ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2 - 3293

D.I. Blokhintsev

MACROSCOPIC CAUSALITY

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

1967.

E2 - 3293

D.I. Blokhintsev

MACROSCOPIC CAUSALITY

Chapter I

S-Matrix

1. Introduction

In 1942 W.Heisenberg [1] suggested his famous program of development of quantum field theory whioh was based on the idea to describe the elementary phenomena by means of the S-matrix, instead of the wave funotion.

The programm has not lost its importance in further development of theoretical and experimental physics. De facto methods based on the concept of wave function gave explicit metnods well before those based on the investigation of the S-matrix analytioal properties.

The death of the wave funotion seems to be obvious. However, it is too early to rejoice at this fact because the S-matrix apparatus has no continuation to the region of small intervals, to the very heart of elementary events.

Theoretical sohemes working only with the S-matrix resemble a faotory where there are only two departments: the department for reception of raw material and the department for packing finished artioles; whereas, the department for processing raw material is absent.

Analytic continuation of the S-matrix from the mass and energy surface allows us to look a little into this processing department, into the "very production". But the analytic continuation methods are not able to give a complete picture of the physical phenomena in the world of elemantary particles.

We realize that our present-day possibilities are very restricted, but there are two facts which speak in favour of the S-matrix methods;

a) There is, as yet, no one physical phenomenon in the world of elementary particles and in their interaction which could not be described in terms of the S-matrix.

b) The S- matrix may belong, at least formally, to the observables.

c) Therefore, we have every reason to consider the S-matrix as a theoretical construction which will conserve its importance in future theory.

In the light of such an aspect the investigation of the S-matrix is a rather reasonable trend in theoretical physics.

2. Main Properties of the S-Matrix

The most important properties of the S-matrix which are expected to be kept in future theory are the following: $S = 1 + i T$

1) The unitarity of the S-matrix:

$$S S^{+} = I \,. \tag{1}$$

This requirement leads to physically transparent relations, such as "optical theorem", which show the connection between various processes, namely, putting $S = 1 + i T$, we have from (1)

$$2 \, \mathrm{Im} \, T_{\alpha\alpha} = (T T^{+})_{\alpha\alpha} = \sum_{\beta} |T_{\alpha\beta}|^{2} . \tag{2}$$

These relations follow from the unitarity condition.

2) The relativistic invariance of the S-matrix. This requirement may be written in the form

$$\mathcal{L} S(q) = U S(\Lambda^{-1} q) U^{-1} = S(q') , \tag{3}$$

where q are the dynamical variables transforming under the Lorentz transformation $\mathcal{L}$

$$q' = \Lambda q \;, \tag{3}$$

$U(\Lambda)$ is the unitary matrix of this transformation.

The relativistic invariance may be violated only if future theory will be based on a geometry different from the Einstein–Minkovsky geometry.

3) Finally, the causality of the S-matrix.

In the method working with the wave function, Ψ from the Schrödinger equation

$$i \frac{\delta \Psi}{\delta \sigma(x)} = g \, W(x) \, \Psi \tag{4}$$

follows the condition [2]

$$[W(x), W(y)] = 0 \;, \tag{5}$$

for $(x-y)^2 < 0$ (i.e. for the space-like interval). As far as the interaction energy W is a local function of the fields $\varphi(x)$ then the condition (5) obeys the requirement

$$[\varphi(x), \varphi(x)] = 0 \;, \tag{6}$$

for $(x-y)^2 < 0$. This is the <u>microcausality</u> condition. This condition may be also extended to the S-matrix, if the latter is considered as a functional of the local field $\varphi(x)$

$$S = S\{\varphi(x)\} . \tag{7}$$

Then the microcausality may be formulated in the form:

$$\frac{\delta^2 S}{\delta\varphi(x)\,\delta\varphi(y)} S^{-1} = 0 , \tag{8}$$

for $(x-y)^2 < 0$.[3]

Here we consider in detail only the third requirement imposed on the S-matrix, the requirement of causality.

This is explained by the fact that the assumption on the existence of local fields appears to be the most weak point of current theory.

3. Causality and the S-matrix

Our task is to formulate the requirements of causality directly imposed on the S-matrix without recourse to the concept of field.

At first sight, such a formulation of the problem has the following unavailable contradition. The S-matrix transforms the state Ψ_{in} given at $t_1 = -\infty$ into the state Ψ_{out} studied at $t_2 = +\infty$

$$\Psi_{out} = \hat{S}\Psi_{in} . \tag{9}$$

These states are not localized in the space-time and therefore there are no preconditions for the formulation of a causal connection.

This fact may be also formulated as follows: the S-matrix is defined in the space of momentum-energy variables, in the many-dimensional Lobachevsky space $\mathcal{R}(p)$ (see Appendix I)[4], whereas for the description of the causal connection, the space-time variable defined in the many-dimensional Minkovsky space is needed.

Owing to this fact causality may be formulated in the language of the S-matrix with hat degree of definiteness which is compatible with the possibility of using simultaneously both spaces $\mathcal{R}(p)$ and $\mathcal{R}(x)$. As applied to the S-matrix, causality is, therefore, called by us macroscopic causality since it is just in macroscopic physics that the space $\mathcal{R}(p,x) = \mathcal{R}(p) \times \mathcal{R}(x)$ may be used.

Our next problems are: I. to establish necessary preconditions for the formulation of macrocausality conditions, and then 2) to formulate the conditions of microcausality.

Now we turn to the first precondition, without which further analysis is impossible.

4. Space-Time Description and the S-Matrix

The S-matrix transforms the states specified at $t_1 = -\infty$ in the state at $t_2 = +\infty$. What does the limit $\pm\infty$ means here?

The answer is the following: if the time of collision (time of particle interaction) is τ, and T is a long time interval then in the S-matrix theory T should be assumed to be infinitely long interval, provided that $\frac{\tau}{T}$ is kept to be finite and $\left(\frac{\tau}{T}\right)^2$ is neglected as being an infinitely small value, consequently, we neglect the remainder x)[5,6] $O\left(\frac{\tau^2}{T^2}\right)$:

$$\frac{\tau}{T} \gg \frac{\tau^2}{T^2} . \tag{10}$$

This requirement may be expressed in the language of distance. If v is the relative velocity of particles then the distance between them, corresponding to the time interval T, will be $R = vT$ Hence, we keep the quantities of the order $\frac{a}{R}$, where a is a certain length (" radius of sphere action", see appendix), and neglect the quantities of the order $\frac{a^2}{R^2}$ (the remainder $O\left(\frac{a^2}{R^2}\right)$);

$$\frac{a}{R} \gg \frac{a^2}{R^2} . \tag{10'}$$

Thus we may operate with finite time intervals

$$t_1 = -T \quad \text{and} \quad t_2 = +T . \tag{11}$$

---

x) Here we do not consider long-range interactions, like the Coulomb one, where τ is an indefinite quantity.

if only the condition (10) (or (10')) will be fulfilled. Restricting ourselves to finite time intervals, we consider the possibility of description of a collision process by means of packets localized in space and time.

To this end, in the next section, we will consider relativistic wave packets and formulate the conditions of macroscopic causality, using these packets.

Chapter II

The Wave Packets

5. Formulation of the Problem

The scattering matrix S for real "in" - and "out" states should obey certain causality conditions. However, these conditions may be formulated only if "in" states are given in the form of localized wave packets instead of plane waves.

In this connection it is necessary to consider possibilities of construction of narrow wave packets for relativistic particles, which do not spread essentially during the time $T = \frac{R}{v}$ much longer than during the collision time τ (here R is the distance between wave packets, and v -their relative velocity).

Thus, we are looking for wave packets which satisfy the conditions:

$$R \gg \Delta \gg \lambda \tag{1}$$

(λ the typical wave length, Δ is the dimension of wave packets, R the distance between them) and

$$|\Delta(T) - \Delta(-T)| \ll \Delta(-T) . \tag{1'}$$

The smaller is the wave length $\lambda\!\!\!^{-}$ the more precise are the conditions for the formulation of macroscopic causality for the S-matrix.

