2} \qquad (24)$$

$$\times\left.\frac{(2j+2s)!!\,(2j-1)!!\,|m|!!\,(|m|+2s-1)!!}{(|m|+2s)!!\,(|m|-1)!!\,(2j)!!\,(2j+2s-1)!!}\right\}^{1/2}$$

Since we consider unitary representations, we can choose the invariant norm in the Hilbert space of functions $\Phi^j_M(a,\theta,\phi)$ to be positively definite. The condition

$$N_1(|m|,p,s) = \frac{\Gamma(s+1+ip)}{\Gamma(|m|+s+1+ip)} \qquad (25)$$

just realizes such a choice with

$$N_1(-|m|,p,s) = N_1(|m|,-p,s) = N^*_1(|m|,p,s)\,.$$

Finally, we shall write down an expression for the nonrelativistic limit of the functions $< a, \theta, \phi | p \frac{m}{2} jM >$ Taking into account $tha = \frac{|\vec{v}|}{c}$ after simple calculations we get

$$\lim_{\substack{a \to 0 \\ (\frac{v}{c}) \to 0}} < a, \theta, \phi | p \frac{m}{2} jM > = > \sum_{l=1}^{2s+1} \sum_{\mu=-s}^{s} B_l^s (j, m, p) (\ell_1 s j M | \ell_1 s m_\ell \mu) \times$$

$$\times \frac{N(\ell_1)}{\sqrt{2}\,\Gamma(\ell_1 + \frac{3}{2})} \left(\frac{a}{2}\right)^{\ell_1} \phi_\mu^s$$

We notice therefore that in the states with a fixed four-dimensional angular momentum in the nonrelativistic limit the largest contribution is given by the wave with a minimum value of $\ell_1 = \ell_{1_{min}} = \frac{|m|}{2} + s - l + 2$

4. Definition of the Scalar Product

We pass now to the definition of the scalar product in the Hilbert space of functions $< a, \theta, \phi | p \frac{m}{2} jM >$. For this purpose we consider the functions $\Psi^{j(p)}_{M(\frac{m}{2})}(a, \theta, \phi)$ realizing unitary irreducible representations of the complete (with space reflections) Lorentz group. The functions $\Psi^{j(p)}_{M(\frac{m}{2})}(a; \theta, \phi)$ are bispinors, which are formed out of the functions $\Phi^j_M (a, \theta, \phi)$: $\Psi = \binom{\phi}{\chi}$, where the spinors ϕ and χ transform according to representations (s, o) and (o, s) respectively:

$$\Psi^{j(p)}_{M(\frac{m}{2})}(a, \theta, \phi) = \binom{\phi}{\chi} = \begin{pmatrix} < a, \theta, \phi | p \frac{m}{2} jM > \\ (-1)^{j+s+m} < a, \pi-\theta, \pi+\phi | p -\frac{m}{2} jM > \end{pmatrix} \qquad (26)$$

Here the operation of the space inversion is defined in the following manner:

$$\Psi'(a, \theta, \phi) = i\,\Gamma^0\, \Psi(a, \pi - \theta, \pi + \phi),$$

where

$$\Gamma^{0} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and I is a unit matrix of the dimension $(2s+1)\mathrm{x}(2s+1)$. The expression (26) can be obtained from an explicit representation of the generators of the complete Lorentz group:

$$\mathfrak{L}_{i} = \begin{pmatrix} L^{0}_{i} + L^{s}_{i} & 0 \\ 0 & L^{0}_{i} + L^{s}_{i} \end{pmatrix}$$

$$K_{i} = \begin{pmatrix} K^{0}_{i} - iL^{s}_{i} & 0 \\ 0 & K^{0}_{i} + iL^{s}_{i} \end{pmatrix}. \quad (27)$$

