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# EXTENSION OF SU(6)$_W$ TO NONCOLLINEAR PROCESSES

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## I. Introduction

After great expectations, nowadays the SU(6) symmetry is regarded with a scepsis because of difficulties in the relativization problem [1]. Nevertheless for collinear processes there exists the relativistic $SU(6)_W$ group, a modified version of SU(6), which gives rather reasonable predictions [2-6]. However, the forward (or backward) scattering can not be precisely measured in experiment. There is an evident need to have an extension to noncollinear processes.

In this paper groups are found, which are isomorphic to SU(6) and are compatible with the Lorentz invariance and crossing symmetry. They leave invariant the free equations and are applicable to binary reactions without being confined to collinearity. These groups are called by us the $SU(6)_x$ groups [x]. All groups, we are interested in, reduce to $SU(6)_W$ for collinear configurations, and therefore we obtain an extension of $SU(6)_W$ to noncollinear processes. This approach is convenient also for treating the $SU(6)_W$ in an arbitrary frame of reference.

As is known, attempts to merge the SU(6) group with the Poincaré group failed. The introduction of $SU(6)_x$ bears no relation to these attempts. It is based only on the spin part of the total angular momentum, which is singled out in a special Lorentz-covariant manner ensuring the invariance of the free equations ( x - spin independence).

What do we pay for compatibility of $SU(6)_x$ with the Lorentz invariance? The $SU(6)_x$ transformations for each particle are allowed to de-

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[x] We have considered an example of $SU(6)_x$ in [7], but the crossing symmetry was violated there.

pend on the 4-momenta of all particles that participate in the binary reaction. Stress that the same situation can be seen even in the collinear group $SU(6)_W$ /3,8/.

Because of this, the $SU(6)_W$ and $SU(6)_x$ symmetries could be justified only as dynamical ones, which approximately describe some features of amplitudes.

Lagrange formalism is here impossible, and one can obtain the invariant amplitudes only in the framework of the S -matrix approach. Then the crossing symmetry and unitarity must be imposed from outside. The transformations are chosen so that to provide from the very beginning the crossing symmetry. However here we put aside the unitarity problem, it should be investigated separately. In any case, unlike $SU(6,6)$, the $SU(6)_x$ theory does not suffer from difficulty with the superfluous momenta and it deals with the conventional free equations of motion.

Some people suggest a hierarchy of symmetries, e.g. the chain: $SU(6)$ for one-particle states, $SU(3)\otimes SU(3)$ for collinear configurations and $SU(3)$ for coplanar ones /6,9/. In this paper it is shown that a symmetry of $SU(6)$ type may be kept for all these processes. At the same time this symmetry is certainly a broken one what follows, at least, from the large mass differences in multiplets. Only the broken symmetry may be compared with the experimental data (recall that the success of the $SU(3)$ symmetry is due to a happy conjecture about the form of its violation). In the present paper we deal only with the exact symmetry. An investigation of its reasonable breakdown will be the next step.

The quark $SU(6)_x$ transformations are discussed in Sec. 2. In Sec.3 such a family of groups is singled out, which is compatible with the requirement of total crossing symmetry. In Sec. 4 the 36- and 56-plets of $SU(6)_x$ are discussed and their $SU(3)\otimes SU(2)$ content is given. The rule for writing down the invariant amplitudes is formulated in Sec. 5. Several examples are given and, in particular, the general amplitude of meson-baryon binary reactions is written down. It is shown in Sec. 6 that for collinear configurations the amplitudes turn out to be $SU(6)_W$ invariant.

## 2. $SU(6)_x$ Transformations for Quarks

In the static SU(6) group the infinitesimal quark transformations

$$\delta\phi = \frac{i}{2}\{\omega^{a}\lambda_{a} + (a_{k} + a_{k}^{a}\lambda_{a})\sigma_{k}\}\phi \qquad (1)$$

are based on the algebra of the Pauli matrices $\sigma_k$ and the Gell-Mann matrices $\lambda_a$ ( $\omega_a$ , $a_k$ and $a_k^a$ are the transformation parameters). The transformation (1) commutes with the Dirac equation only in the rest system. For a nonzero spatial momentum the spin matrices are to be modified. One can take relativistic spin matrices for quarks ( x -spin matrices) in the form[x/]

$$s^{x}_{k} = -i e^{k}_{\mu}\gamma_{\mu}\gamma_{5}\frac{-i\gamma p + m}{2m} \qquad (p_0 = \sqrt{\vec{p}^{\,2} + m^2}), \qquad (2)$$

where $p$ and $m$ are the 4-momentum and the quark mass, respectively, and $e^{k}_{\mu}$ are three 4-vectors (threeleg) which are orthogonal to each other and to the momentum $p$

$$p_{\mu}e^{k}_{\mu} = 0, \qquad e^{j}_{\mu}e^{k}_{\mu} = \delta_{jk} \qquad (j, k = 1, 2, 3). \qquad (3)$$

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x/ One can choose the matrices $s^x_1$, $s^x_2$, $s^x_3$ also in the Barnes form/3/ $-ie^{1}_{\mu}\gamma_{\mu}\gamma_{5}$, $-ie^{2}_{\mu}\gamma_{\mu}\gamma_{5}$, $e^{1}_{\mu}\sigma_{\mu\nu}e^{2}_{\nu}$ or in the form $e^{2}_{\mu}\sigma_{\mu\nu}e^{3}_{\nu}$, $e^{3}_{\mu}\sigma_{\mu\nu}e^{1}_{\nu}$, $e^{1}_{\mu}\sigma_{\mu\nu}e^{2}_{\nu}$. However, as applied to positive energy solutions of the Dirac equation, they reduced to matrices (2) due to the identity (A.6) of Appendix 1. The adavntage of the matrices (2) in their linearity in $e_{\mu}$

The spin matrices $e_\mu \gamma_\mu \gamma_5$ (with $p_\mu e_\mu = 0$ ) were used earlier in the formal theory of reactions[10] and they are connected with the well-known relativistic spin operator[11] (in fact, $e_\mu \gamma_\mu \gamma_5 \approx e_\mu w_\mu$).It is easy to verity that, due to (3), the matrices commute with the Dirac equation

$$[ s^x_k , i\gamma p + m ] = 0$$

and satisfy the algebra of Pauli matrices[x]

$$[ s^x_i , s^x_j ] = 2 i \epsilon_{ijk} s^x_k . \qquad (4)$$

Note also that

$$\{ s^x_i , s^x_j \} = 2 \delta_{ij} \frac{-i\gamma p + m}{2m} . \qquad (4)$$

The matrices $s^x_k$ generate x-spin transformations

$$\delta u (p) = \frac{i}{2} a_k s^x_k u(p) . \qquad (5)$$

Combining the x-spin group with the SU(3) one can construct relativistic $SU(6)_x$ transformations, the $SU(6)_W$ ones being a particular case. For the positive energy Dirac spinors we have

$$\delta u(p) = \frac{i}{2} \{ \omega^a \lambda_a + ( a_k + a^a_k \lambda_a ) s^x_k \} u(p) \qquad (6)$$

and, as it is easy to verify, the transformations (6) form a group. Since the algebras of the matrices $s^x_k$ and $\sigma_k$ are identicai, the generators

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[x] When checking, the relation

$$(\gamma e^i)(\gamma e^j)(\gamma e^k)\frac{\gamma p}{i m} = \epsilon_{ijk} \, \mathrm{Det}\left| e^1 e^2 e^3 \frac{p}{i m} \right| = \epsilon_{ijk} \gamma_5 \qquad (i \neq j \neq k)$$

is to be used.

$\lambda_a$, $s^x_k$ and $\lambda_a s^x_k$ satisfy the SU(6) algebra. Transformations for other multiplets are direct products of the quark ones. All conventional free equations (the Dirac, Bargmann-Wigner, Proca, Rarita-Schwinger equations) are invariant under such transformations in contrast to the situation in SU(6,6) and so on (see[5,6]).

The choice of a basis $e^k_\mu$ is the most important problem in constructing the $SU(6)_x$. Let us consider an amplitude of some binary reaction

$$1+2 \to 3+4 \qquad (p_1 + p_2 = p_3 + p_4). \tag{7}$$

For simplicity all four particles will be assumed to be quarks.

First of all we note that there are no amplitudes invariant under transformations (6) with all conceivable threelegs $e^k_\mu$ simultaneously. Really, such an invariance is equivalent to the invariance under the group, considered in[12,8] and based on the relativistic spin operator $w_\mu$ (or $s_{\mu\nu}$). Having infinite number of parameters this group gives an infinite number of limitations, and in this case non-zero amplitudes are absent[8].

At the same time there are always non-vanishing amplitudes invariant under (6), if vectors $e^k_\mu$ for each particle are fixed. In this case the transformations (6) form a finite parametric group, which imposes a finite number of restrictions. The group is based also on the relativistic spin operator, which enters transformations via its projections on fixed threelegs of all particles ($s^x_k$ is, in fact, $e^k_\mu w_\mu$).