The matter is that in many papers devoted to the problem of the relativistic particle localization it is asserted that a spinor particle cannot be exactly localized since the states of positive energy do not form a complete set of functions. Therefore the eigenfunction $\delta(x-x')$ of the coordinate operator $\hat{x}$ cannot be expanded in the eigenfunctions corresponding only to positive energy states.

The same is related to spinless particles obeying the Klein equation.

We shall show that if quadratically integrable wave packets are used instead of the δ function then particles can be localized in positive energy states with any degree of accuracy.

6. Fermions

Firstly, we consider the case of Dirac particles. Let us take one-particle state, represented by a quadratically integrable wave function:

$$\psi(\vec{x}, t, \alpha) = \int c(\vec{p})\, u(\vec{p}, \alpha)\, e^{i(\vec{p}\vec{x} - Et)}\, d^3p \,, \tag{2}$$

where $E = +\sqrt{m^2 + \vec{p}^{\,2}}$, $u(\vec{p}, \alpha)$ – Dirac spinor, and

$$\int |c(p)|^2 d^3p = 1 \,, \tag{3}$$

$$Sp\, u^* u = 1 \,. \tag{4}$$

Now we calculate the mean square value of a coordinate, for instance, of z. Assuming that at t=0, $\bar{z}$ =0 we have obtained

after simple calculations:

$$\overline{\Delta z^2} = \overline{z^2} - \int \left|\frac{\partial c(\vec{p})}{\partial p_z}\right|^2 d^3p + \int |c(p)|^2\, Sp\left(\frac{\partial u^*}{\partial p_z}\frac{\partial u}{\partial p_z}\right) d^3p \,. \tag{5}$$

The last term is characteristic of the relativistic case.

Now we represent $C(\vec{p})$ in the form:

$$c(p) = f(\vec{\xi}) / p^{0\,3/2} \,, \quad \vec{\xi} = \frac{\vec{p}}{p^0} \,, \tag{6}$$

where p^0 is the quantity describing the momentum dispersion in the considered state:

$$\Delta p_z^2 \simeq p^{0\,2} \,. \tag{7}$$

The first integral in eq. (5) gives:

$$I_1 = \int \left|\frac{\partial c}{\partial p_z}\right|^2 d^3p = \frac{\alpha}{p^{0\,2}} \,. \tag{8}$$

The second integral is

$$I_2 = 4\pi \int_0^{\infty} |f(\xi)|^2 \xi^2 d\xi\, M(\xi) \,, \tag{8'}$$

where

$$M(\xi) = \int Sp\left(\frac{\partial u^*}{\partial p_z}\frac{\partial u}{\partial p_z}\right)\frac{d\Omega}{4\pi} \tag{8''}$$

and $M(\xi)$ is equal to

$$M(\xi) = \begin{cases} \dfrac{1}{4m^2} \,; & \xi \ll m/p^0 \\[2ex] \dfrac{1}{4m^2}\,\dfrac{4}{3}\,\dfrac{m^2}{p^{0\,2}}\,\dfrac{1}{\xi^2} \,; & \xi \gg \dfrac{m}{p^0} \,. \end{cases} \tag{9}$$

(See Appendix 3).

Therefore we have that at t=0

$$\overline{\Delta z^2} = \alpha \frac{\hbar^2}{\Delta p_z^2} + \beta \frac{\hbar^2}{m^2 c^2} \tag{10}$$

if $\Delta p_z^2 << m^2c^2$. For $\Delta p_z^2 >> m^2c^2$ we have

$$\overline{\Delta z^2} = \alpha' \frac{\hbar^2}{\Delta p_z^2}, \tag{10'}$$

where α, β, α' are of the order of unity. It is well seen that although in eq. (10) an additional term $\hbar^2/m^2c^2$ appears as if pointing out that the Dirac particle cannot be localized more exactly than within $\Delta z \sim \frac{\hbar}{mc}$ but, in fact, it is of no importance since at $\Delta p_z^2 \to \infty$ eq. (10) transforms into eq. (10').

Notice however that at $\Delta p_z^2 \to \infty$ the considered state is not described by the function:

$$\psi_{z'}(z) = \delta(z - z') \tag{11}$$

since this function is not quadratically integrable but the considered state is described by quadratically integrable functions. This quadratically integrable function $\psi_{z'}(z, p^0)$ localized about $z = z'$, is related to the function (11) as follows:

$$z\, \psi_{z'}(z, p^0) = z'\, \psi_{z'}(z, p^0) + \Delta(z - z', p^0) \tag{12}$$

$$\Delta(z - z', p^0) = p^{0\,1/2}\left[(z - z')\, \psi_{z'}(z, p^0)/p^{0\,1/2}\right] \tag{12'}$$

in this case

$$\psi_{z'}(z, p^0)/p^{0\,1/2} \to \delta(z - z_0)$$

at $p^0 \to \infty$. Therefore if the function $\psi_{z'}(z)$ is considered as an "ideal" eigenfunction of the operator of the coordinate z then the function $\psi_{z'}(z, p^0)$ approximates it so that $\Delta(z - z', p^0) \to 0$ at $p^0 \to \infty$ (see Appendix 4).

7. Bosons

Now we turn to the spinless particles, and consider again one-particle state. The field $\varphi(x)$ may be represented in the form:

$$\varphi(x) = \int A(\vec{k})\, U_k(x)\, d^3k, \quad U_k = \frac{e^{ikx}}{\sqrt{\omega}}, \tag{13}$$

where $kx = \vec{k}\vec{x} - \omega t$, $\omega = +\sqrt{m^2 + k^2}$

(see Appendix 5).

The density $\rho(x)$ is

$$\rho(x) = \frac{1}{2}\left[\Omega \varphi^* \varphi + \varphi^* \Omega \varphi\right], \quad \Omega = +\sqrt{m^2 - \nabla^2} \tag{14}$$

and, generally speaking, is non-definite even for positive-energy states $\omega = \sqrt{m^2 + k^2}$.

In this case it is also impossible to represent the δ-function as a superposition of waves U_k with $\omega > 0$.

Now let us consider localized states with integrable density ρ. We calculate the quantity $\overline{z^2}$ at $t = 0$ under the condition:

$$\int \rho(x)\, d^3x = 1. \tag{15}$$

We have

$$\overline{\Delta z^2} = \overline{z^2} = \frac{1}{2}\int z^2 (\Omega \varphi^* \varphi + \varphi^* \Omega \varphi)\, d^3x. \tag{16}$$

After simple calculations we find that:

$$\overline{z^2} = \int \left|\frac{\partial A}{\partial k_z}\right|^2 d^3k - \frac{1}{4}\int |A|^2 \frac{k_z^2}{\omega^4}\, d^3k. \tag{16'}$$

This expression is non-definite, therefore the density $\rho(\vec{x},0)$ cannot be treated as a density of any probability.

It might be expected that such "anomalies" in the bahaviour of $\rho(\vec{x})$ arise only when the density $\rho(\vec{x})$ is concentrated within $\Delta x \sim \frac{\hbar}{mc}$. But this not is the case: $\rho(x)$ may assume negative values also when $\Delta x \sim \frac{\hbar}{mc}$ (see Appendix 5B and 5C). Taking A in the form

$$A(\vec{k}) = A(\omega) = f\left(\frac{\omega}{\omega_0}\right)\frac{1}{\omega_0^{3/2}}, \quad \omega/\omega_0 = \xi,$$

we find

$$\overline{x^2} = \frac{1}{\omega_0^2}\int_{\xi > m}\left[|f'|^2 - \frac{1}{4\xi^2}|f|^2\right]\frac{\xi_x^2}{\xi^2}\,d^3\xi \tag{16}$$

It is not difficult to choose such function f that $\left[f'^2 - \frac{1}{4\xi^2}|f|^2\right] > 0$. Then it is seen, that at $\omega_0 \to \infty$, $\overline{x^2} = 0$ and we come to the state with a well localized density, i.e. density which at t=0 is concentrated within an arbitrary small region $\Delta z \sim \frac{\hbar}{\omega_0} \to 0$ (see Appendix 5).

Thus, as far as the possibility of localization concerns, the situation is quite similar to that which takes place for the Dirac particle, however, the $\rho(x,t)$ for spinless particle might not be interpreted as the density of the probability to detect the particle near the point $\vec{x}$ at time t.

