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ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ

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CP-VIOLATION AS CONSEQUENCE OF GEOMETRIZATION OF MAXWELL FIELD

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1. Introduction

When the parity nonconservation was discovered in 1956[1] the serious fault of the theoretical conception of the space-time symmetries was exposed. Indeed, till 1956 the invariance under the discrete space-time transformation (the space reflection and the time reversal) was thought to be the evident consequence of the most general properties of the space-time, which are described by the postulates of the special theory of relativity (see e.g.[2]). The observed violation seemed therefore to be inconsistent with the basic, well established properties of the space-time.

There was found however a beatiful way around this difficulty. Landau, Lee and Yang and Wigner[3-5] have suggested, that the time operation of the space reflection is not P but the combined inversion CP and thus all interactions were CP invariant. In virture of CPT theorem, the breaking of which would lead to a thorough revision of the basis of the relativistic quantum mechanics (see e.g.[6]) all interactions appeared to be T-invariant and the symmetry of the space time was restored.

The hypothesis of CP-invariance agreed with numerous experiments and soon everybody considered it as one of the fundamental laws of Nature. So the discovery in 1964 of the decay $K_L \to \pi^+ \pi^-$, which is forbidden by CP-invariance, was a sensation. Simple ways to save CP were rejected after the detailed experimental investigation of the K-decays (see rewiews[8-12]). It became clear, that our conception of discrete symmetries of the space-time needs a substantial change (see, for example, [10] and [12]).

It may be noted, however, that there are, in principle, some possibilities of going around this difficulty. One of these bears a resemblance to the CP-hypothesis and was indicated by Lee and Yang in 1957 in connection with discussions of possible T-violation. They supposed, that our world is "doubled" with respect to some new quantum number, which corresponds to some new degree of freedom of particles. Thus each particle has its counterpart, a "mirror particle", which differs from the former only in this new quantum number. Then the symmetry of the world is restored if one admits that the true operation of space reflection (or time reversal) is the product of CP (or T) and the operation of transition from the usual particles to the "mirror" ones. The recent detailed discussion of this hypothesis[14] has led to a conclusion, that the usual particles may interact only very weakly with the "mirror" ones. Therefore this simple way to save the habitual notions may prove quite imaginary.

Another possibility of the new interpretation of discrete symmetries was considered by T.D.Lee and Wick[15]. They introduce different definitions of the discrete transformations in different interactions and in fact they give up the geometrical interpretation of discrete symmetries, considering them as dynamical[12] ones. In addition there is some ambiguity in the definition of the new operators and general principles to avoid this ambiguity, seem to be lacking.

Thus, if even we consider these ways to be open, we can say without exaggeration that the problem of discrete space-time symmetries is now in almost the same position as it was in 1957. Therefore any attempts of the geometrical interpretation of the discrete symmetries seem to be quite in good time. Such attempts were not numerous (we found only several works[16] about geometric interpretation of P-violation[x]).

One possibility was pointed out in [17] and was discussed later in[18,19]. In these papers the geometric approach to the theory of weak interaction was considered, in which the weak interaction appears as a consequence of the space-time curvature "inside" the particles. P-nonconservation arises then in virtue of simple geometric assumption. In subsequent papers [20-22] we tried to find a similar geometric interpretation of CP violation. In the ground of this interpretation rests a conjecture of some link between the electromagnetic field and the space-time torsion. Some additional physical hypothesis on the form of this connection were admitted by us to predict several effects in weak-electromagnetic interaction (with the coupling constant $\approx Ge$, $G \approx \frac{10^{-5}}{m^2}$ is the weak interaction constant, and e is the elementary electric charge). In the works, mentioned above, we discussed in detail the possibilities of the detection of such effects and the difference in predictions of geometric model and other models of CP violation (see e.g. [23-26]) but the geometric interpretation itself was only mentioned.

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[x]See also [45], where a mechanical model of P-violation in spaces with torsion is sketched.

The present paper is devoted to an attempt to construct a consistent geometric theory of electromagnetic (EM) field, which is based on the interpretation of EM field as a torsion of the space-time. Of course, our final intention is to understand the connection between weak and electromagnetic interaction (see[18, 19,22]) but this time we shall not take into account the space-time curvature, and therefore, shall not try to construct the unified theory of weak interaction and electromagnetism[x]. Note that in what follows we widely exploit the methods which were used by Einstein in his attempts to create the unified theory of gravitation and electromagnetism[xx], but we totally give up the idea of the link between EM field and gravitational field.