We define now a norm in the Hilbert space of functions $\Psi^{j(p)}_{M(\frac{m}{2})}(a,\theta,\phi)$ in the following way[x/]:

$$(\Psi^{j(p)}_{M(\frac{m}{2})}(a,\theta,\phi), \Psi^{j(p)}_{M(\frac{m}{2})}(a,\theta,\phi)) = \int_{0}^{\infty}\int_{0}^{\pi}\int_{0}^{2\pi} da\, d\theta\, d\phi\, \mathrm{sh}^{2}a \sin\theta\, \Psi^{+j(p)}_{M(\frac{m}{2})}(a,\theta,\phi)\Gamma^{0}\Psi^{j(p)}_{M(\frac{m}{2})}(a,\theta,\phi), \quad (28)$$

where Ψ^{+} is a hermitian conjugated of Ψ. Since the functions $\Psi^{j(p)}_{M(\frac{m}{2})}(a,\theta,\phi)$ form the basis for the irredicible representations of the Lorentz group, then the orthogonality relation should be

---

x/ This scalar product is invariant under the complete Lorentz group. We notice that the scalar product for the functions $\langle a,\theta,\phi | p\, \frac{m}{2}\, j M\rangle$ invariant only under the proper Lorentz group

$$(\Phi^{j}_{M}(a,\theta,\phi), \Phi^{j}_{M}(a,\theta,\phi))_{1} = \int_{0}^{\infty}\int_{0}^{\pi}\int_{0}^{2\pi} da\, d\theta\, d\phi\, \mathrm{sh}^{2}a \sin\theta\, \Phi^{+j}_{M}(a,\theta,\phi)\Phi^{j}_{M}(a,\theta,\phi) \quad (28')$$

leads to the divergent integral.

$$(\Psi^{j(p)}_{M(\frac{m}{2})}(a,\theta,\phi),\ \Psi^{j'(p')}_{M'(\frac{m'}{2})}(a,\theta,\phi)) = \delta_{mm'}\,\delta_{jj'}\,\delta_{MM'}\,\delta(p-p')$$

We remark that the density $\Psi^+\Gamma^0\Psi$ is not positively definite. The scalar product, given by the expression (28), is always positive, since the functions $\Psi^{j(p)}_{M(\frac{m}{2})}(a,\theta,\phi)$ realize unitary representation of the complete Lorentz group.

5. The Relation to the Solutions of the Free Field Equations

Since the bispinors $\Psi^j_M = \binom{\phi}{\chi}$ realizing a unitary irreducible representation of the complete homogeneous Lorentz group, have been built as a direct product of the finite-dimensional representations of the special type $(s,o)+(0,s)$ and the infinite-dimensional representations on the hyperboloid, then these representations may be extended to irreducible representations of the Poincare group. Taking into account that all the possible relativistic equations for free fields with fixed mass κ and spin s are derived from the considerations, that the solutions would transform according to the irreducible representations of the Poincare group /6/ (κ, s), it is obvious that the above constructed functions $\Psi^{j(p)}_{M(\frac{m}{2})}(a,\theta,\phi)$ are the solutions of such equations and form the complete set of functions in which arbitrary solution may be expanded.

As an illustration of this statement we choose the relativistic equations, solutions of which are the spinors, containing a minimum number of components, necessary to describe the spin s, and transforming according to the representation of the type $(s,o)+(o,s)$ /7/. Such equations do not require subsidiary conditions necessary to eliminate superfluous spinor components /8/.

These equations have the form

$$(\mathcal{P}^{(s)}(p) - \kappa^{2s})\Psi = 0, \tag{29}$$

where

$$\mathcal{P}^{(s)}(p) = -i^{2s}\gamma^{\mu_1\mu_2\cdots\mu_{2s}} p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{2s}}$$

$$p_{\mu_1} = -i\frac{\partial}{\partial q_{\mu_1}}$$

and the generalized Dirac γ -matrices are defined in the following manner:

$$\gamma^{\mu_1 \mu_2 \cdots \mu_{2s}} = -i^{2s} \begin{pmatrix} 0 & t^{\mu_1 \mu_2 \cdots \mu_{2s}} \\ \bar{t}^{\mu_1 \mu_2 \cdots \mu_{2s}} & 0 \end{pmatrix}$$

here

$$\bar{t}^{\mu_1 \mu_2 \cdots \mu_{2s}} = \pm\, t^{\mu_1 \mu_2 \cdots \mu_{2s}}$$

The sign + is taken for an even number of spacelike indices and the sign - for an odd number.