An invariant amplitude depends on vectors $e^k_\mu$, unless they coincide for different particles. It is evidently impossible to choose a common basis for all particles[x], as for each particle $e^k_\mu$ are orthogonal to its momentum, and the momenta of various particles are different (in the collinear case two vectors may be common for all particles, but in a general case only one may, the normal to the reaction plane). However,

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[x] Phenomenological analysis of amplitudes often involves some basis constructed out of the momenta (see[13]). We use four bases simultaneously.

an amplitude will not be Lorentz-invariant, if it contains any outside vectors.

Therefore for noncollinear binary reactions <u>the vectors must be expressed in terms of the momenta of particles</u> that participate in the reaction.

Stress, that $SU(6)_x$ does not represent the merging of the $SU(3)$ and Lorentz groups: only the spin part of the total angular momentum enters transformations (6), and it is singled out in some specific relativistic manner to ensure commutation with the free equations of motion. Therefore $SU(6)_x$ expresses the spin-independence, which is understood in a certain relativistic sence.

Finally let us take into account explicitly that each particle in reaction (7) has its own basis and that the basis vectors are expressed in terms of the 4-momenta $p_1$, $p_2$, $p_3$ and $p_4$. To this end we denote three vectors ($k = 1,2,3$) for $n$-th particle ($n = 1,2,3,4$) by $e^{nk}_{\mu}(p_1 p_2 p_3 p_4)$. Then for quarks in reaction (7) the $SU(6)_x$ transformations are written down as

$$\delta u(p_n) = \frac{1}{2}\{ i\omega^d \lambda_d + e^{nk}_{\mu}(p_1 p_2 p_3 p_4)\gamma_{\mu}\gamma_5(a_k + a^d_k \lambda_d)\} u(p_n). \qquad (8)$$

Quarks with different 4-momenta and in various binary reactions transform according to different equivalent representations of the $SU(6)_x$ group. A representation is characterized by the set of the 4-momenta of particles that participate in the binary reaction. The same is true for all other supermultiplets. In $SU(6)_x$ there are upper and lower indices as in $SU(6)$. By definition the quantity $u(p_n)$ which transforms according to (8), has the lower index A. The index A unites the Dirac spinor index $\alpha$ with 4-values and the quark $SU(3)$ index $a$ ($a = 1,2,3$). Respectively, the quantity $\bar{u}(p_n)$ has the upper index, $\bar{u}^A(p_n)$, and the transformation law for it is

$$\delta \bar{u}(p_n) = -\frac{1}{2}\bar{u}(p_n)\{ i\omega^d \lambda_d + e^{nk}_{\mu}(p_1 p_2 p_3 p_4)\gamma_{\mu}\gamma_5(a_k + a^d_k \lambda_d)\}. \qquad (8)$$

## 3. Crossing Symmetry and Co-Ordination of Threelegs

The requirement of the crossing symmetry of amplitudes imposes essential restrictions on the choice of threelegs $e_{\mu}^{nk}$. As a consequence, threelegs for all particles turn out to be co-ordinated in such a way that if the threeleg for one of particles is found, then it defines threelegs for other particles, that participate in the reaction.

It is natural to identify one of the vectors, say $e_{\mu}^{n1}(p_1 p_2 p_3 p_4)$, with the normal to the binary reaction plane

$$e_{\mu}^{n1}(p_1 p_2 p_3 p_4) = X^{n}(stu\,1234)\,\epsilon_{\mu\nu\lambda\rho}\, p_{1\nu} p_{2\lambda} p_{3\rho}, \tag{9}$$

where

$$s = -(p_1 + p_2)^2,\ t = -(p_1 - p_3)^2,\ u = -(p_1 - p_4)^2$$

are the Mandelstam variables, numbers 1,2,3,4 stand instead of the masses $m_1$, $m_2$, $m_3$, $m_4$ of particles, and $X^n$ are the functions to be defined. For generality all masses are considered to be different.

Second vectors of each particle may be written down as follows

$$e_{\mu}^{12}(p_1 p_2 p_3 p_4) = \epsilon_{\mu\nu\lambda\rho} \frac{p_{1\nu}}{i\,m_1}[A_1(stu\,1234)p_{2\lambda} + B_1(stu\,1234)p_{3\lambda}]\, e_{\rho}^{11}$$

$$e_{\mu}^{22}(p_1 p_2 p_3 p_4) = \epsilon_{\mu\nu\lambda\rho} \frac{p_{2\nu}}{i\,m_2}[A_2(stu\,1234)p_{1\lambda} + B_2(stu\,1234)p_{4\lambda}]\, e_{\rho}^{21} \tag{10}$$

$$e_{\mu}^{32}(p_1 p_2 p_3 p_4) = \epsilon_{\mu\nu\lambda\rho} \frac{p_{3\nu}}{i\,m_3}[A_3(stu\,1234)p_{4\lambda} + B_3(stu\,1234)p_{1\lambda}]\, e_{\rho}^{31}$$

$$e_{\mu}^{42}(p_1 p_2 p_3 p_4) = \epsilon_{\mu\nu\lambda\rho} \frac{p_{4\nu}}{i\,m_4}[A_4(stu\,1234)p_{3\lambda} + B_4(stu\,1234)p_{2\lambda}]\, e_{\rho}^{41}.$$

Here their orthogonality to the corresponding momenta and to the first vector is expressed in an explicit form.

The third vector for $n$-th particle is

$$e^{n3}_{\mu}(p_1 p_2 p_3 p_4) = \epsilon_{\mu\nu\lambda\rho} \frac{p_{n\nu}}{i\, m_n} e^{n1}_{\lambda} e^{n2}_{\rho}. \qquad (11)$$

If previous vectors are normalized, this vector will be normalized too.

Let us require for our invariant amplitudes to be totally crossing symmetric. Recall that the crossing symmetry means the following. Let $A_{pppp}(p_1 p_2 p_3 p_4)$ be an amplitude of the reaction (7). Representing the amplitude according to the LSZ method, we derive the identities

$$A_{pppp}(p_1 p_2 p_3 p_4) = \pm A_{\alpha p \alpha p}(-p_3 p_2 - p_1 p_4) \qquad (12)$$

$$A_{pppp}(p_1 p_2 p_3 p_4) = \pm A_{pppp}(p_1 p_2 p_3 p_4). \qquad (13)$$

(The sign is determined by statistics). In the right-hand side of eq.(12) there arises an amplitude with two antiparticles in the nonphysical domain. By means of analytical continuation $p_3 \leftrightarrow -p_1$ one can pass to the amplitude $A_{\alpha p \alpha p}(p_1 p_2 p_3 p_4)$ for the physical process $1_\alpha + 2_p \to 3_\alpha + 4_p$ with two antiparticles. In the identity (13) both amplitudes are in the physical domain. The final amplitude is deduced by interchange $p_3 \leftrightarrow p_4$.

One always can modify the transformations so that they should leave invariant the crossing amplitudes. The modified transformations also form a representation of $SU(6)_x$, the only difference being in the form of vectors $e^k_\mu$ (see Table). If the vectors for the original reaction are chosen (the first row), then the vectors for all crossing reactions are defined too (the subsequent rows). In this sense the crossing amplitudes remain $SU(6)_x$ invariant.

## T a b l e

### Threelegs for Transformations which Leave Invariant the Crossing Amplitudes

| Crossing | Vectors | | | |
|---|---|---|---|---|
| | 1st particle | 2nd particle | 3d particle | 4th particle |
| 1. | $e^{1k}(p_1 p_2 p_3 p_4)$ | $e^{2k}(p_1 p_2 p_3 p_4)$ | $e^{3k}(p_1 p_2 p_3 p_4)$ | $e^{4k}(p_1 p_2 p_3 p_4)$ |
| 2. $p_3 \leftrightarrow p_4$ | $e^{1k}(p_1 p_2 p_4 p_3)$ | $e^{2k}(p_1 p_2 p_4 p_3)$ | $e^{4k}(p_1 p_2 p_4 p_3)$ | $e^{3k}(p_1 p_2 p_4 p_3)$ |
| 3. $p_1 \leftrightarrow p_2$ | $e^{2k}(p_2 p_1 p_3 p_4)$ | $e^{1k}(p_2 p_1 p_3 p_4)$ | $e^{3k}(p_2 p_1 p_3 p_4)$ | $e^{4k}(p_2 p_1 p_3 p_4)$ |
| 4. $p_2 \leftrightarrow -p_3$ | $e^{1k}(p_1 -p_3 -p_2 p_4)$ | $-e^{3k}(p_1 -p_3 -p_2 p_4)$ | $-e^{2k}(p_1 -p_3 -p_2 p_4)$ | $e^{4k}(p_1 -p_3 -p_2 p_4)$ |
| 5. $p_2 \leftrightarrow -p_4$ | $e^{1k}(p_1 -p_4 p_3 -p_2)$ | $-e^{4k}(p_1 -p_4 p_3 -p_2)$ | $e^{3k}(p_1 -p_4 p_3 -p_2)$ | $-e^{2k}(p_1 -p_4 p_3 -p_2)$ |
| 6. $p_1 \leftrightarrow -p_3$ | $-e^{3k}(-p_3 p_2 -p_1 p_4)$ | $e^{2k}(-p_3 p_2 -p_1 p_4)$ | $-e^{1k}(-p_3 p_2 -p_1 p_4)$ | $e^{4k}(-p_3 p_2 -p_1 p_4)$ |
| 7. $p_1 \leftrightarrow -p_4$ | $-e^{4k}(-p_4 p_2 p_3 -p_1)$ | $e^{2k}(-p_4 p_3 p_2 -p_1)$ | $e^{3k}(-p_4 p_2 p_3 -p_1)$ | $-e^{1k}(-p_4 p_2 p_3 -p_1)$ |

We consider all crossings which reshuffle a given set of particles. If a particle is replaced by some antiparticle there arises the sign minus in the table.