It is worth noting that the mathematical formalism which we use here, differs from the one used in[20-22]. In fact the introduction of the nonsymmetric metric tensor is a purely formal procedure and throws quite a pure light on the geometry of the space-time. The geometry is uniquely determined if one define the curvature and torsion tensors (see e.g.[29-35]), which may be expressed in terms of the affine connexion. In the general theory of relativity the Euclidean space is generalized up to Riemannian space with the symmetric connection which defined the zero torsion. However we think the simplest generalization of the notion of Minkowsky space is the space with zero curvature, pseudoeuclidean metric and the torsion different from zero (nonsymmetric connexion). We shall show that the investigation of such spaces leads to the geometric interpretation of the free electromagnetic field. The simple geometric restriction imposed on the torsion give the generalized Maxwell equations which reduce to the usual Maxwell equations in the limit of the weak field. Then we consider the Dirac equation in this space and show that it automatically contains the CP-odd interactions of the form which we have postulated earlier[20-22]. It is possible that the detailed form of this interaction for different particles may be found only in the unified theory of the weak and electromagnetic interactions.

2. The basic properties of the space with absolute parallelism

In this section we briefly describe the theory of the spaces with absolute parallelism following mainly Cartan[29], who first considered these spaces, and Einstein[27,28] who used these spaces in one of the variants of the unified field theory. We concentrate our attention on the facts which will be essential in what

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[x] Some connection between these phenomena will be always kept in mind, however.

[xx] See[27]. In the book[28] one may find other works devoted to the same problem.

follows. The mathematical details may be found in the mentioned works by Cartan and in the books by Eisenhart[30] and Schroedinger[31, x]. We stress once more that the model under consideration has nothing to do with the unified theories of gravitaion and electromagnetism, and is based rather on the attempt to unite the phenomena of electromagnetism and weak interactions on the geometric grounds.

The space with absolute parallelism is defined locally by the condition that the result of the parallel displacement of the vector from any point x to the other arbitrary point y is independent of the path by which the displacement goes. This condition is equivalent to the possibility of constructing the system of linearly independent vectors $h^i_a(x)$ (a is the number of the vector, $h^i_a(x)$ is the projection of a-th vector on the i-th axis of some given system of coordinates in the point x ; $a, i = 0,1,2,3$ [xx]) and the system h in the point y is obtained from the one in the point x in the way of parallel displacement (see [27-34]). The parallel displacement is expressed in terms of the affine connexion Γ^i_{jk}. The contravariant components of any vector $A^i(x)$ get the following increments while parallel displacement from the point x^k to the nearly point $x^k + \delta x^k$

$$\delta A^i(x) = -\Gamma^i_{jk}(x) A^j(x) \delta x^k \tag{2.1}$$

and the covariant ones $A_j(x)$:

$$\delta A_j(x) = \Gamma^i_{jk}(x) A_i(x) \delta x^k \tag{2.2}$$

(the repeated indices are supposed to be summed up).

Thus, we obtain the following equation for vectors of the ennuple

$$h^i_{(a),k} \equiv \partial_k h^i_{(a)} \equiv \frac{\partial}{\partial x^k} h^i_{(a)} = -h^j_{(a)} \Gamma^i_{jk} \tag{2.3}$$

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[x] Some information concerning the spaces with torsion and, in particular, the spaces with absolute parallelism may be found in the well known books on differential geometry (see e.g.[32-35]).

[xx] We shall always consider the four-dimensional space-time although the method may be used in the case of the spaces having any number of dimensions and any metrics.

Introducing the normalized minors $h_{(a)j}$ of the matrix $h^{i}_{(a)}$ which are defined by the equations[x]

$$h^{i}_{(a)}(x)\, h^{(a)}_{j}(x) = \delta^{i}_{j} \tag{2.4}$$

we get from (2.3) the expression for

$$\Gamma^{i}_{jk} = h^{i}_{(a)}\, h^{(a)}_{j,k} = - h^{(a)}_{j}\, h^{i}_{(a),k} \tag{2.5}$$

(the last equality follows from Eq. (2.4)). In the space with the absolute parallelism Eq. (2.3) should be integrable, so it follows

$$0 = h^{i}_{(a),jk} - h^{i}_{(a),kj} = \partial_{k}(\Gamma^{i}_{\ell j}\, h^{\ell}_{(a)}) - \partial_{j}(\Gamma^{i}_{\ell k}\, h^{\ell}_{(a)})\ . \tag{2.6}$$

Using once more Eq. (2.3) we find that the Riemannian curvature tensor $R^{i}_{jk\ell}$ vanishes

$$R^{i}_{jk\ell} \equiv -\Gamma^{i}_{jk,\ell} + \Gamma^{i}_{j\ell,k} - \Gamma^{s}_{jk}\, \Gamma^{i}_{s\ell} + \Gamma^{s}_{j\ell}\, \Gamma^{i}_{sk}\ . \tag{2.7}$$

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[x]Here and in what follows δ^{i}_{j} denotes the usual kroneker symbol, whereas $\delta^{ij} = \delta_{ij}$ is the diagonal matrix with the elements $\delta_{00} = 1, \delta_{11} = \delta_{22} = \delta_{33} = -1$. We rise and lower the indices (a) with the aid of the metric tensor δ^{ab}. For instance $h^{(a)}_{j} = \delta^{ab} h_{(b)j}$ etc.