$(2s + 1)$ - rowed matrices $t^{\mu_1 \mu_2 \cdots \mu_{2s}}_{\sigma \sigma'}$ are the immediate generalization of the Pauli matrices and obey the following restrictions:

1. $t^{\mu_1 \mu_2 \cdots \mu_{2s}}_{\sigma \sigma'}$ are symmetrical for all indices μ ,
2. $g_{\mu_1 \mu_2} t^{\mu_1 \mu_2 \cdots \mu_{2s}} = 0$,
3. $t^{\mu_1 \mu_2 \cdots \mu_{2s}}_{\sigma \sigma'}$ is a tensor in the sense that

$$D^{(s)}(\Lambda) t^{\mu_1 \mu_2 \cdots \mu_{2s}} D^{(s)}(\Lambda)^{+} = \Lambda^{\mu_1}_{\ \nu_1} \Lambda^{\mu_2}_{\ \nu_2} \cdots \Lambda^{\mu_{2s}}_{\ \nu_{2s}} t^{\nu_1 \nu_2 \cdots \nu_{2s}} ,$$

where

$D^{(s)}(\Lambda)$ is a $(2s+1)$ -rowed matrix, corresponding to the Lorentz transformation Λ in the representation $(s, 0)$. It may be shown, that

$$\mathcal{P}^{(s)}(p) = m^{2s} \begin{pmatrix} 0 & \exp(-2a\vec{n}\vec{L}^{(s)}) \\ \exp(2a\vec{n}\vec{L}^{(s)}) & 0 \end{pmatrix} ,$$

where

$$\vec{n} = \frac{\vec{p}}{|\vec{p}|} , \qquad \mathrm{sh}\, a = \frac{|\vec{p}|}{m} .$$

We look for such a solution of (29), that the spinor $\phi = \langle s,\theta,\phi | p \frac{m}{2} jM \rangle$ realizes irreducible unitary representation of the proper Lorentz group, In this case it may be shown that the solution of (29) would be the bispinor (26). In the Appendices an explicit form of functions – the solutions of the equation (29) for spins $s = \frac{1}{2}$ and $s = 1$ is written down.

Generally speaking, the equation (29) as the equation of the $2s$ order has $2s$ independent solutions for each spinor component.

Actually, since

$$\gamma^{\mu_1 \mu_2 \cdots \mu_{2s}} \gamma^{\nu_1 \nu_2 \cdots \nu_{2s}} p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{2s}} p_{\nu_1} p_{\nu_2} \cdots p_{\nu_{2s}} = (p^2)^{2s},$$

then acting on the equation (29) by the operator $(-i)^{2s} \gamma^{\mu_1 \mu_2 \cdots \mu_{2s}} p_{\mu_1} p_{\mu_2} \cdots p_{\mu_{2s}} + \kappa^{2s})$ we get

$$[(p^2)^{2s} - (\kappa^2)^{2s}] \Psi = 0 . \qquad (30)$$

Hence we see that the $2s$ solutions correspond to $2s$ roots of the equation $(p^2)^{2s} - \kappa^2)^{2s} = 0$.

However, we pick out the unique "physical solution", corresponding to a free particle with the mass κ, by the "subsidiary condition" $p^2 = \kappa^2$ which fixes the phase and means that the basis is built on the upper sheet of the real hyperboloid.

In conclusion we make use of the opportunity to thank D.P.Zhelobenko, V.I.Ogievetsky and I.V.Polubarinov for illuminating discussions.