So, having defined transformations for some reaction we know transformations for all crossing reactions. If one independently writes down invariant amplitudes for each row of Table, the crossing symmetry will be fulfilled up to the form factors, attached to invariant structures. It is easy to choose these form factors in the crossing symmetrical form.

However new relations for threelegs arise if identical particles participate in the reaction. For example, if the particles 3 and 4 are identical then the crossing $p_3 \leftrightarrow p_4$ does not change the reaction. Hence, in this case the following equalities are to be satisfied.

$$e^{nk}_{\mu}(p_1 p_2 p_3 p_4) = e^{nk}_{\mu}(p_1 p_2 p_4 p_3)\ (n=1,2), \quad e^{3k}_{\mu}(p_1 p_2 p_3 p_4) = e^{4k}_{\mu}(p_1 p_2 p_4 p_3). \tag{14}$$

Further if the particles 2 and 4 are identical and neutral then after crossing $p_2 \leftrightarrow -p_4$ and analytical continuation we obtain the original reaction. Hence here

$$e^{nk}_{\mu}(p_1 p_2 p_3 p_4) = e^{nk}_{\mu}(p_1 - p_4 p_3 - p_2), \qquad (n = 1,3) \tag{15}$$

etc. To have no troubles with such situations we shall require for vectors in each column of the Table to be equal up to the sign. We shall call such a basis the universal one. The universal threeleg is defined only by the momenta of particles, participating in the reaction, and it is independent of any other properties of particles.

If after crossing a given particle does not pass into an antiparticle, then the corresponding vectors must simply remain unchanged. For example, we equate the vectors in 1-5 rows of the first column. If a crossing turns a particle into some antiparticle, then vectors are to be equal only up to sign. So, after the crossing 6 we set for the first particle

$$-e^{3k}(-p_3 p_2 - p_1 p_4) = \pm e^{1k}(p_1 p_2 p_3 p_4). \tag{16}$$

Therefore, in the framework of the universality there are the two following possibilities x/.

1) All vectors ( $k = 1,2,3$) in each column are equated with those signs, as they stand. Then $x$ - spin transformations of both quark and antiquark turn out to be identical (if the kinematics is the same). More difinitely the antiquark transformation law is written down as

$$\delta u_C(p_n) = \frac{1}{2}[-i\omega^d \lambda^T_d + e^{nk}_{\mu}(p_1 p_2 p_3 p_4)\gamma_\mu \gamma_5 (a_k + a^d_k \lambda^T_d)] u_C(p_n) \tag{17}$$

( $T$ markes transposed matrices). In terms of the Dirac equation solutions $u_C(p_n) = C\bar{u}^T(-p_n)$, where $C$ is the charge conjugation matrix. Note that the quantity $u^T_C(p_n)C^{-1}\gamma_5$ transforms like $\bar{u}(p_n)$.

2) For $k = 1,2$ the vectors in each column are equated, omitting

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x/ We do not consider more cumbersome situations, where basic vectors for antiparticles are linear combinations (rotation) of ones for particles. This is a way to look for further generalizations of SU(6).

minus if it stand in the Table. As to the third vectors ( $k=3$ ) they must be equated just as in the case 1) due to group considerations. Now the x-spin transformations for antiquarks

$$\delta u_C(p_n)=\frac{1}{2}\{-i\omega^d\lambda_d^T+[-e_\mu^{n1}(p_1p_2p_3p_4)(\alpha_1+\alpha_1^d\lambda_d^T)-$$

$$-e_\mu^{n2}(p_1p_2p_3p_4)(\alpha_2+\alpha_2^d\lambda_d^T)+e_\mu^{n3}(p_1p_2p_3p_4)(\alpha_3+\alpha_3^d\lambda_d^T)]\gamma_\mu\gamma_5\}u_C(p_n) \quad (18)$$

differ from those for quarks (8). Here we see a play of signs that is the same as in $SU(6)_W$. However now it is a consequence of the crossing symmetry. Note that it is difficult to apply directly the charge conjugation to transformations due to the momenta involved (even in the collinear case). Just as in $SU(6)_W$, $u_C(p)$ is not a quantity with upper or lower indices. However the quantity

$$u_{\approx C}(p_n)=u_C^T(p_n)C^{-1}e_\mu^{n3}\gamma_\mu \quad (u_C(p_n)=C\bar{u}^T(-p_n)) \quad (19)$$

transforms like $\bar{u}(p_n)$ i.e. it has the upper index: $u^A_{\approx C}(p_n)$. It satisfies the same Dirac equation as $\bar{u}(p_n)$ does

$$u_{\approx C}(p_n)(i\gamma p_n+m)=0.$$

Now we pass to the realization of an universal basis in the case 2) of interest. We shall see that in the case 1) there are severe contradictions with experiment, e.g. the pion-nucleon scattering with the parity conservation is forbidden.

## Restrictions on the Basis in the $SU(6)_*$ (case 2)

In the case 2) the total crossing symmetry gives the following restrictions on the threelegs. A vector associated with the normal to the reaction plane is the same for all particles. It is written down as

$$n_\mu = e_\mu(p_1 p_2 p_3 p_4) = N_1 (s-t)(t-u)(u-s)\epsilon_{\mu\nu\lambda\rho} p_{1\nu} p_{2\lambda} p_{3\rho}, \tag{20}$$

where $N_1$ is a positive normalization factor.

For the second vector of the first particle we obtain

$$e^{12}_\mu(p_1 p_2 p_3 p_4) = \epsilon_{\mu\nu\lambda\rho} \frac{p_{1\nu}}{i m_1} [A(stu\,1234) p_{2\lambda} - A(tsu\,1324) p_{3\lambda}] n_\mu, \tag{21}$$

where $A(stu\,1234) = A_1(stu\,1234)$. The function $B_1$ in (10) turned out to be expressed in terms of A. Transformations for the first particle are determined by one function $A(stu\;1234)$ with the properties

$$A(uts\,1432) = -A(stu\,1234) \tag{22}$$

( an antisymmetry under permutation of pairs of arguments 1,5 and 3,7) and

$$A(stu\,1234) + A(ust\,1423) + A(tus\,1342) = 0 \tag{23}$$

(cyclic permutations of pairs of arguments 1,5; 2,6 and 3,7). One can represent a function which satisfies (22) and (23) in the form

$$A(stu\,1234) = f(stu\,1234) - f(uts\,1432) + f(sut\,1243) - f(ust\,1423), \tag{24}$$

where $f$ is a completely arbitrary function. The simplest example of function A is

$$A(stu\,1234) = (s-u)\phi(stu), \tag{25}$$

where $\phi(stu)$ is an arbitrary totally symmetric function.

For particles 2,3 and 4 the second vector is obtained from the vector (21) by means of permutations

$$-(12)(34),\ -(13)(24),\ (14)(23), \tag{26}$$

respectively. These permutations and the identity one form an alternating representation of the Klein four-group. The operation $-(13)(24)$ means that one must mutually transpose the momenta $p_1$ with $p_3$ and $p_2$ with $p_4$ (and correspoding masses) and change the common sign. The foregoing refers to the first vector too, but being chosen in the form (20) it is invariant under permutations (26). The third basic vectors are constructed according to (11). For the particle I we have

$$e^{13}_{\mu}(p_1 p_2 p_3 p_4) = A(stu\,1234)\left[p_{2\mu} + \frac{p_{1\mu}(p_1 p_2)}{m_1^2}\right] - A(tsu\,1324)\left[p_{3\mu} + \frac{p_{1\mu}(p_1 p_3)}{m_1^2}\right] \tag{27}$$

The third vectors for the particles 2,3 and 4 can be obtained by applying to (27) the permutations

$$(12)(34),\ (13)(24),\ (14)(23) \tag{28}$$

of the Klein four-group.

For the second and third vectors of the particle 1 the normalization condition is

$$(e^{1k})^2 = A^2(stu\,1234)\left[\frac{(p_1 p_2)^2}{m_1^2} - m_2^2\right] + A^2(tsu\,1324)\left[\frac{(p_1 p_3)^2}{m_1^2} - m_3^2\right] -$$

$$-2A(stu\,1234)A(tsu\,1324)\left[\frac{(p_1 p_2)(p_1 p_3)}{m_1^2} + p_2 p_3\right] = 1 \qquad (k=2,3). \tag{29}$$

Due to (22) and (23) this expression is symmetric under any permutation of pairs $s, m_2; t, m_3$ and $u, m_4$ with each other. Thus, the total crossing symmetry makes the threelegs of all particles in the binary reaction be defined in terms of one function $A(stu\,1234)$ with the properties (22) and (23).