The quantity $\rho(\vec{x},t)$ should be considered as a purely "field" quantity representing a spinless particle in space-time.

8. Spread of Wave Packets

Now we consider the bahaviour of relativistic wave packets in the course of time. All the above discussed states localized at t=0 are spreading: the quantities $\overline{\Delta x^2}, \overline{\Delta y^2}, \overline{\Delta z^2}$ increase. However, this increase is such that under certain conditions it may be said that the relativistic packet is moving during a rather long time T conserving its characteristic size.

In other words, the change in the packet size during time T may be small as compared with its initial size even for long time intervals. Here a long time interval implies such interval that $R = CT \gg \Delta x, \Delta y, \Delta z$, where $\Delta x, \Delta y, \Delta z$ are taken at t=0, C is the velocity of light.

It is easy to show that the packet width $\Delta_\parallel$ measured in the direction parallel to the packet motion increase with t according to the law:

$$\Delta_\parallel^2(t) = \Delta_\parallel^2(0) + \frac{\bar{\lambda}}{\Delta_\parallel^2(0)}\frac{m^4}{E^4}v^2t^2 \tag{17}$$

and the width $\Delta_\perp$ measured in the direction perpendicular to the packet motion increases according to the law

$$\Delta_\perp^2(t) = \Delta_\perp^2(0) + \frac{\bar{\lambda}}{\Delta_\perp^2(0)}v^2t^2 = \Delta_\perp^2(0) + \frac{\hbar^2}{m^2c^2}\frac{1}{\Delta_\perp^2(0)}\frac{m^2}{E^2}t^2. \tag{17'}$$

Here $\bar{\lambda}$ is the particle wave length, $v = \frac{\partial E}{\partial p}$ is the packet velocity, m is the particle rest mass, $\Delta^2(0)$ is the value of $\Delta^2(t)$ at t=0 (see Appendix 7). From these equations it following that

$$\left|\frac{\Delta^2(t) - \Delta^2(0)}{\Delta^2(0)}\right| \ll 1 \tag{18}$$

if

$$R = ct < \frac{\Delta^2(0)}{\lambda} \,.$$

Now we come back to the conditions (1),(1') and combine them with the result (1). We find the inequalities:

$$\Delta \frac{\Delta}{\lambda} > R >> \Delta >> \lambda \qquad (19)$$

which can be realized for any t under the condition that $\lambda \to 0$ (i.e. $v \to c$).

This important condition of a possible long existence of a localized relativistic packet is exclusively the result of the relativistic effect: increase of the particle mass with increasing velocity. Thus we see that the present day theory in a formal way (because there is no practical way to construct arbitrary narrow s split) permits the oneparticle states which are localized in space with any degree of accuracy $\Delta \to 0$ for the time intervals $T < \frac{\Delta^2}{\lambda c} \to \infty$ (at $\lambda \to 0$). This gives the possibility to formulate conditions of macroscopic causality directly for S-matrix, taken on the mass- and energy surfaces.[6,7]

Chapter III

Condition of Macroscopic Causality

9. Description of Collisions by Wave Packets

In the foregoing we have shown that possibility of constructing the wave packets which keep their dimensions during the time $T = R/v$ (v - is the relative velocity of packets, R- is the distance between packets). The condition which restricts the distance R reads

$$\frac{\Delta^2}{\lambda} >> R >> \Delta >> \lambda \,, \qquad (1)$$

where Δ - is the dimention of the wave packet. Under these conditions the wave packet retains its dimensions during the time $T = \frac{R}{v}$ and represents the state of a particle with the momentum $p = \hbar/\lambda$ localized in the region Δ .

Fig.1 shows the description of the particle collision by wave packets. In a non-transparent screen $x'x''$ there are two diagrams A and B which are opened during a short time $t_1 = -T$ so that there appear localized wave packets $u_1(x_1)$ and $u_2(x_2)$, removed apart at the distance AB=R. These packets are moving along the lines $u_1 u_1'$ and $u_2 u_2'$, increasing in a certain degree their dimensions. The packets $u_1'(x_1)$ and $u_2'(x_2)$ are the same packets in the time $t_2 = T$.

When the condition (1) is fulfilled the dimensions of these packets little differ from those of the initial packets u_1 and u_2 .

In the region S the wave packets begin to interact between them. This region is a source of secondaries and scattered waves.

Now we turn to mathematical description of the collision of these packets, using the S-matrix theory. We write the S-matrix in the form:

$$\langle f/S/i \rangle = \delta_{fi} - (2\pi)^4 i\, \delta^4(\mathcal{P}_f - \mathcal{P}_i) \langle f/T/i \rangle, \qquad (2)$$

where as usual (i) denote the quantum numbers of the in- state Ψ_i and (f) are those for the out-state Ψ_f . $\mathcal{P}_i$ is the total momentum in the in-state, $\mathcal{P}_f$ is the same quantity for the out--state . The matrix element $\langle f/T/i \rangle$ can be represented in a more detailed form:

$$\langle f|T|i\rangle = \frac{\langle P_m P_{m-1} \dots P_{n+1}|I|P_n P_{n-1} \dots P_1\rangle}{\sqrt{2P_m^0\, 2P_{m-1}^0 \dots 2P_1^0}}, \tag{3}$$

where $\langle P_m, P_{m-1} \dots P_{n+1}|I|P_n, P_{n-1} \dots P_1\rangle$ is the invariant function of the momenta $P_m, P_{m-1} \dots P_1$; $P_m^0, P_{m-1}^0 \dots P_1^0$ are their fourth components. Further

$$\mathcal{P}_f = P_m + P_{m-1} + \dots + P_{n+1} \tag{4}$$

$$\mathcal{P}_i = P_n + P_{n-1} + \dots + P_1 \; . \tag{4'}$$

In what follows for the sake of simplicity, we shall restrict ourselves to the simplest case of the collision of two scalar particles. In this case the in-state Ψ_i is represented by two wave packets $u_1(x_1)$ and $u_2(x_2)$ of the above-considered type. For the scalar particles these packets can be represented in the form of the integrals:

$$u(x) = \frac{1}{(2\pi)^{3/2}} \int \tilde{u}(\vec{p}) \exp[ipx] \frac{d^3p}{2p^0}, \tag{5}$$

where $p^0 = +\sqrt{\vec{p}^2 + m^2}$. The wave function of the initial state in the momentum representation will be of the form:

$$\Psi_{in}(P_2 P_1) = \frac{\tilde{u}_2(\vec{P}_2)\, \tilde{u}_1(\vec{P}_1)}{\sqrt{2P_2^0\, 2P_1^0}} \; . \tag{6}$$

From (3) (5) (6) we get :

$$\Psi_{out}(P_m, P_{m-1}, \dots P_3) = -(2\pi)^4 i \int \delta^4(\mathcal{P}_f - \mathcal{P}_i) \times$$

$$\times \frac{\langle P_m, P_{m-1}, \dots P_3|I|P_2 P_1\rangle}{\sqrt{2P_m^0\, 2P_{m-1}^0 \dots 2P_3^0}}\, \tilde{u}_2(\vec{P}_2)\, \tilde{u}_1(\vec{P}_1) \frac{d^3P_2\, d^3P_1}{\sqrt{2P_2^0\, 2P_1^0}} \tag{7}$$

and for m=4 (the elastic collision)

$$\Phi_{out}(P_4, P_3) = \Phi_{in}(P_4, P_3) - (2\pi)^4 i \int \delta^4(P_4 + P_3 - P_2 - P_1) \times$$

$$\times \frac{\langle P_4, P_3|I|P_2 P_1\rangle}{\sqrt{2P_4^0\, 2P_3^0}}\, \tilde{u}_2(\vec{P}_2)\, \tilde{u}_1(\vec{P}_1) \frac{d^3P_1\, d^3P_2}{2P_1^0\, 2P_2^0} \tag{8}$$

10. Space-Time Description

Now we go over to the co-ordinate representation.[7] For this we multiply the left-hand side of (8) by

$$\frac{1}{(2\pi)^{3/2(m-2)}} \frac{\exp[i(P_m x_m + P_{m-1} x_{m-1} + \dots + P_3 x_3)]}{\sqrt{2P_m^0\, 2P_{m-1}^0 \dots 2P_3^0}}$$

and integrate over $d^3P_m\, d^3P_{m-1} \dots d^3P_3$. Further by (5) we express $\tilde{u}(\vec{P})$ in terms of $u(x)$:

$$\frac{\tilde{u}(P)}{2P^0} = \frac{1}{(2\pi)^{3/2}} \int u(x) \exp[-iPx]\, d^3x \; . \tag{5}$$