APPENDIX I

Spin 1/2

In this case the equation (29) reduces to the Dirac equation $(\mathcal{P}^{(1/2)}(p) - \kappa)\Psi = 0$, where $\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \chi_1 \\ \chi_2 \end{pmatrix}$,

or, taking into account (1), we have

$$\chi_1 = (\text{ch}a - \text{sh}a \cos\theta)\,\phi_1 + \text{sh}a \sin\theta\, e^{-i\phi}\,\phi_2$$

$$\chi_2 = \text{sh}a \sin\theta\, e^{i\phi}\,\phi_1 + (\text{ch}a + \text{sh}a\cos\theta)\,\phi_2 \qquad (1.1)$$

Spinor $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ is chosen in the form

$$\langle a, \theta, \phi \mid p\, \tfrac{1}{2}\, j\, M \rangle = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \frac{G(\frac{1}{2})}{\text{sh}^{1/2} a} \left(j + \tfrac{1}{2}\; \tfrac{1}{2}\; j\, M \mid j + \tfrac{1}{2}\; \tfrac{1}{2}\; M - \tfrac{1}{2}\; \tfrac{1}{2}\right).$$

$$(1.2)$$

$$\times \frac{(-1)^{M-1/2}}{\sqrt{2\pi}} \sqrt{\frac{(j+1)(j+1-M)!}{(j+M)!}} \left[P^{-(j+1)}_{-1+ip}(\text{ch}a) \begin{pmatrix} P^{M-1}_{j+\frac{1}{2}}(\cos\theta)\, e^{i(M-\frac{1}{2})\phi} \\ \frac{1}{(j-M+1)}\, P^{M+\frac{1}{2}}_{j+\frac{1}{2}}(\cos\theta)\, e^{i(M+\frac{1}{2})\phi} \end{pmatrix} + \right.$$

$$\left. + \frac{1}{(j+1-ip)}\, P^{-j}_{-1+ip}(\text{ch}a) \begin{pmatrix} \frac{(j+M)}{(j-M+1)}\, P^{M-\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta)\, e^{i(M-\frac{1}{2})\phi} \\ \frac{(-1)}{(j-M+1)}\, P^{M+\frac{1}{2}}_{j-\frac{1}{2}}(\cos\theta)\, e^{i(M+\frac{1}{2})\phi} \end{pmatrix} \right]$$

Having put the expression (I.2) into (I.1) we get the following expression for the spinor χ :

$$\chi(a, \theta, \phi) = (-)^{j+\frac{3}{2}} \langle a, \pi-\theta, \pi+\phi \mid p - \tfrac{1}{2}\; j\, M \rangle .$$

The wave functions are normalized by the condition[x]

$$(\Psi^{j(p)}_{M(\frac{1}{2})}, \Psi^{j'(p')}_{M'(\frac{1}{2})}) = |G(\tfrac{1}{2})|^2 \frac{(1+4p^2)|\Gamma(-\frac{1}{2}+ip)|^2}{|\Gamma(j+2+ip)|^2} \delta_{jj'} \delta_{MM'} \delta(p'-p). \tag{1.3}$$

APPENDIX II

Spin I

In this case the equation (29) takes on the form

$$(\mathcal{P}^{(1)}(p) - \kappa^2)\Psi = 0 \qquad \Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

or, taking into account (1), we have:

$$\begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{pmatrix} = \left[1 + 2 \begin{pmatrix} -\mathrm{sh}\,a\cos\theta & \frac{\mathrm{sh}\,a\sin\theta}{\sqrt{2}} e^{-i\phi} & 0 \\ \frac{\mathrm{sh}\,a\sin\theta}{\sqrt{2}} e^{i\phi} & 0 & \frac{\mathrm{sh}\,a\sin\theta}{\sqrt{2}} e^{-i\phi} \\ 0 & \frac{\mathrm{sh}\,a\sin\theta}{\sqrt{2}} e^{i\phi} & \mathrm{sh}\,a\cos\theta \end{pmatrix} \right.$$

$$\left. \times \begin{pmatrix} \mathrm{ch}\,a - \mathrm{sh}\,a\cos\theta & \mathrm{sh}\,a\sin\theta\, e^{-i\phi} & 0 \\ \frac{\mathrm{sh}\,a\sin\theta}{\sqrt{2}} e^{i\phi} & \mathrm{ch}\,a & \frac{\mathrm{sh}\,a\sin\theta}{\sqrt{2}} e^{-i\phi} \\ 0 & \frac{\mathrm{sh}\,a\sin\theta}{\sqrt{2}} e^{i\phi} & \mathrm{ch}\,a + \mathrm{sh}\,a\cos\theta \end{pmatrix} \right] \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \tag{II.1}$$