## Restrictions on the Basis in the Case 1)

In the case 1) the particle 1 threeleg is given again by the formulae (20)-(23) and (27). However in this case transitions to the particles, 2,3 and 4 are performed by the permutations (28) of the Klein four-group for all three vectors ( $k = 1,2,3$). This makes the particle and antiparticle $x$ -spin transformations be the same. But here the common direction of the first vectors for both initial particles is opposite to that for final ones, unlike the case 2) where the first vector is the same for all particles. Because of this in the case 1) there are no invariant parity conserving amplitudes for binary meson-baryon reactions.

## 4. Classification of $SU(6)_x$ Multiplets

Above we confined ourselves mainly to the quarks. The mesons and baryons are described by the higher representations of $SU(6)_x$ which are the direct products of the quark ones.

Mesons. The mesons can be described by tensor $M_A^B(p)$ which transforms like $u_A(p) \otimes \bar{u}^B(p)$:

$$\delta M_A^B(p) = \frac{1}{2}[i\omega^d \lambda_d + e_\mu^{\cdot k} \gamma_\mu \gamma_5 (a_k + a_k^d \lambda_d), M(p)]_A^B . \tag{30}$$

The quantity $M_A^B(p)$ satisfies the equations

$$(i\gamma p+\mu)_{\alpha\alpha'} M^{B}_{\alpha'a}(p)=0; \quad M^{\beta' b}_{A}(p)(i\gamma p+\mu)_{\beta'\beta}=0$$

which are invariant under (30).

The supermultiplet of $0^-$ and $1^-$ mesons is decomposed with respect to $SU(2)\times SU(3)$ as follows

$$M^{B}_{A}(p)=\frac{1}{\sqrt{2}}\{[-\phi^{b}_{a}(p)+i\gamma_{\mu}\gamma_{5}b^{b}_{\mu a}(p)]\frac{-i\gamma p+\mu}{2\mu}ie^{\cdot 3}_{\nu}\gamma_{\nu}\gamma_{5}\}_{\alpha\beta}, \tag{31}$$

where $e^{\cdot 3}_{\nu}$ is the third basis vector for a given particle, and $\phi^{b}_{a}(p)$ and $b^{b\,\nu}_{\mu a}(p)$ are the wave functions of the $0^-$ and $1^-$ nonets

$$(p^2+\mu^2)\phi^{b}_{a}=0,\ (p^2+\mu^2)b^{b}_{\mu a}=0,\ p_{\mu}b^{b}_{\mu a}=0 .$$

The last equations are invariant under transformations $\phi$ and $b$ which follow from (30) and (31).

In spite of the presence of vector $e^{\cdot 3}_{\mu}$ in (31), the multiplet is in fact defined without any reference to the basis. Really, we suppose that the meson wave functions transform like $u(p)\otimes u_{c}(p)$ as if the mesons were composed of quarks and antiquarks. Hence we find for the meson multiplet

$$\frac{1}{\sqrt{2}}\{-\phi^{b}_{a}(p)+i\gamma_{\mu}\gamma_{5}b^{b}_{\mu a}(p)\}\frac{-i\gamma p+\mu}{2\mu}i\gamma_{5}C . \tag{32}$$

This quantity transforms from the left and from the right according to the quark law (8) and the transposed antiquark law (18), respectively. Multiplying by $C^{-1}e^{\cdot 3}_{\mu}\gamma_{\mu}$ (see (19)) from the right, we obtain more convenient quantity (31) with one upper and one lower indices.

Taking into account the Lorentz condition we can expand $b^{b}_{\mu\alpha}$ in the basis vectors

$$b^{b}_{\mu\alpha} = e^{\bullet k}_{\mu} V^{b}_{k\alpha} , \quad V^{b}_{k\alpha} = e^{\bullet k}_{\mu} b^{b}_{\mu\alpha} \quad (k = 1, 2, 3) \qquad (33)$$

so that

$$M^{B}_{A}(p) = \frac{1}{\sqrt{2}} (\phi^{b}_{\alpha} + s^{x}_{k} V^{b}_{k\alpha}) s^{x}_{3} =$$

$$(34)$$

$$= \frac{1}{\sqrt{2}} \left( \frac{-i\gamma p + \mu}{2\mu} V^{b}_{3\alpha} + i s^{x}_{1} V^{b}_{2\alpha} - i s^{x}_{2} V^{b}_{1\alpha} + s^{x}_{3} \phi^{b}_{\alpha} \right),$$

where $s^{x}_{k}$ are spin matrices (2) for a given particle. Let us introduce components

$$V_{+1} = \frac{V_1 - i V_2}{\sqrt{2}} , \quad V_0 = V_3 , \quad V_{-1} = - \frac{V_1 + i V_2}{\sqrt{2}}$$

instead of $V_k$. They correspond to the projections 1, 0 ,−1 of the usual spin provided a quantization axis is directed along $e^{\bullet 3}_{\mu}$. Then it is seen from (34), that $-V_{+1}, \phi, V_{-1}$ form the $x$-spin triplet with the projections +1, 0, −1 , and $V_0$ turns out to be the singlet. Here we deal with the spin rearrangement which is the same as in the $SU(6)_W$. In the latter group the 36 sometimes is represented analogously as /14/ $M^{B}_{A} = (P + \sigma_k V_k)\sigma_3$ if all momenta are in the $Z$-direction in some coordinate system. Stress that $M^{B}_{A}$ has the non-zero trace

$$M^{A}_{A} = \sqrt{2}\, V^{\alpha}_{3\alpha} = \sqrt{6}\, e^{\bullet 3}_{\mu} b_{\mu} , \qquad (35)$$

where $b_\mu$ is the $1^-$ singlet of SU(3). As $b_\mu$ enters the remaining part of the $M_A^B$, in the $SU(6)_x$ we deal with the 36-plet (just as in the $SU(6)_W$).

Note also that 36-plet for $0^+$ and $1^+$ particles has the form

$$M_A^B(p) = \frac{1}{\sqrt{2}} [ - \phi_\alpha^b + i\gamma_\mu \gamma_5 b_{\mu a}^b ] \frac{-i\gamma p + \mu}{2\mu} . \qquad (36)$$

It splits into 35 and 1 unlike the multiplet (31) of interest.

<u>Baryons.</u> The baryon 56-plet $B_{ABC}$ transforms like the direct product of three quark representations, being totally symmetric in indices $A \equiv (\alpha, a)$, $B \equiv (\beta, b)$ and $C \equiv (\gamma, c)$. The three Dirac equations are satisfied

$$(i\gamma p + m)_{\alpha\alpha'} B_{\alpha' a BC} = (i\gamma p + m)_{\beta\beta'} B_{A\beta' bC} = (i\gamma p + m)_{\gamma\gamma'} B_{AB\gamma' c} = 0$$

(the Bargmann-Wigner equation).

The 56 of the four-momentum $p_\mu$ is decomposed with respect to $SU(2) \otimes SU(3)$ in a manifestly relativistic manner as follows

$$B_{ABC}(p) = \frac{1}{\sqrt{2}} \psi_{\mu\alpha abc}(p) (\gamma_\mu \frac{i\gamma p + m}{2m} C)_{\beta\gamma} + \frac{1}{\sqrt{18}} \{ \epsilon_{abd} \psi^d_{\gamma c}(p) (\gamma_5 \frac{i\gamma p + m}{2m} C)_{\alpha\beta} +$$

$$+ \epsilon_{cad} \psi^d_{\beta b} (\gamma_5 \frac{i\gamma p + m}{2m} C)_{\gamma\alpha} + \epsilon_{bcd} \psi^d_{\alpha a} (\gamma_5 \frac{i\gamma p + m}{2m} C)_{\beta\gamma} \}, \qquad (37)$$

where $C$ is the charge conjugation matrix, $\psi$ is the Dirac spinor, and $\psi_\mu$ is the spin 3/2 wave function:

$$(i\gamma p+m)\psi(p)=0,\ (i\gamma p+m)\psi_\mu(p)=0,\ p_\mu\psi_\mu=0,\ \gamma_\mu\psi_\mu=0.$$