Then from (7) we get

$$\Phi_{out}(x_m, x_{m-1}, \dots x_3) = -(2\pi)^4 i \int g(x_m, x_{m-1}, \dots x_3|x_2 x_1) \times u_2(x_2)\, u_1(x_1)\, d^3x_2\, d^3x_1 \tag{9}$$

and in similar way from (8)

$$\Phi_{out}(x_4 x_3) = \Phi_{in}(x_4 x_3) - (2\pi)^4 i \int g(x_4, x_3|x_2 x_1)\, u_2(x_2)\, u_1(x_1)\, d^3x_2\, d^3x_1 \tag{9'}$$

In this case we have

$$g(x_m, x_{m-1}, \dots x_3|x_2, x_1) =$$

$$= \int \delta^4(P_m + P_{m-1} + \dots + P_3 - P_2 - P_1) \langle P_m, P_{m-1}, \dots P_3|I|P_2, P_1\rangle \times \tag{10}$$

$$\frac{\exp[i(P_m x_m + \dots + P_3 x_3 - P_2 x_2 - P_1 x_1)]}{2P_m^0\, 2P_{m-1}^0 \dots 2P_3^0}\, d^3P_m\, d^3P_{m-1} \dots d^3P_3$$

or

$$g(x_m, x_{m-1}, \dots x_3 / x_2 x_1) = -\frac{4\partial^2 g_0(x_m, x_{m-1} \dots x_3 / x_2 x_1)}{\partial t_2 \partial t_1}, \qquad (11)$$

where g_0 is the invariant function of the co-ordinates

$$g_0(x_m, x_{m-1}, \dots x_3 / x_2, x_1) = \int \delta^4(p_m + p_{m-1} + \dots + p_3 - p_2 - p_1) \times$$
$$\langle p_m, p_{m-1}, \dots p_3 / I / p_2, p_1 \rangle \exp[i(p_m x_m + \dots + p_3 x_3 - p_2 x_2 - p_1 x_1)] \frac{d^3p_m \, d^3p_{m-1} \dots d^3p_1}{2p_m^0 \, 2p_{m-1}^0 \dots 2p_1^0} \qquad (12)$$

We notice that due to the presence of the δ-function under the integral in g and g_0 these functions are translation-invariant and depend only on the difference of the variable $x_m, x_{m-1} \dots x_1$.

II. Conditions of Macroscopic Causality

Now we may formulate the principle of macrocausality:

a) the wave packets $u_2(x_2)(\Delta x_2 \sim L)$ and $u_1(x_1)(\Delta x_1 \sim L)$ removed apart at the distance

$$|\vec{x}_2 - \vec{x}_1| = |x| > L >> \bar{\lambda} \qquad (13)$$

contribute to Φ_{out} provided only that

$$x^2 = (t_2 - t_1)^2 - (\vec{x}_2 - \vec{x}_1)^2 > 0 \; . \qquad (14)$$

b) Further $\Phi_{out} = 0$ if the co-ordinates of the particles $x_m, x_{m-1}, \dots, x_3$ created in the collision lie out of the future light cone with respect to the points x_2, x_1

$$(x_s - x_2)^2 > 0 \qquad (x_s - x_1)^2 > 0 \qquad (15)$$

$$t_s > t_2 \qquad t_s > t_1 \qquad (15')$$

$s = m, m-1, \dots 3$. Thus the function $g(x_m, x_{m-1}, \dots, x_3 / x_2 x_1)$ must consequently vanish outside the above-mentioned space-time regions, however, only asymptotically, i.e. for

$$R \to \infty, \; (t_s - t_2), \; (t_s - t_1) \to \infty \; . \qquad (16)$$

From the physical point of view these conditions are identical with the requirements of classical macroscopic causality and imply the assumption that all the particles in the final state Φ_{out} can be produced (or change their state) later than the initial packets exchange the field quanta (see Fig.2).

The usual local theory satisfies, of course, the above stated requirement of macrocausality.

This requirement will be satisfied also by any scattering matrix in which the microcausality is violated only in a small localized space-time region.

12. Some Properties of the Co-Ordinate Representation [7]

We represent the S-matrix in the form

$$\hat{S} = e^{i\hat{\eta}} = \sum_{s=0}^{\infty} \frac{i^s}{s!} \hat{\eta}^s , \qquad (1)$$

where $\hat{\eta}$ is the phase operator. This operator is an Hermitian one

$$\hat{\eta} = \hat{\eta}^+ . \qquad (2)$$

The condition (2) provides the unitarity of the operator. In the matrix form, in the asymptotic space $R(p)$ eqs.(1) and (2) read:

$$\langle p|\hat{S}|p'\rangle = \sum_{s=0}^{\infty} \frac{i^s}{s!} \langle p|\hat{\eta}^s|p'\rangle \qquad (1')$$

$$\langle p|\hat{\eta}|p'\rangle = \langle p'|\hat{\eta}|p\rangle^* \qquad (2')$$

the elements $\langle p|\hat{\eta}|p'\rangle$ being of the form

$$\langle p|\hat{\eta}|p'\rangle = \frac{\delta^4(\mathcal{P}-\mathcal{P}')\,\eta(p,p')}{\sqrt{2p^0 \ldots 2p^{0\prime}}} , \qquad (3)$$

where $\mathcal{P} = \sum p$, $\mathcal{P}' = \sum p'$ are the total momenta in the initial and final states. The form (3) ensures the validity of the law of multiplication

$$\langle p|\hat{\eta}^2|p'\rangle = \int \langle p|\hat{\eta}|p''\rangle\, d\omega(p'')\, \langle p''|\hat{\eta}|p'\rangle , \qquad (4)$$

where

$$d\omega(p) = \frac{d^3p}{2p^0} \qquad (5)$$

is the volume element in the Lobachevsky's space

$$p^0 = +\sqrt{\vec{p}^2 + m^2} .$$

Now we consider the S-matrix in the co-ordinate space $R(x)$.

The matrix element $\hat{\eta}$ is now written in the form

$$\langle x|\hat{\eta}|x'\rangle = \langle x|\hat{\eta}^+|x'\rangle = \langle x'|\hat{\eta}|x\rangle^* . \qquad (6)$$

The multiplication of this matrix is defined by the law:

$$\langle x|\hat{\eta}^2|x'\rangle = \int \langle x|\hat{\eta}|x''\rangle\, \Delta^+(x''-x''') \times \langle x'''|\hat{\eta}|x'\rangle\, d^4x''\, d^4x''' , \qquad (7)$$

where $\Delta^+(x)$ is the positive-frequency singular function

$$\Delta^+(x) = \int \frac{e^{ipx}\, d^3p}{2p^0} , \quad p^0 = \sqrt{\vec{p}^2 + m^2} . \qquad (8)$$

This function has the property

$$[\Delta^+(x-y)]^+ = \Delta^+(x-y) . \qquad (9)$$

In the transition to the momentum representation

$$\langle p|\hat{\eta}|p'\rangle = \int \frac{e^{ipx}}{\sqrt{2p^0}} \langle x|\hat{\eta}|x'\rangle \frac{e^{ip'x'}}{\sqrt{2p^{0\prime}}}\, d^4x\, d^4x' \qquad (10)$$

this law of multiplication automatically reproduces the law of multiplication in the Lobachevsky's space and returns us to the elements of the $\hat{\zeta}$ matrix represented in the form (4). In this case the appearance of the function $\delta^4(\mathcal{P}-\mathcal{P}')$ ensuring the validity of the conservation law of the total momentum $\mathcal{P}=\sum p$, $\mathcal{P}'=\sum p'$ is due to the translation invariance of the matrix elements. Owing to $\langle x/\hat{\zeta}/x'\rangle$ the fact that the space-time is homogeneous, these matrix elements are functions of only the differences of the variables x and x'. In view of the transformation (10) we note that the elements of the phase operator $\langle x/\hat{\zeta}/x\rangle$ may have a spectral expansion going beyond the limits of the space $\mathcal{R}(p)$.