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[x] Having put the explicit expression for $|G(\frac{1}{2})|^2$ in (1.3) it is not difficult to make oneself sure that the normalizing coefficient does not de pend on j and therefore Ψ may always be normalized on δ -function.

We look for the solution of the form:

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} = \langle a,\theta,\phi \mid p\,1\,j\,M\rangle = \frac{G(1)}{\mathrm{sh}^{1/2} a}\,(j+1\;\; 1\,j\,M \mid j\,1\,M-1\,1)\,\frac{(-1)^{M-1}}{\sqrt{2\pi}}$$

$$\times\sqrt{\frac{(2j+3)(j-M+2)!}{2(j+M)!}}\Big[\, P^{-(j+\frac{3}{2})}_{-\frac{3}{2}+ip}(\mathrm{ch}\,a)\begin{pmatrix} P^{M-1}_{j+1}(\cos\theta)\, e^{i(M-1)\phi} \\ \dfrac{\sqrt{2}}{(j-M+2)}\, P^{M}_{j+1}(\cos\theta)\, e^{iM\phi} \\ \dfrac{1}{(j-M+1)(j-M+2)}\, P^{M+1}_{j+1}(\cos\theta)\, e^{i(M+1)\phi}\end{pmatrix} -$$

$$-\frac{(2j+1)}{j(j+2-ip)}\, P^{-(j+\frac{1}{2})}_{-\frac{3}{2}+ip}(\mathrm{ch}\,a)\begin{pmatrix} \dfrac{-(j+M)}{(j-M+2)}\, P^{M-1}_{j}(\cos\theta)\, e^{i(M-1)\phi} \\ \dfrac{-\sqrt{2}\,M}{(j-M+1)(j-M+2)}\, P^{M}_{j}(\cos\theta)\, e^{iM\phi} \\ \dfrac{1}{(j-M+1)(j-M+2)}\, P^{M+1}_{j}(\cos\theta)\, e^{i(M+1)\phi}\end{pmatrix} + \qquad (\mathrm{II}.2)$$

$$+\frac{(j+1)}{j(j+2-ip)(j+1-ip)}\, P^{-(j-\frac{1}{2})}_{-\frac{3}{2}+ip}(\mathrm{ch}\,a)\begin{pmatrix} \dfrac{(j+M)(j+M-1)}{(j-M+1)(j-M+2)}\, P^{M-1}_{j-1}(\cos\theta)\, e^{i(M-1)\phi} \\ \dfrac{-\sqrt{2}\,(j+M)}{(j-M+1)(j-M+2)}\, P^{M}_{j-1}(\cos\theta)\, e^{iM\phi} \\ \dfrac{1}{(j-M+1)(j-M+2)}\, P^{M+1}_{j-1}(\cos\theta)\, e^{i(M+1)\phi}\end{pmatrix}\Big]$$

Substituting the expression (II.2) in (II.1) after simple transformations we get

$$\chi(a,\theta,\phi) = (-1)^{j+1}\,\langle a,\pi-\theta,\pi+\phi \mid p-1\,j\,M\rangle .$$

The wave functions are normalized by the condition x/

$$(\Psi^{j(p)}_{M(1)},\Psi^{j'(p')}_{M'(1)}) = |G(1)|^2\,\frac{4(2j+1)(p^4+p^2)\,|\Gamma(-1+ip)|^2}{j\,|\Gamma(j+3+ip)|^2} \qquad (11.3)$$

$$\times\ \delta_{jj'}\ \delta_{MM'}\ \delta(p'-p).$$

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x/ See foot-note on p. 21.