The expressions $(\gamma_\mu \frac{i\gamma p+m}{2m} C)_{\beta\gamma}$ and $(\gamma_5 \frac{i\gamma p+m}{2m} C)_{\beta\gamma}$ are symmetric and antisymmetric in $\beta\gamma$, respectively. This provides the symmetry properties of $B_{ABC}$. In particular, the first term is totally symmetric in $\alpha\beta\gamma$ (and, therefore, in abc). Such a representation of 56 was used in the collinear case [4]. It follows from (37) that $\bar{B}^{ABC}$ is written down as

$$\bar{B}^{ABC}=\frac{1}{\sqrt{2}}\bar{\psi}^{abc}_{\mu\alpha}(C^{-1}\frac{i\gamma p+m}{2m}\gamma_\mu)_{\beta\gamma}-\frac{1}{\sqrt{18}}\Sigma\,\epsilon^{abd}\,\bar{\psi}^{c}_{\gamma d}(C^{-1}\frac{i\gamma p+m}{2m}\gamma_5)_{\alpha\beta} \quad (38)$$

with summation over three cyclic permutations of A,B,C. Further, for the antibaryon 56-plet we find (in accordance with (19))

$$B^{ABC}_{\tilde{\tilde{c}}}(p)=-\frac{1}{\sqrt{2}}\psi^{abc}_{\tilde{\tilde{c}}\,\mu\alpha}(C^{-1}\frac{i\gamma p+m}{2m}(\overset{\bullet}{e}{}^{3}\gamma)\gamma_\mu(\overset{\bullet}{e}{}^{3}\gamma))_{\gamma\beta}+$$

$$+\frac{1}{\sqrt{18}}\Sigma\,\epsilon^{abd}\,\psi^{c}_{\tilde{\tilde{c}}\,\gamma d}(C^{-1}\frac{i\gamma p+m}{2m}\gamma_5)_{\beta\alpha}, \quad (39)$$

where

$$\psi_{\tilde{\tilde{c}}}(p)=\psi^T_c(p)\,C^{-1}(\overset{\bullet}{e}{}^{3}_\nu\gamma_\nu);\quad \psi_c(p)=C\bar{\psi}^T(-p)$$

$$(40)$$

$$\psi_{\tilde{\tilde{c}}\,\mu}(p)=\psi^T_{c\mu}(p)\,C^{-1}(\overset{\bullet}{e}{}^{3}_\nu\gamma_\nu);\quad \psi_{c\mu}(p)=C\bar{\psi}^T_\mu(-p).$$

Here the unitary indices are omitted and for the spinor indices the matrix notation is used.

Note, that the original antibaryon wave function does not contain $e_{\mu}^{\cdot 3}$ and transforms like $u_c \otimes u_c \otimes u_c$. When writing down the invariant amplitudes it is expedient to use the quantities with the lower or upper indices. This leads us to (39).

In conclusion of this section we note, that other $SU(6)_x$ representations may be treated quite analogously.

## 5. Construction of Invariant Amplitudes

Possessing its own basis, each particle in a given reaction transforms according to its own law. Nevertheless we can easy formulate a general rule for writing down the $SU(6)_x$ invariant amplitudes for any reactions with arbitrary number of particles. An invariant amplitude is obtained by contracting the upper indices with the lower ones, some "metric matrix" being inserted to transform the corresponding threelegs into one another. For example, the contraction of the indices B and C in the product $M_A^B(p_1) M_C^D(p_2)$ is written down as

$$M_A^{\beta b}(p_1) S_{\beta\beta'}(12) M_{\beta' b}^{D}(p_2), \qquad (41)$$

where $S(1,2)$ is the matrix which transforms the threeleg of particle 2 into that of particle 1. Due to $S(1,2)$ the product (41) transforms as if there were only indices A and D, since the variations $M(p_1)$ over $\beta b$ and $M(p_2)$ over $\beta' b$ cancel. An invariant amplitude is obtained by contracting in such manner all quark indices. Let us give some examples. The amplitude of the singlet-quark scattering is

$$F\, \bar{u}^{\alpha' a}(3)\, S_{\alpha' \alpha}(3,1)\, u_{\alpha a}(1)\ \ \bar{\phi}(4)\, \phi\,(2). \qquad (42)$$

For the quark-quark scattering we have

$$F_1 \bar{u}^{\beta' b}(4) S_{\beta'\beta}(4,2) u_{\beta b}(2) \bar{u}^{\alpha' a}(3) S_{\alpha'\alpha}(3,1) u_{\alpha a}(1) + \tag{43}$$

$$+ F_2 \bar{u}^{\beta' b}(4) S_{\beta'\beta}(4,1) u_{\beta b}(1) \bar{u}^{\alpha' a}(3) S_{\alpha'\alpha}(3,2) u_{\alpha a}(2).$$

The general amplitude of the binary meson-baryon reaction $B(1)+M(2) \to B(3)+M(4)$ is written down as

$$\bar{B}^{\alpha a, \beta b, \gamma c}(3) S_{\alpha\alpha'}(3,1) S_{\beta\beta'}(3,1) S_{\gamma\gamma'}(3,1) B_{\alpha' a, \beta' b, \gamma' c}(1) \cdot$$

$$\cdot \left\{ F_1 \bar{M}^{\epsilon e}_{\delta d}(4) S_{\epsilon\epsilon'}(4,2) M^{\delta' d}_{\epsilon' e}(2) S_{\delta'\delta}(2,4) + G_1 \bar{M}^{D}_{D}(4) M^{E}_{E}(2) \right\} +$$

$$+ \bar{B}^{\alpha a, \beta b, \gamma c}(3) S_{\alpha\alpha'}(3,1) S_{\beta\beta'}(3,1) B_{\alpha' a, \beta' b, \delta d}(1) \cdot$$

$$\cdot \left\{ F_2 S_{\gamma\gamma'}(3,4) \bar{M}^{\epsilon e}_{\gamma' c}(4) S_{\epsilon\epsilon'}(4,2) M^{\delta' d}_{\epsilon' e}(2) S_{\delta'\delta}(2,1) + \tilde{\tilde{F}}_2 S_{\gamma\gamma'}(3,2) M^{\epsilon e}_{\gamma' c}(2) S_{\epsilon\epsilon'}(2,4) \cdot \bar{M}^{\delta' d}_{\epsilon' e}(4) S_{\delta'\delta}(4,1) + \right. \tag{44}$$

$$\left. + G_2 S_{\gamma\gamma'}(3,4) \bar{M}^{\delta' d}_{\gamma' c}(4) S_{\delta'\delta}(4,1) M^{E}_{E}(2) + \tilde{\tilde{G}}_2 S_{\gamma\gamma'}(3,2) M^{\delta' d}_{\gamma' c}(2) S_{\delta'\delta}(2,1) \bar{M}^{E}_{E}(4) \right\} +$$

$$+ F_3 \bar{B}^{\alpha a, \beta b, \gamma c}(3) S_{\alpha\alpha'}(3,1) B_{\alpha' a, \delta d, \epsilon e}(1) S_{\beta\beta'}(3,4) \cdot$$

$$\cdot \bar{M}^{\delta' d}_{\beta' b}(4) S_{\delta'\delta}(4,1) S_{\gamma\gamma'}(3,2) M^{\epsilon' e}_{\gamma' c} S_{\epsilon'\epsilon}(2,1).$$

In eqs. (42)-(44) F and G are arbitrary form factors, which depend on the Mandelstam variables s , t and u . To illustrate the procedure we keep all indices. We can analogously write down amplitudes for the baryon-baryon processes, for 2-meson annihilation. It is easy to construct also the amplitudes of processes, in which an arbitrary number of particles is produced. The metric matrix S , however, depends on threelegs, and amplitudes will be Lorentz-invariant if and only if threelegs are constructed out of the momenta of particles in the reaction.

For binary reactions such a realization of threelegs was obtained in Sec. 3, one vector in all threelegs being the normal to the reaction plane. In this case it is easy to derive[x)] the matrix $S(3,1)$ which transforms the first particle vectors $e^{12}_{\mu}, e^{13}_{\mu}, e^{14}_{\mu} = \frac{p_{3\mu}}{i m_3}$ into the third particle vectors $e^{32}_{\mu}, e^{33}_{\mu}, e^{34}_{\mu} = \frac{p_{3\mu}}{i m_3}$, , respectively, and which does not change the normal $n_{\mu} = e^{11}_{\mu} = e^{31}_{\mu}$. It may be represented in the form

$$S(3,1) = \frac{1}{2}\left(\sqrt{e^{3\rho}_{\mu}\, e^{1\rho}_{\mu}} + \frac{i\, e^{3\sigma}_{\nu}\, \sigma_{\nu\lambda}\, e^{1\sigma}_{\lambda}}{\sqrt{e^{3\rho}_{\mu}\, e^{1\rho}_{\mu}}}\right), \qquad (45)$$

where summation over $\rho, \sigma = 1,2,3,4$, is implied. Via threelegs the matrix $S(3,1)$ depends on the momenta of all particles in reaction. Of course, we may replace the numbers 3 and 1 by any others. When we say that $S(3,1)$ turns out basis into other, we imply its property

$$S(3,1)\,\gamma_{\mu}\, e^{1\rho}_{\mu} = \gamma_{\mu}\, e^{3\rho}_{\mu}\, S(3,1) \qquad (\rho = 1,2,3,4). \qquad (46)$$

In particular, for $\rho = 1$ and $\rho = 4$

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x/ For derivation see Appendix 2. There one can find also the general matrix $S(3,1)$ which transforms two arbitrary fourlegs into one another.

$$[S(3,1), \gamma_\mu n_\mu] = 0, \quad S(3,1)\frac{\gamma p_1}{i m_1} = \frac{\gamma p_3}{i m_3} S(3,1). \tag{46}$$

Further, note the relation

$$\gamma_4 S^+(3,1)\gamma_4 = S^{-1}(3,1) = S(1,3) \tag{47}$$

and the group property

$$S(3,2)S(2,1) = S(3,1). \tag{48}$$

Due to these properties we can essentially simplify amplitude in each particular case. For example, after putting (31) and (37) into (44) we obtain the amplitude for all $0+\frac{1}{2} \to 0+\frac{3}{2}$ reactions in the form

$$\frac{1}{6} F_3 \bar{\psi}_\mu^{abc}(3) e_\mu^{33} S(3,1)\, i e_\nu^{13} \gamma_\nu \gamma_5 \psi_b^{d'}(1) \epsilon_{a'cd'} (\bar{\phi}_a^{a'}(4)\phi_b^{b'}(2) + \bar{\phi}_a^{b'}(4)\phi_b^{a'}(2)). \tag{49}$$

Hence, the well-known $SU(6)_W$ relation for collinear events[5]

$$\sigma_{\pi^- p \to \pi^- \Delta^+} : \sigma_{\pi^- p \to \pi^0 \Delta^0} : \sigma_{\pi^- p \to \pi^+ \Delta^+} = 2:9:24 \tag{50}$$

is valid for all angles in the exact $SU(6)_x$ symmetry.