For further consideration it is more convenient not to distinguish between the variables $x=(x_1, x_2 \ldots x_m)$ related to the final state and the variables $x'=(x_1', x_2' \ldots x_n')$ related to the initial state and denote all by

$$x=(x_1, x_2, \ldots x_m, x_{m+1}, x_{m+2}, \ldots x_N).$$

The matrix element

$$\langle x/\hat{\zeta}/x'\rangle \equiv \zeta(x_1, x_2, \ldots x_i, \ldots x_k, \ldots x_N) \tag{11}$$

will be the function of the differences $x_i - x_k$.

In local theory microcausality is displayed by the appearance in this matrix element of definite singularities which are located on the light cones:

$$(x_i - x_k)^2 = 0 \tag{12}$$

or at the points

$$x_i = x_k. \tag{12'}$$

13. Acausal S-Matrix

Instead of the local matrix (11), we consider now a non-local acausal matrix in which the singularities characteristic of the local matrix are excluded completely or partially.[5] This new matrix $\hat{\zeta}$ is considered by us as a function of a four-dimensional time-like unit vector n. We denote the elements of this matrix by

$$\langle x/\hat{\zeta}/x', n\rangle = \zeta(x_1, x_2, \ldots x_i \ldots x_k \ldots x_N, n) \tag{13}$$

$$n^2 = 1. \tag{13'}$$

We introduce the functions of the space point $\rho(\xi, n)$ by means of which we want to eliminate (completely or partially) singularities characteristic of local theory.

Basing on certain considerations which will be presented below we call these functions "pseudo-sources".

For the sake of definiteness, these functions are assumed to be the functions of the invariant x)

$$R^2 = \frac{1}{2}\left[(\xi n)^2 - \xi^2\right] \tag{14}$$

---

x) This assumption is not, of course, obligatory.

which, in the frame of reference where $n = (1,0,0,0)$, degenerates in a three-dimensional sphere:

$$R^2 = \vec{\xi}^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \quad . \tag{15}$$

Thus, we suppose that

$$\rho(\xi, n) = f\left(\frac{R^2}{a^2}\right), \tag{16}$$

where a is a certain lenght characteristic of the scale of the space region $(\sim a^3)$ inside which causality is violated (it is assumed that the function f rather rapidly tends to zero at $R/a \to \infty$)

Suppose that the matrix element (II) has a singularity in the variable $x_i - x_k$; we eliminate this singularity by averaging the element (II) over the function of the pseudo-source:

$$\begin{aligned} &\mathfrak{z}(x_1, x_2, \dots x_i - x_k, \dots x_N, n) = \\ &= \int \mathfrak{z}(x_1, x_2, x_3 \dots x_i - x_k - \xi, \dots x_N)\, \rho(\xi, n)\, d^4\xi \; . \end{aligned} \tag{17}$$

If the vertex of the light cone $(x_i - x_k)^2 = 0$ is considered as a source of singularity then the averaging (17) means the replacement of the point source by the extended source which has the volume of the order of $\sim a^3$. If the singularity is at the point $(x_i = x_k)$ then this point is replaced by the volume $\sim a^3$.

Thus the averaging reduces to the replacement of the point sources by the extended ones; that is why we called the function

$\rho(\xi, n)$ - "pseudo-source". When averaging microcausality is violated only in a small space region which is localized in the volume a^3 (which is, naturally, assumed to be small). Owing to this fact, macrocausality is violated, at all, by averaging (17).

Notice that the Fourier transform of the pseudo-source $\rho(\xi, n)$ (16) is of the form

$$\tilde{\rho}(q, n) = \tilde{f}\left(\tilde{R}^2 / a^2\right), \tag{18}$$

where

$$\tilde{R}^2 = \frac{1}{2}\left[(qn)^2 - q^2\right] \; . \tag{18'}$$

From (17) it is seen that the Fourier transform of the acausal matrix $\mathfrak{z}(x, n)$ (17) is simply the product of the Fourier transform of the local matrix $\mathfrak{z}(x)$ (11) and that of the pseudo-source (18):

$$\langle p | \hat{\mathfrak{z}}(n) | p' \rangle = \langle p | \mathfrak{z} | p' \rangle \cdot \tilde{\rho}(q, n) \; . \tag{19}$$

Hence, it follows that if the Fourier transform of the matrix $\mathfrak{z}(x)$, $\tilde{\mathfrak{z}}(p)$ is the Hermitian matrix, then the Fourier transform of the matrix $\mathfrak{z}(x, n)$, $\tilde{\mathfrak{z}}(p, n)$ will be also Hermitianm provided that the vector n either is indepednet of the vectors, or is their symmetrical function.

The Hermitioity of the acausal phase matrix $\tilde{\zeta}(P,n)$ provides the unitary of the acausal scattering matrix $\hat{S}(n)$:

$$\hat{S}(n) = e^{i\hat{\zeta}(n)} . \tag{20}$$

Thus it is shown, that one may oonstruot the scattering matrix S in whioh: a) microoausality is violated, but b) maorocausality and c) unitarity are satisfied.

14. Remarks on the Functions $\rho(x,n)$

Now we oonsider the problem of the choioe n by means of which pseudo-sources $\rho(x,n)$ were construoted. In principle, there are two possibilities: a) the veotor n is not oonneoted with the system of interacting particles. In this case we oall the vector n external. The assumption about the existence of such an external vector single out with neoessity some frame of reference (or some frames of referenoes) and thereby violate the usual interpretation of relativistic invariance.

Such singled out frame of referenoe may turn out to be the physioal vaccum system.[8] We shall not discuss in what follows to what degree such assumptions are compatible with the known physical facts. In any case, the situation is not trivial, at all. It seems that only comparison of the results on colliding beams with those on fixed target may clear up these problems.

b) Turn to the second possibility when the vector n is a function of the dynamical variables of the system of interesting particles (their momenta). Now we call the veotor n <u>internal</u>.

In this case we may totally oonserve relativistio invarianoe in its usual interpretation.

It is only causality that will be violated, and only in a small space, of the order of a^3 which is connected with the interaction partiole region. The most natural way of introducing the internal vector is to identify it with the total momentum P vector of the system more exactly, with the unit vector $P/\sqrt{P^2}$:

$$n = P/\sqrt{P^2} . \tag{21}$$

This may be done because the S-matrix has no non-zero matrix elements between states with different total momentum P and P', $P' \neq P$:

$$(P|\hat{S}|P') = 0 . \tag{22}$$

The same relates to the $\hat{\zeta}$ matrix

$$(P|\hat{\zeta}|P') = 0 , \quad \text{for } P \neq P' .$$

Moreover, if the matrix element $\zeta(x, x_2 \ldots x_N)$ is divided into complexes with smaller number of variables:

$$\zeta(x_1, x_2 \ldots x_k, x_{k+1}, \ldots x_N) = \zeta'(x_1, x_2 \ldots x_k) \times \times \zeta''(x_{k+1}, x_{k+2}, \ldots x_N) \tag{23}$$

then such a complex in the momentum representation (due to trans - lation invariance) has again no matrix elements with $P \neq P'$.

Therefore the momentum P may be ascribed to the total matrix as well as to individual complexes $\xi(x), \xi'(x), \xi''(x)$ etc. It is clear that if one or several particles of a complex are removed from the others at a large distance this complex vanishes for finite sphere of particle interaction).

Therefore one should not think that the removed particle may affect the others only because its momentum is involved in the total momentum; when a particle is removed apart from the others, the corresponding matrix element must tend to zero x) . Thus, in eq. (19) the vector n may be associated with the total momentum of the system P (according to eq. 21).

Now we turn to the analytical properties of the acausal matrix S. From (19) it follows that in the example under consideration the analytical properties of the S matrix are new as compared to those of the S matrix in local theory, and defined by the analytical properties of the Fourier transform $\tilde{\rho}(q,n)$ of the pseudo-source $\rho(\xi,n)$.

The new singularities of the S matrix are identical with those for the function $\tilde{\rho}(q,n)$. The problem of the nature of allowed singularities is not yet sufficiently investigated.