Sometimes it is useful to bear in mind, that the $S(3,1)$ between spinors with the momenta $p_1$ and $p_3$ (for example, between $u(1)$ and $\bar{u}(3)$) may be reduced to

$$S(3,1) \to f_1 + i f_2 \gamma_\mu (p_2 + p_4)_\mu , \tag{51}$$

where $f_1$ and $f_2$ are some functions of $s$ and $t$. In practice only their ratio $f_1 : f_2$ is essential which is equal to the ratio of the quantity

$$m_1 m_3 \{ (2s - m_1^2 + 2m_2^2 - m_3^2) A(stu\,1234) + (2s - m_1^2 + 2m_4^2 - m_3^2) A(stu\,3412) - \tag{52a}$$
$$-2(s-u+m_2^2-m_4^2) A(tsu\,1324) - 2(s-u-m_2^2+m_4^2) A(tsu\,3142) \}$$

to the quantity

$$m_3(s - m_2^2 + m_1 m_3) A(stu\,1234) + m_1(s - m_4^2 + m_1 m_3) A(stu\,3412) + \tag{52b}$$
$$+ [t - (m_1 + m_3)^2][m_1 A(tsu\,3142) + m_3 A(tsu\,1342)].$$

The ratio $f_1 : f_2$ may be also written in other forms (see Appendix 3). Stress, that up to the common multiplier eq. (51) is expressed directly in terms of the function A( stu 1234). Such a reduction of the metric matrix $S$ is always possible. For example, in eq. (44) we are able to replace all $S$. However, the expression (51) does not possess simple properties, which are inherent to $S(3,1)$. Therefore, such replacements are expedient only on the final steps of calculations.

Note that using some matrices $S$ (see Appendix 2) we may transform the wave functions of all particles to a some standard basis $e_\mu^{st.\rho}$ ( $\rho$ = 1,2,3,4) (cf. /7/), e.g. for quark $u'(p_n) = S(st.,n)u(p_n)$. The standard basis may be chosen in such a manner that $e_\mu^{st.4} = (0,0,0,1)$ and $e_\mu^{nk}\gamma_\mu\gamma_5$ for $k = 1,2,3$ turn into $\sigma_1\gamma_4$, $\sigma_2\gamma_4$, $\sigma_3$, respectively. The "primed" wave functions transform as

$$\delta u'(p_n) = \frac{i}{2}\{\omega^d\lambda_d + \gamma_4\sigma_1(\alpha_1 + \alpha_1^d\lambda_d) + \gamma_4\sigma_2(\alpha_2 + \alpha_2^d\lambda_d) + \sigma_3(\alpha_3 + \alpha_3^d\lambda_d)\}u'(p_n)$$

over each quark index. In these terms the transformations coincide formally with the Lipkin-Meshkov ones, and amplitudes are constructed out by the direct contraction of indices[x/].

Inconsistency of case 1). Stress that the metric matrix (45) connects the bases, which differ from each other only in a rotation without reflection. In the case 1) the direction of the vectors $e^{\bullet 1}_{\mu}$ for the initial particles is opposite to that for the final particles. Therefore the matrix $S(3,1)\, e^{13}_{\mu}\, \gamma_{\mu} \gamma_{5}$ serves as the metric matrix between the initial and final particles. It includes the reflection. Hence, in the case 1) the parity is violated in the meson-baryon reactions. So, in the simplest example of the singlet-quark scattering the invariant amplitude is pseudoscalar: $\bar{u}(3)\, S(3,1)\, e^{13}_{\mu}\, \gamma_{\mu} \gamma_{5}\, u(1)$. Thus, the requirement of the parity conservation excludes the case 1).

## 6. The Collinear Case

Even after imposing the total crossing symmetry there remains a family of the $SU(6)_x$ groups. Each of them is characterized by its own $A$ (stu 1234). It is remarkable that for the collinear configurations all these groups give the same amplitudes as in the $SU(6)_W$ symmetry. This can be seen from the following considerations.

a) In the collinear case each matrix $S$ reduces to the unit matrix since now only 2 four momenta are independent. Really if we replace all $S$ according to (51), then the matrix $(\gamma, p_2 + p_4)$ reduces to the unit matrix due to the Dirac equation and the identity $\bar{u}(p)\gamma_{\mu} u(p) = \frac{p_{\mu}}{im} \bar{u}(p) u(p)$.

b) Furthermore, for the collinear configurations the vectors (23) are the linear combinations of two independent momenta $p$ and $q$. Therefore putting the wave function of the 36-plet (31) into the amplitude we have the possibilities: either the matrix $e^{\bullet 3}\gamma$ reduces to the unit matrix, and the 36 can be represented by the quantity

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x/ Since $S(3,1) = S(3, st.)\, S(st,1)$, these amplitudes are, of course, identical to the above ones.

$$\Phi_A^B(p) = \frac{1}{\sqrt{2}} \{ -[\phi_a^b + i\gamma_\mu \gamma_5 b_{\mu a}^b] \frac{-i\gamma p + m}{2\mu} i\gamma_5 \}_{\alpha\beta} ,$$

or $\overset{\bullet}{e}{}^3 \gamma$ turns into $\gamma q$. Let us give one example. For the collinear meson-baryon processes there are 7 amplitudes which correspond to (44). We can replace $\overset{\bullet}{e}{}^3 \gamma$ by the unit matrix in the four amplitudes with the form factors $F_i$, and these amplitudes are the same as in /4/. Three amplitudes with the form factors $G_1$ ( which is absent in /4/ ) contain the traces $M_A^A(p) \approx \Phi_{\alpha\alpha}^{\beta\alpha}(p)(\gamma q)_\beta^\alpha$ Here $\overset{\bullet}{e}{}^3 \gamma$ turns into $\gamma q$. These amplitudes arise because in $SU(6)_W$ we deal with 36 instead of 35. Among them only the amplitude with $G_2$ is of interest. It describes the neutral vector meson production.

Therefore, for all bases the collinear $SU(6)_x$ leads to the amplitudes, which are the same as in the $SU(6)_W$.

Let us discuss briefly the transformations of the collinear $SU(6)_x$. In this case the third vectors turn out to be the definite combinations of two independent momenta. So, for a particle of the momentum p

$$\overset{\bullet}{e}{}_\mu^3 = N_3 (p_\mu - \frac{q_\mu p^2}{pq}).$$

At the same time an indeterminacy arises for the vectors $e_\mu^{n1}$ and $e_\mu^{n2}$ when passing to the collinear case. It should be evaluated, provided that

$$e_\mu^{1k} = e_\mu^{2k} = e_\mu^{3k} = e_\mu^{4k}, \; p e^{1k} = q e^{1k} = 0 \qquad (k=1,2),$$

i. e. these vectors become identical for all particles /3/.

Threefolds (vertices) correspond to the collinear case, and are treated according to $SU(6)_W$.

As to twofolds, none of three vectors $\overset{\bullet}{e}{}_\mu^k$ ( $k = 1,2,3$) can be expressed in terms of the momenta, and therefore the bases must be the same for both wave functions.

Thus , being identical with the $SU(6)_W$ for the collinear configurations, the $SU(6)_x$ is the generalization of the former to the non-collinear case. It is of interest, that the requirement of the total crossing symmetry selects only those groups, which coincide with $SU(6)_W$ in the collinear limit.

## 7. Conclusion

Thus, for the general binary reactions we can construct the family of the symmetry groups $SU(6)_x$, which leave invariant the free equations and pass into $SU(6)_W$ in the collinear case. A lot of consequences ( e. g. (50)) do not depend on the choice of the group within the family. These consequences would be verified first of all. Afterwards to fix the threeleg one could use the predictions, which depend on the basis, e. g. those concerning the polarization phenomena. In spite of an agreement in some cases ( e. g., the Johnson-Treiman relations), it would be naive to expect agreement of the exact $SU(6)_x$ symmetry with experiment. Really, the $SU(6)_x$ contains the strongly violated $SU(3)$. Moreover, the nucleon-isobar mass difference manifests additional violations. The success of the $SU(3)$ is due to the lucky conjectured form of its violations. Only after an examination of the structure of the $SU(6)_x$ violations one will be able to give reasonable predictions.

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## APPENDIX 1

For bilinear combinations of the spinors which satisfy the Dirac equations $(i\gamma p + m_1)\, u(p) = 0$ and $\bar{u}(q)\,(i\gamma q + m_2) = 0$ the following identities are valid

$$(m_1 m_2 - qp)\bar{u}(q)\gamma_\mu u(p) = -i(m_1 q_\mu + m_2 p_\mu)\bar{u}(q)u(p) + \epsilon_{\mu\nu\lambda\rho} p_\nu q_\lambda \bar{u}(q)\gamma_\rho\gamma_5 u(p) \quad (A.1)$$

$$(m_1 m_2 + qp)\bar{u}(q)\gamma_\mu\gamma_5 u(p) = -i(m_1 q_\mu - m_2 p_\mu)\bar{u}(q)\gamma_5 u(p) - \epsilon_{\mu\nu\lambda\rho} p_\nu q_\lambda \bar{u}(q)\gamma_\rho u(p) \quad (A.2)$$

$$(m_1 m_2 - qp)\bar{u}(q)\sigma_{\mu\nu} u(p) = i\epsilon_{\mu\nu\lambda\rho} p_\lambda q_\rho \bar{u}(q)\gamma_5 u(p) - i(p_\mu q_\nu - p_\nu q_\mu)\bar{u}(q)u(p) + \quad (A3)$$

$$+\epsilon_{\mu\nu\lambda\rho}(m_1 q_\lambda + m_2 p_\lambda)\bar{u}(q)\gamma_\rho\gamma_5 u(p)$$

$$(m_1 m_2 + qp)\bar{u}(q)\sigma_{\mu\nu} u(p) = -i\epsilon_{\mu\nu\lambda\rho} p_\lambda q_\rho \bar{u}(q)\gamma_5 u(p) + i(p_\mu q_\nu - p_\nu q_\mu)\bar{u}(q)u(p) - \quad (A.4)$$

$$-(m_1 q_\mu - m_2 p_\mu)\bar{u}(q)\gamma_\nu u(p) + (m_1 q_\nu - m_2 p_\nu)\bar{u}(q)\gamma_\mu u(p) \quad (A.5)$$

$$\bar{u}(q)u(p) = -i\frac{\alpha m_2 p_\sigma + \beta m_1 q_\sigma}{m_1 m_2} u(q)\gamma_\sigma u(p);\ \bar{u}(q)\gamma_5 u(p) = i\frac{\alpha m_2 p_\sigma - \beta m_1 q_\sigma}{m_1 m_2}\bar{u}(q)\gamma_\sigma\gamma_5 u(p), \quad (A.6)$$

where $\alpha + \beta = 1$ . To derive (A.1)–(A.4) we write $\bar{u}\Gamma u = \bar{u}\frac{\gamma q}{im_2}\Gamma\frac{\gamma p}{im_1}u$ and decompose $\gamma q \Gamma \gamma p$ in the complete set of the Dirac matrices. The identities (A.5) follow from the relation

$$\bar{u}\Gamma u = \bar{u}\left(\alpha\Gamma\frac{\gamma p}{im_1} + \beta\frac{\gamma q}{im_2}\Gamma\right)u .$$

It is often convenient to choose arbitrary parameters $\alpha$ and $\beta$ in the forms $\alpha = \beta = \frac{1}{2}$

$$\text{or}\quad \alpha = \frac{m_1}{m_1 + m_2},\ \beta = \frac{m_2}{m_1 + m_2}.$$

Note also the linear identity

$$\epsilon_{\mu\nu\lambda\rho}\frac{p_\lambda}{m}\gamma_\rho\gamma_5 u(p) = (\delta_{\mu\mu'} + m^{-2}p_\mu p_{\mu'})(\delta_{\nu\nu'} + m^{-2}p_\nu p_{\nu'})\sigma_{\mu'\nu'}\ u(p). \quad (A.6)$$

For other identities, in particular, for spin $\frac{3}{2}$ see [8].

## APPENDIX 2. Derivation of the Metric Matrix S(31)

For vectors it is easy to write down a Lorentz transformation which turns the four orthonormal vectors $e^{1\lambda}_{\mu}$ into any other ones $e^{3\lambda}_{\mu}$ ($\lambda = 1,2,3,4$):

$$a_{\mu\nu} = e^{3\lambda}_{\mu} e^{1\lambda}_{\nu} \quad (e^{3\lambda}_{\mu} = a_{\mu\nu} e^{1\lambda}) . \tag{A.7}$$

This form of $a_{\mu\nu}$ remains valid in the presence of reflections. In derivation of matrix S we suppose that the two bases differ from each other in a rotation without reflections. Let us start with the threedimensional rotations. One can find the matrix $S = e^{\frac{i}{2}\vec{\sigma}\vec{\omega}}$ in terms of matrix $\| a_{mn} \|$ with the help of the condition

$$e^{\frac{i}{2}\vec{\sigma}\vec{\omega}} \sigma_m e^{-\frac{i}{2}\vec{\sigma}\vec{\omega}} = \sigma_n a_{nm}, \; a_{nm} = e^{3\ell}_{n} e^{1\ell}_{m} \quad (\ell, m, n = 1,2,3). \tag{A.8}$$

Decomposing the left hand side in matrices $\sigma_n$ we find

$$a_{nm} = 2\cos|\vec{\omega}| \left( \delta_{nm} - \frac{\omega_n \omega_m}{\vec{\omega}^2} \right) + \frac{\omega_n \omega_m}{\vec{\omega}^2} + \frac{\sin|\vec{\omega}|}{|\vec{\omega}|} \epsilon_{nmk} \omega_k . \tag{A.9}$$

Contracting $nm$ and multiplying (A.9) by $\epsilon_{kmn}$ we obtain

$$a_{mm} = 2\cos|\vec{\omega}| + 1, \quad \omega_k = -\frac{|\vec{\omega}|}{2\sin|\vec{\omega}|} \epsilon_{kmn} a_{nm} . \tag{A.10}$$

Hence we find $\vec{\omega}$ in terms of $a_{mn}$ and finally

$$S(3,1) = e^{\frac{i}{2}\vec{\sigma}\vec{\omega}} = \cos\frac{|\vec{\omega}|}{2} + i\frac{\vec{\sigma}\vec{\omega}}{|\vec{\omega}|}\sin\frac{|\vec{\omega}|}{2} = \frac{1}{2}\left(\sqrt{1+a_{mm}} + \frac{i\sigma_i \epsilon_{ijk} a_{jk}}{\sqrt{1+a_{mm}}}\right). \tag{A.11}$$

When verifying the property $\vec{\sigma}\,\vec{e}^{\,3k}\, S(3,1) = S(3,1)\,\vec{\sigma}\,\vec{e}^{\,1k}$, or what is the same, (A.8) the following identity is useful

$$(1 + a_{rr})\left(\frac{a_{mn} + a_{nm}}{2} - \delta_{mn}\right) = \frac{1}{2}(a_{mr} - a_{rm})(a_{rn} - a_{nr}). \qquad (A.12)$$

In our 4-dimensional case the normal remains unchanged, i.e. a rotation is in fact a 3-dimensional one. It can be represented in the form

$$S = e^{\frac{1}{4}\sigma_{\mu\nu}\,\omega_{\mu\nu}} = e^{\frac{1}{2}\omega_i\,\sigma_i},$$

where matrices

$$\sigma_i = \frac{1}{2}\epsilon_{ijk}\, e^{ij}_{\mu}\, e^{ik}_{\nu}\, \sigma_{\mu\nu} \quad (i,j,k = 2,3,4)$$

satisfy the algebra of the Pauli matrices. The matrix S is determined by the equality $S\sigma_{\mu\nu}S^{-1} = \sigma_{\mu'\nu'}\, a_{\mu'\mu} a_{\nu'\nu}$ with $a_{\mu\nu}$ (A.7). This equality can be reduced to (A.8) with $a_{nm} = e^{1n}_{\mu}\, e^{3m}_{\mu}$ (note the property $e^{3m}_{\mu} = e^{1n}_{\mu}\, a_{nm}$). The same considerations as in the euclidian case lead us to eq. (A.11) which is identical to eq. (45). In verification of (46) the following identity for rotations about the normal is useful

$$a_{\rho\rho}\left(\frac{a_{\mu\nu} + a_{\nu\mu}}{2} - \delta_{\mu\nu} + e^{\cdot 1}_{\mu}\, e^{\cdot 1}_{\nu}\right) = \frac{1}{2}(a_{\mu\lambda} - a_{\lambda\mu})(a_{\lambda\nu} - a_{\nu\lambda}). \qquad (A.13)$$

It follows from (A.12). For arbitrary 4-rotations the following relation can be proved

$$S = e^{\frac{1}{4}\sigma_{\mu\nu}\,\omega_{\mu\nu}} = \frac{1+\gamma_5}{2}\left(\cos r_- + \frac{1}{4}\sigma_{\mu\nu}\,\omega_{\mu\nu}\,\frac{\sin r_-}{r_-}\right) + \frac{1-\gamma_5}{2}\left(\cos r_+ + \frac{1}{4}\sigma_{\mu\nu}\,\omega_{\mu\nu}\,\frac{\sin r_+}{r_+}\right) \qquad (A.14)$$

where $r_{\pm} = \sqrt{\dfrac{\omega_{\mu\nu}(\omega_{\mu\nu} \pm \check{\omega}_{\mu\nu})}{8}}$ with $\check{\omega}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}\,\omega_{\lambda\rho}$.