In paper [5] a particular case is considered when these singularities are poles located symmetrically on the imaginary axis

x/ This fact is not connected with the acausalitz considered by us: in the usual local theorz the scattering matrix also depends on the total momentum of all the particles involved in the reaction and when one of them is removed apart the corresponding matrix element tends to zero.

In conclusion it should be noted that the properties of the functions $\rho(\xi,n)$ are introduced as a method for constructing the acausal matrix, as an example.

The refusal from microcausality in a small scale will give rise to serious geometrical consequences: the concept of four-dimensional pseudo-Euclidean co-ordinates of the point t, x, y, z may loss in this situation not only its meaning (if it is not already the case in modern field theory) but also a purely formal meaning. Therefore it is more reasonable to consider our constructive method as a formal way to map a "magic circle" (the scale of this circle is defined by the elementary length a) inside which a situation may occur which radically differs from that given by the modern theory.

Conclusion

The S matrix is, in principle, a physically observable quantity. This suggests an idea that one concept of S matrix will survive the modern local theory and will be a part of future theoretical conceptions. However, this "future" S matrix will, apparently, obey the routine conditions a) unitarity, b) causality.

In the present investigation we admitted condition a) as obligatory and focused out attention of condition b). We showed that causality, as applied to the S matrix, contains the contradiction which is based on the complementarity of the space-time description and the momentum energy description. This contradiction may be reduced only in the framework of essentially weakened requirements of causality i.e. in the framework of the macrocausality.

These weekened requirements make it impossible to catch the particles in violation of the local microcausality, even if the latter is violated.

Therefore the limits of macrocausality turn out to be "tolerable" and include a very large number of acausal theories.

APPENDIX I

The Lobachevsky's space is the space of a constant of negative curvature K. In the three dimensional case this space is described by the metric

$$dS^2 = dp_1^2 + dp_2^2 + dp_3^2 - \frac{(p_1 dp_1 + p_2 dp_2 + p_3 dp_3)^2}{m^2 + p_1^2 + p_2^2}, \qquad (1)$$

where dS is the interval between two infinitely close points with the co-ordinates (p_1, p_2, p_3) and $(p_1+dp_1,\ p_2+dp_2,\ p_3+dp_3)$. The quanta $K = -\frac{1}{m^2}$ determines the space curvature, m is the radius of curvature.

To the metric (1) there corresponds the element of the vOlume:

$$dw = \| Det\, g_{ik} \| =$$

$$= \frac{dp_1 dp_2 dp_3}{|p^0|} = \frac{dp_1 dp_2 dp_3}{\sqrt{m^2 + p_1^2 + p_2^2 + p_3^2}} .$$

Here g_{ik} is the metric tensor of the form (1). As will be seen from § 9, in the matrix elements of the operators and in the wave functions in the momentum representation there appear multipliers, like $\frac{1}{\sqrt{2p^0}}$, which are, at first sight, noncovariant. However, after the matrices have been multiplied, these factors lead to the appearance of the expressions which ensure the covariance: This expression is the element of the volume in the Lobachevsky's space $\mathcal{R}(p)$.

APPENDIX 2

Action Sphere a

Some idea about the comparison of the terms $\frac{a}{R}$ and $\frac{a^2}{R^2}$ may be obtained from the theory of elastic scattering.

Let $\psi(x)$ be the wave function and $V(x)$ the interaction energy. Then

$$\psi(x) = \psi_0(x) + \int g(x-x')\, V(x')\, \psi(x')\, d^3x', \qquad (1)$$

where $\psi_0 = e^{i\kappa x}$ is the incident wave with the momentum κ, $g(x-x')$ is the Green function

$$g(x-x') = \frac{1}{4\pi}\, \frac{e^{i\kappa|x-x'|}}{|x-x'|}\, . \qquad (2)$$

Expanding $g(x-x')$ in the inverse powers of $R=|x|$, we find

$$\psi(x) = \psi_0(x) + \frac{A}{4\pi}\,\frac{e^{i\kappa R}}{R} + \frac{B}{4\pi}\,\frac{e^{i\kappa R}}{R^2} + O(1/R^3). \qquad (3)$$

(4)

R may be assumed to be a large quantity, if

$$\frac{A}{R} >> \frac{B}{R} \qquad \text{i.e. if} \qquad R >> \frac{B}{A}\, . \qquad (5)$$

APPENDIX 3

The spinor $u_r(P,\alpha)$ can be written for $E>0$ in the form

$r=1$

$u(1) = N$

$u(2) = 0$

$u(3) = \frac{P_z N}{m+E}$

$u(4) = \frac{\Pi N}{m+E}$

$r=2$

$u(1) = 0$

$u(2) = N$

$u(3) = \frac{\Pi^* N}{m+E}$ (1)

$u(4) = -\frac{P_z N}{m+E}$

$\Pi = P_x + iP_y$

$$N = \frac{1}{\sqrt{2}}\left(1+\frac{m}{E}\right)^{1/2}$$

Hence, it is seen that the traces of bilinear combinations for r=1 and 2 are identical. A simple calculation gives

$$Sp\left(\frac{\partial u^*}{\partial P_z}\frac{\partial u}{\partial P_z}\right) = \frac{1}{8}\left(1+\frac{m}{E}\right)^{-1}\frac{m^2P_z^2}{E^6} + \frac{1}{2}\left(1+\frac{m}{E}\right)^{-1}\frac{1}{E^2}\Big\{1 + \frac{P_z^4}{\left(1+\frac{m}{E}\right)^2E^4} + \frac{1}{4}\frac{m^2P_z^4}{\left(1+\frac{m}{E}\right)^2E^6} - \frac{2P_z^2}{\left(1+\frac{m}{E}\right)E^2} - \frac{mP_z^2}{\left(1+\frac{m}{E}\right)E^3} + \frac{mP_z^4}{\left(1+\frac{m}{E}\right)^2E^5}\Big\} + \frac{1}{2}\frac{(E^2-m^2-P_z^2)P_z^2}{E^6}\left(1+\frac{m}{E}\right)^{-3} \times \left(1+\frac{1}{2}\frac{m}{E}\right)^2 \geq 0 \qquad (2)$$

Noting that

$$\int p_z^2 d\Omega = \frac{4}{3}\pi p^2 , \int p_z^4 d\Omega = \frac{4}{5}\pi p^4 \tag{3}$$

we find

$$M = \frac{1}{4\pi}\int Sp\left(\frac{\partial u^*}{\partial p_z}\frac{\partial u}{\partial p_z}\right)d\Omega = \begin{cases} \frac{1}{4m^2} ; & p \ll mc \\ \frac{1}{3}\frac{1}{p^2} ; & p \gg mc \end{cases} \tag{4}$$

or

$$M\left(\xi, \frac{m}{p^0}\right) = \begin{cases} \frac{1}{4m^2} ; & \xi \ll \frac{m}{p_0} \\ \frac{1}{4m^2}\,\frac{4}{3}\,\frac{m^2}{(p^0)^2}\,\frac{1}{\xi} ; & \xi \gg \frac{m}{p^0} \end{cases} \tag{5}$$

The integral of $M(\xi)$ is of the form

$$I_2\left(\frac{m}{p^0}\right) = \int_0^\infty |f(\xi)|^2 \xi^2 d\xi\, M\left(\xi, \frac{m}{p^0}\right) . \tag{6}$$

This integral tends to zero at $\frac{m}{p^0} \to 0$ since the region where $M(\xi) = \frac{1}{4m^2}$ reduces as $\frac{m}{p^0}$ decreases. At $\frac{m}{p^0} \to \infty$ it is finite and equal to $\frac{1}{4m^2}$.

APPENDIX 4

Let us consider the connection between the wave function representing the state localized about the point $x = x'$ and the δ-function. We denote this function by $\psi_{x'}(x,a)$ where $a \simeq \frac{1}{p^0}$. It can be of the form

$$\psi_{x'}(x,a) \simeq e^{-\frac{(x-x')^2}{2a^2}} . \tag{7}$$

This function leads to $\overline{(x-x')^2} = a^2$ so that at $a \to 0$, $(x-x')^2 \to 0$ the function $\psi_{x'}(x,a)/\sqrt{a}$ has a limit $\delta(x-x')$ at $a \to 0$

$$\psi_{x'}(x,a)/\sqrt{a} \underset{a \to 0}{\longrightarrow} \psi_{x'}(x) = \delta(x-x') . \tag{8}$$

Therefore

$$x\,\psi_{x'}(x,a) = x'\,\psi_{x'}(x,a) + \sqrt{a}\left[(x-x')\frac{\psi_{x'}(x,a)}{\sqrt{a}}\right] . \tag{9}$$

The last term tends to zero at $a \to 0$ owing to (8) and the relation $(x-x')\delta(x-x') = 0$.