This relation is deduced by the decomposition of $\exp(\frac{1}{4}\sigma_{\mu\nu}\omega_{\mu\nu})$ into the product of two 3-rotations,

$$e^{\frac{1}{4}\sigma_{\mu\nu}\omega_{\mu\nu}} = \exp[\frac{1}{8}\omega_{\mu\nu}\sigma_{\mu\nu}(1+\gamma_5)]\exp[\frac{1}{8}\omega_{\mu\nu}\sigma_{\mu\nu}(1-\gamma_5)].$$

Further, it follows from condition $S\gamma_\mu S^{-1} = \gamma_\nu a_{\nu\mu}$ that

$$a_{\mu\nu} = (\cos r_- \cos r_+ + \omega_{\lambda\rho}\omega_{\lambda\rho}\frac{\sin r_- \sin r_+}{8 r_- r_+})\delta_{\mu\nu} + \frac{\sin r_+ \sin r_-}{2 r_+ r_-}\omega_{\mu\lambda}\omega_{\lambda\nu} + \tag{A.15}$$

$$+ \frac{\sin r_- \cos r_+}{2 r_-}(\omega_{\mu\nu} - \check{\omega}_{\mu\nu}) + \frac{\sin r_+ \cos r_-}{2 r_+}(\omega_{\mu\nu} + \check{\omega}_{\mu\nu})$$

and, finally, that

$$S = \frac{1}{a_{\lambda\lambda}}(\frac{1+\gamma_5}{2}\cos r_- + \frac{1-\gamma_5}{2}\cos r_+)(a_{\mu\mu} + i\sigma_{\mu\nu} a_{\mu\nu}) \tag{A.16}$$

$$16\cos^2 r_\pm = (a_{\mu\mu})^2 + a_{\mu\nu}(a_{\mu\nu} - a_{\nu\mu}) \mp 2 a_{\mu\nu}\check{a}_{\mu\nu} \tag{A.17}$$

$$4\cos r_- \cos r_+ = a_{\mu\mu} .$$

The expression (A.16) represents a general rotation (without reflection) of spinors in terms of the rotation matrix for vector $\| a_{\mu\nu} \|$ . With the help of (A.17) one can obtain the identity

$$16(a_{\mu\mu})^2 = [(a_{\mu\mu})^2 + a_{\mu\nu}(a_{\mu\nu} - a_{\nu\mu})]^2 - 4(a_{\mu\nu}\check{a}_{\mu\nu})^2 . \tag{A.17}$$

Note the useful identity which expresses the symmetric part of an arbitrary orthogonal matrix in terms of its antisymmetric part and its trace

$$a_{\lambda\lambda}(a_{\mu\nu} + a_{\nu\mu} - \frac{1}{2}\delta_{\mu\nu}a_{\rho\rho}) = (a_{\mu\lambda} - a_{\lambda\mu})(a_{\lambda\nu} - a_{\nu\lambda}) + \frac{1}{2}\delta_{\mu\nu}a_{\lambda\rho}(a_{\lambda\rho} - a_{\rho\lambda}). \tag{A.18}$$

This identity was used also in /15/ .

The matrix S (A.16) can serve as a "metric matrix" for many particle amplitudes when bases of particles do not have any common vector, and we can not confine ourselves to the 3-dimensional rotations.

## Appendix 3 Other Forms for the Ratio $f_1 : f_2$

Invariant amplitude can be derived directly from the requirement of the invariance (without use of the metric matrix). As an example we consider the singlet-quark scattering. For generality the masses of initial and final quarks and singlet are supposed to be different. The general amplitude is written down as

$$M = \bar{u}(p_3)[f_1 + i f_2(\gamma, p_2 + p_4)] u(p_1). \qquad (A.19)$$

After transformations (8) we have

$$\delta M = \bar{u}(p_3)[(\alpha f_1 + \beta f_2)\gamma_5 + (r f_1 + \rho f_2)(\gamma, p_2 + p_4)\gamma_5] u(p_1),$$

where $\alpha, \beta, r$ and $\rho$ are definite functions of $A$ (stu 1234) and masses. Two homogeneous equations $\alpha f_1 + \beta f_2 = 0$ and $r f_1 + \rho f_2 = 0$ turn out to be compatible: vanishing of the determinant is equivalent to the equality of the norms (19) of vectors $e^{33}_\mu$ and $e^{13}_\mu$. In this way we obtain the ratio $f_1 : f_2$ in two forms: one of them is the ratio of (52a) to (52b) and the second one is the ratio of the quantity

$$m_3(s - m_2^2 - m_1 m_3) A(\text{stu } 1234) - m_1(s - m_4^2 - m_1 m_3) A(\text{stu } 3412) +$$

$$+ [t - (m_1 - m_3)^2][m_3 A(\text{tsu } 1324) - m_1 A(\text{tsu } 3142)]$$

to the quantity

$$m_1 m_3 [A(stu\,1234) - A(stu\,3412)].$$

The latter form of the ratio is inconvenient as it contains an indeterminacy when passing to the equal masses. We can receive third form for this ratio if we evaluate $\bar{u}(p_3)S(3,1)u(p_1)$ with the use of the explicit form of basic vectors and the identity (A.4). In this way the ratio $f_1 : f_2$ turns out to be the ratio of the quantity

$$A(stu\,1234)A(stu\,3412)[s^2 - s(m_2^2 - 2m_1m_3 + m_4^2) - m_1m_3(m_1^2 - m_2^2 + m_3^2 - m_4^2) - m_1^2m_3^2 + m_2^2m_4^2] + A(stu\,1234)A(tsu\,3142)[t-(m_1+m_3)^2]\cdot$$

$$\cdot(s - m_2^2 - m_1m_3) + A(tsu\,1324)A(stu\,3412)[t-(m_1+m_3)^2](s - m_4^2 - m_1m_3) +$$

$$+ A(tsu\,1324)A(tsu\,3142)[(t - m_1^2 - m_3^2)^2 - 4m_1^2m_3^2] - 4m_1m_3$$

to the quantity

$$[(s - m_2^2 + m_1m_3)m_3 + (s - m_4^2 + m_1m_3)m_1]A(stu\,1234)A(stu\,3412) +$$

$$+[t-(m_1+m_3)^2][m_3A(stu\,3412)A(tsu\,1324) + m_1A(stu\,1234)A(tsu\,3142)].$$

One can directly verify the equivalence of this ratio with the ratio (52a) to (52b). The verification is rather tedious and uses essentially the equality of norms (29) of vectors $e^{13}$ and $e^{33}$.

## REFERENCES


1. R.H. Dalitz. XIII International Conference on High Energy Physics at Berkeley, preprint Oxford, 1966.

2. H.I. Lipkin, S. Meshkov. Phys. Rev. Lett., 14, 670, 1965.
J.C. Carter, J.J. Coyne, S. Meshkov, D. Horn, M.Kugler, H.J. Lipkin. Phys. Rev. Lett., 15, 373 (1965).
3. K.J. Barnes. Phys. Rev. Lett., 14, 798, 1965; Phys. Rev., 139, B947 (1965).
4. R. Delbourgo, M.A. Rashid, A. Salam, J. Strathdee. High Energy Physics and Elementary Particles, Intern. Atomic Energy Agency, Vienna 1965.
5. V.G. Kadyševsky, R.M, Muradjan, Ja. A. Smorodinsky. Fortshr. d. Physik, 13, 599 (1965).
A. Pais. Rev. Mod. Phys., 38, 215 (1966).
6. H. Ruegg, W. Rühl, I.S. Santhanam. Preprint CERN IV, 66/1106/5.
7. V.I. Ogievetsky, I.V. Polubarinov. JETP Letters, 4, 325, 1966.
A.N. Zaslavsky, V. Tybor. Preprint P2-3087, Dubna, 1966.
8. V.I. Ogevetsky, I.V. Polubarinov. Yader. fiz., 4, 1231, 1966 ; Preprint P -2696 , Dubna, 1966.
9. H. Ruegg, D.V. Volkov. Nuovo Cim., 43, 84, 1966.
10. L. Mishel, A.S. Wightman. Phys. Rev., 98, 1190, 1955.
11. V. Bargman, E.P. Wigner. Proc. Nat. Acad. Sci. U.S.A., 34, 211, 1948;
Yu.M. Shirokov. JETP, 21, 748, 1951.
12. A. Salam. Phys. Lett., 13, 354, 1964.
13. Ya. A. Smorodinsky. IX Intern. Conference on High Energy Phys. at Kiev, 1959; p. 271, Moscow, 1960.
14. M. Cini. Phys. Lett., 19, 251, 1965.
15. B.A. Arbuzov, A.T. Filippov. Preprint E2-3089, Dubna, 1967.

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