APPENDIX5

A. Usually the Fourier representation for the scalar field is written in the form

$$\varphi(x) = \int \frac{c(\vec k) e^{ikx}}{2\omega} d^3k = \int \frac{c(\vec k)}{\sqrt{2\omega}} U_{\vec k}(x) d^3k = \int A(\vec k) U_{\vec k}(x) d^3k \qquad (1)$$

$A(\vec k)$ in contrast to $c(\vec k)$ is not a scalar.

B The fact that the quantity $\rho(\vec x, t)$ is non-definite is seen ffom the following, example. We put

$$\frac{c(\vec k)}{\omega} = \frac{c_1 e^{-\frac{(\vec k - \vec k_1)^2}{2b^2}}}{\omega} + \frac{c_2 e^{-\frac{(\vec k - \vec k_2)^2}{2b^2}}}{\omega} . \qquad (2)$$

We find $\rho(\vec x, 0)$

$$\varphi(x) = c_1 \int \frac{e^{-\frac{(\vec k - \vec k_1)^2}{2b^2} + i\vec k \vec x}}{\omega} d^3k + c_2 \int \frac{e^{-\frac{(\vec k - \vec k_2)^2}{2b^2} + i\vec k \vec x}}{\omega} d^3k .$$

For simplicity we assume that $|\vec k_1 - \vec k_2| >> b$.

Further

$$\Omega \bar\varphi = c_1^* e^{-\frac{b^2 x^2}{2} - i\vec k_1 \vec x} + c_2^* e^{-\frac{b^2 x^2}{2} - i\vec k_2 \vec x} .$$

From here

$$\rho(\vec x, 0) = \frac{1}{2}(\Omega\bar\varphi\varphi + \bar\varphi\Omega\varphi) = e^{-b^2x^2}\Big[\frac{|c_1|^2}{\omega_1} + \frac{|c_1 c_2|}{\omega_1}\cos(\Delta\vec k \vec x + \varphi) + \frac{|c_1 c_2|}{\omega_2}\cos(\Delta\vec k \vec x + \varphi) + \frac{|c_2|^2}{\omega_2}\Big], \qquad (4)$$

where $\varphi = \arg\frac{c_2}{c_1}$. Now we believe that $\omega_2 >> \omega_1$, $|c_1|, |c_2|$ are comparable. Then

$$\rho(x, 0) = e^{-\frac{b^2 x^2}{2}} \frac{|c_1|^2}{\omega_1}\{1 + |\frac{c_2}{c_1}|\cos(\Delta\vec k \vec x + \varphi)\} . \qquad (5)$$

It is seen that if $|\frac{c_2}{c_1}| > 1$ then $\rho(\vec x, 0)$ periodically changes the sign.

The density is in this case not too strongly localized (it was assumed that b is small) and in any case the quantity $\Delta x^2 = \frac{1}{b^2}$ is by no means connected with the Compton wave-length $\hbar/mc$.

C. Negative Values of $\overline{\Delta z^2}$

Now we turn to the one-dimensional case. Eq.(16)' reads now

$$\overline{\Delta z^2} = \int_{-\infty}^{+\infty}\Big[\Big(\frac{\partial A}{\partial k}\Big)^2 - \frac{1}{4} A^2 \frac{k^2}{\omega^4}\Big] dk .$$

We put A=1 for $\omega = \sqrt{k^2+1} < \Omega >> 1$

$$A = e^{-\frac{a^2}{2}(\omega - \Omega)^2} \qquad \omega > \Omega$$

(the particle mass is taken to be unity). Going over to the integration over ω we get

$$\frac{1}{2}\overline{\Delta z^2} = \int_\Omega^\infty e^{-a^2(\omega-\Omega)^2} a^4 (\omega - \Omega)^2 \Big(1 - \frac{1}{\omega^2}\Big)^{1/2} d\omega - \frac{1}{4}\int_1^\Omega \Big(1 - \frac{1}{\omega^2}\Big)^{1/2}\frac{d\omega}{\omega^2} - \frac{1}{4}\int_\Omega^\infty \Big(1 - \frac{m^2}{\omega^2}\Big)^{1/2} e^{-a^2(\omega-\Omega)^2}\frac{d\omega}{\omega^2} . \qquad (7)$$

It is sufficient to consider two first integrals.

Assuming $a(\omega - \Omega) = \xi$ we find that the first integral will be of the order a and the second one is simply calculated and for $\Omega >> 1$ is $-\frac{1}{4}\Big(\frac{\pi}{4}\Big) = -\frac{\pi}{16}$ the third integral is far smaller.

At $a << \frac{\pi}{16}$, $\overline{\Delta z^2} \approx -\frac{\pi}{8}\frac{\hbar^2}{m^2c^2}$. In the case of three dimensions, under the normalization condition (15) we have not succeeded in finding an example with $\overline{\Delta z^2} < 0$.

APPENDIX 6

Let us consider a relativistic packet described by the field $\varphi(x,t)$

$$\varphi(\vec{x},t) = \int \frac{C(\vec{K})}{\omega} e^{i(\vec{K}\vec{x}-\omega t)} d^3K \tag{1}$$

$$i\frac{\partial \varphi^*(\vec{x},t)}{\partial t} = \int C^*(\vec{K}) e^{-i(\vec{K}\vec{x}-\omega t)} d^3K . \tag{2}$$

The density $\rho(\vec{x},t)$ is determined by the expression

$$\rho(x,t) = \frac{i}{2}\left(\varphi\frac{\partial \varphi^*}{\partial t} - \varphi^*\frac{\partial \varphi}{\partial t}\right) \tag{3}$$

The localization will be strong if φ or $\frac{\partial\varphi}{\partial t}$ are strongly localized, We choose $C(K)$ in the form

$$C(K) = \mathcal{N} e^{-\frac{(\vec{K}-\vec{K}_1)^2}{2b^2}} , \tag{4}$$

where $\mathcal{N} = \frac{1}{\sqrt{b^3}}$. Then

$$\frac{\partial \varphi(x,t)}{\partial t} = -i\mathcal{N}\int e^{-\frac{(K-K_1)^2}{2b^2} + iKx} d^3K = \tag{5}$$

$$\cong e^{i\vec{K}\vec{x}} \frac{e^{-\frac{x^2}{2a^2}}}{\sqrt{a}} ,$$

where $a \sim \frac{1}{b}$. At $a \to 0$ this function is arbitrary strongly localized about x=0. The connection of such a function with the δ function was considered in Appendix 4.

APPENDIX 7

We calculate the spreading of a relativistic packet , starting from its representation in the form ((I)App.6) and take $C(K)$ in the form ((4)App.6). If b is not too large then the field $\varphi(\vec{x},t)$ can be represented in the form

$$\varphi(x,t) = \frac{\mathcal{N}}{\omega_1} e^{i(\vec{K}\vec{x}-\omega t)} I(\vec{x},t) , \tag{1}$$

where

$$I(\vec{x},t) = \int e^{-\frac{(\vec{K}-\vec{K}_1)}{2b^2} + i(\vec{K}-\vec{K}_1,\vec{x}) - i(\omega-\omega_0)t} d^3K . \tag{2}$$

For definiteness we put $\vec{K}_1 = (K_{x_1}, 0, 0)$.

Then

$$\omega - \omega_1 = \frac{K_x}{\omega_1} q_x + \frac{1}{2\omega_1}(q_x^2 + q_y^2 + q_z^2) - \frac{1}{2}\frac{K_{x_1}^2}{\omega_1^3} q_x^2 , \tag{3}$$

where $q = K - K_1$. A simple calculation yields

$$I(x,t) = A(t) e^{i\alpha(x,t) - \frac{(x-v_1t)^2}{2\Delta_\parallel^2(t)} - \frac{y^2+z^2}{2\Delta_\perp^2(t)}} , \tag{4}$$

where $A(t)$ is a slowly changing quantity α is a real number and the quantities $\Delta_\parallel^2(t)$ and $\Delta_\perp^2(t)$ are

$$\Delta_\parallel^2(t) = \frac{1}{b^2} + \frac{b^2 m^4}{\omega^6} t^2 \tag{5}$$

$$\Delta_\perp^2(t) = \frac{1}{b^2} + \frac{b^2}{\omega^2} t^2 . \tag{5'}$$

Putting $\frac{1}{b^2} = \Delta^2(0)$ these formulas can be rewritten in the form

$$\Delta_\parallel^2(t) = \Delta^2(0) + \frac{\hbar^2}{\Delta^2(0)}\frac{m^4}{E^4} v^2 t^2 \tag{6}$$

$$\Delta_\perp^2(t) = \Delta^2(0) + \frac{\lambda^2}{\Delta^2(0)} v^2 t^2 \quad . \tag{6}$$

Here $\lambda = \frac{\hbar}{p}$ is the particle momentum $v = \frac{p}{E}$ is its velocity. From the first formula it is seen that for m=0 the wave packet does not spread in the longitudinal direction as it must be for particles without rest mass (in this case there is no dispersion of the de Broglie waves). The formulas for $\Delta_\perp^2(t)$ can be also derived from the diffraction theory. The increase of the beam width due to the diffraction is determined by the multiplier 3,

$$\sim e^{-\frac{a^2}{\lambda^2}\sin^2\vartheta}, \tag{7}$$

where a is the diameter of the diaphragm orifice λ is the wave length, ϑ is the angle defining the beam width. The width $\rho = R\sin\vartheta$ where $R = vt$ is the distance to the diaphragm. Therefore

$$\sim e^{-\frac{a^2}{\lambda^2}\sin^2\vartheta} \simeq e^{-\frac{a^2}{\lambda^2}\frac{\rho^2}{v^2t^2}} \simeq e^{-\frac{\rho^2}{\Delta_\rho^2}} \tag{8}$$

so that

$$\Delta_\rho^2 = \frac{\lambda^2}{a^2} v^2 t^2 \tag{9}$$

according to eq. (6) for $\Delta_\perp^2$.

This formula can be also represented in the alternative form

$$\Delta_\rho^2 = \frac{\Lambda_0^2}{a^2} \left(\frac{m}{E}\right)^2 c^2 t^2 , \tag{10}$$

where $\Lambda_0 = \frac{\hbar}{mc}$. In this formula the multiplier $\frac{m}{E}$ characterizing the dalay of the clock is clearly seen.

References:

1. W.Heisenberg, Zs.fur Phys.,120, 513 (1962).
2. Н.Н.Боголюбов, Д.В.Ширков, Введение в теорию квантованных полей, М-Л., 1957.
3. S.Tomonaga, Progr.Theor.Phys.,1, 27 (1946).
4. Международная зимняя школа теоретической физики при Объединенном институте ядерных исследований, т.3, Дубна,1964.
5. D.I.Blokhintsev, G.I.Kolerov, Nuovo Cim.,v.34, 163 (1964).
6. Физика высоких энергий и теория элементарных частиц, Наукова думка, 1966.
7. D.I.Blokhintsev, G.I.Kolerov, Nuovo Cim.,44, N.4 (1966).
8. D.I.Blokhintsev, Phys.Lett.,v.12, N.3 (1964).

Received by Publishing Department
on April 21, 1967.

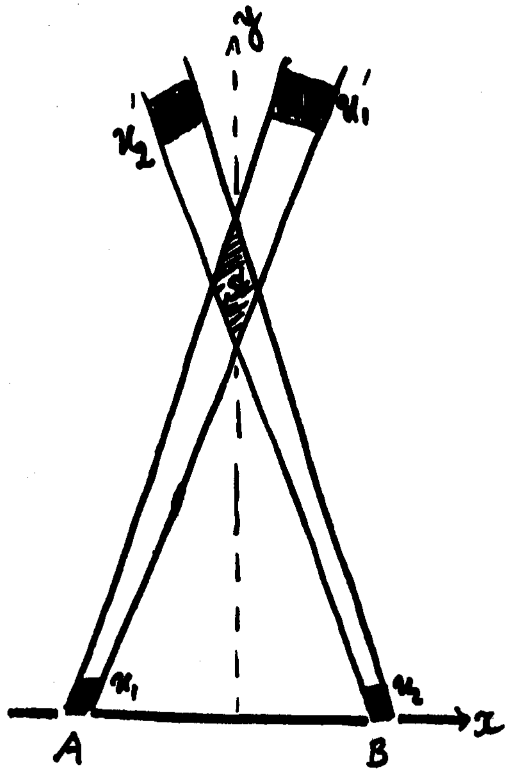

Fig.1. The wave packets u_1 and u_2 are formed at the time $t=-T$ moment by means of the diaphragms A and B; S is the zone of collision at $t \simeq 0$; u_2' and u_1' are the same packets at $t=+T$, but somewhat spreaded.

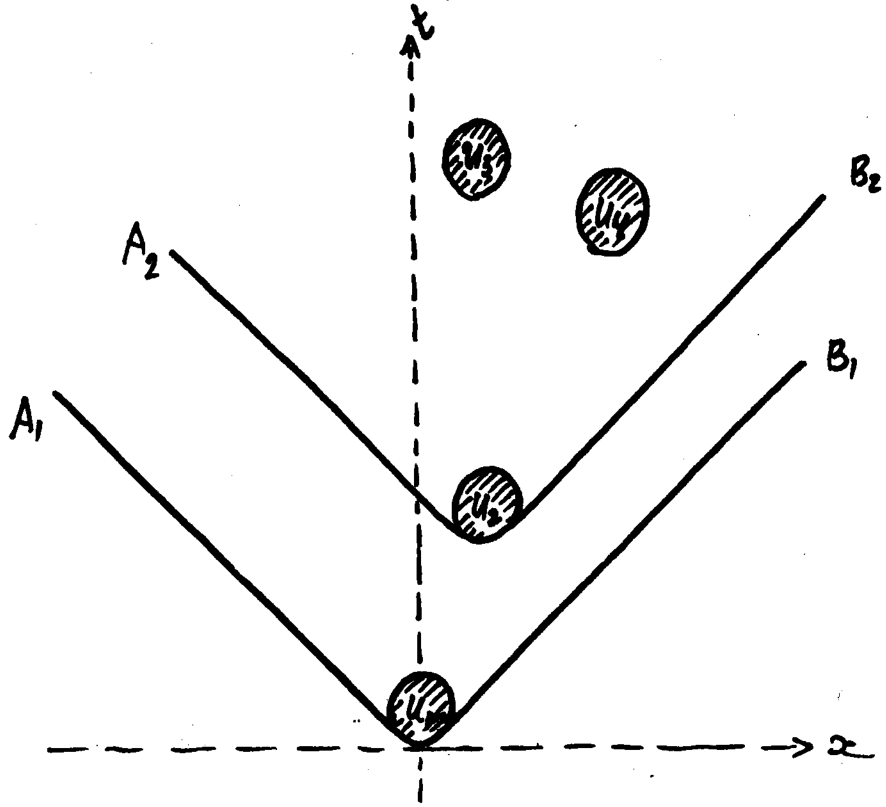

Fig.2. An example of location of the primary wave packets u_1 and u_2 for which the macroscopic causality is valid $A_1 u_1 B_1$ and $A_2 u_2 B_2$ are the light cones.

Added in Proof

page 25, formula (14) should read as

$$R^2 = [2(\xi n)^2 - \xi^2]$$

page 26, formula (15) should read

$$R^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2$$

page 26, line seven instead of "space region" $(\sim a^3)$ read "space time region" $(\sim a^4)$

page 26, line 19,21 instead of $(\sim a^3)$ read $(\sim a^4)$

page 27, line 2 instead of "space region" read "space time region"; instead of a^3, read a^4

page 27, line 4, instead of "is violated" read "is not violated".

page 27, formula (18) should read as

$$R^2 = [2(qn)^2 - q^2]$$