One may show (see e.g.[30,31]) that the last condition is also sufficient for Eq. (2.3) to be integrable. Thus, the affine space has the absolute parallelism if and only if its curvature tensor is indentically zero (Schroendinger[31] calls such spaces integrable). The existence of the ennuple $h^{i}_{(a)}$ which gives the affine connection by Eq. (2.5) is equivalent to this condition. Without loss of generality, we may consider in the following all the ennuples to be orthogonal and normalized ones

$$h^{i}_{(a)} h^{(b)}_{i} = \delta^{b}_{a}; \quad h^{i}_{(a)} h_{(b)i} = \delta_{ab}, \quad h^{(a)i} h^{(b)}_{i} = \delta^{ab}. \tag{2.8}$$

Then, from the geometric sense of the quantities $h^{i}_{(a)}$ and $h_{(a)j}$ we find the following connection between h and the metric tensor g_{ij}

$$g_{ij} = h_{(a)i} h^{(a)}_{j} \quad ; \quad g^{ij} = h^{i}_{(a)} h^{(a)j} \tag{2.9}$$

It should be noted, that the quantities $h^{i}_{(a)}$, $h_{(a)i}$ are not uniquelly defined. In fact both Eqs. (2.5) and (2.9) do not change under the transformation

$$h^{i}_{(a)} \to L^{(b)}_{(a)} h^{i}_{(b)}; \quad h^{(a)}_{i} \to L^{(a)}_{\cdot (b)} h^{(b)}_{i} \tag{2.10}$$

where $L^{(b)}_{(a)}$ is an arbitryry pseudoorthogonal matrix[x] independent of x.

One may get rid of this uncertainty with the aid of the following physical condition. Consider such coordinates which continuously transform into Cartesian coordinates in any finite region of the space, when the space, becames flat (switching off the interaction). Inasmach as in this limiting process, the axes of all the ennuples h are always parallel (in the sense of the absolute parallelism), we come in this Cartesian limit to the ennuples with the axes being parallel to each other (in the usual sense) but in general not parallel to

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[x] The pseudoorthogonal matrix satisfies the following conditions $L^{(c)}_{(a)} \delta_{cd} L^{(d)}_{(b)} = \delta_{ab}$; $L^{(a)}_{(c)} \delta^{cd} L^{(b)}_{(d)} = \delta^{ab}$, where δ^{ab} is the metric tensor of the pseudoeuclidean space (see above.)

the Cartesian axes. With the aid of the transformation (2.10) we may always succeed in choosing the coordinates so, that the ennuple is directed in the same way as the axes of the basic Cartesian coordinates do, i.e. in the limit we have

$$h^{i}_{(\alpha)} = \delta^{i}_{\alpha}\,,\ h_{(\alpha)i} = \delta'_{\alpha i}\,, \tag{2.11}$$

For the general case we might exploit these conditions to eliminate the ambiguity in choosing the ennuple, but here we shall use more simple formal devise, which we describe in the next section.

It is worth noting that the orthogonal ennuple is not quite necessary for the description of the spaces with the absolute parallelism. All the theory may be developed without introducing these objects (see e.g.[32]). We use the ennuple first because the spaces with absolute parallelism are described in its terms in the most simple and natural way, and, secondly, because with the aid of the orthogonal ennuple one may in a simple way introduce spinors in non-euclidean spaces[x]. By the way, there are other methods, quite convenient, to introduce the spinors in noneuclidean spaces (see, in particular [40,41]), which we suppose to consider elsewhere.

In the conclusion of this section we consider the condition, connecting the affine connexion and the metrix. One may come to this condition by demanding the metric structure given by the affine connexion to agree with the metrics, defined by the tensor g_{ij}. In other words, the distance which is defined by metric tensor $ds^2 = g_{ij}\, dx^i\, dx^j$ should be the same distance, which may be defined along any geodesic line with the aid of the affine connexion only. As was shown, for example, in the book by Schroedinger[31], in the general case, for this condition to satisfy it is necessary and sufficient for the symmetric part of the connexion to be represented in the form

$$\Gamma^{i}_{jk} = \frac{\Gamma^{i}_{jk} + \Gamma^{i}_{kj}}{2} = \left\{ {i \atop jk} \right\} + g^{i\ell}\, T_{\ell jk}$$

where $\left\{ {i \atop jk} \right\}$ are the Christoffel brackets and the symmetric in j and k

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[x]See[36-39]. The most thorough results were obtained by V.A.Fock and were set forth in his paper[32].

arbitrary tensor $T_{\ell jk}$ satisfies the condition

$$T_{\ell jk} + T_{j\ell k} + T_{k\ell j} = 0.$$

These conditions do not give any restriction on the antisymmetric part of the affine connexion

$$\Omega^{k}_{ij} = \frac{1}{2}(\Gamma^{k}_{ij} - \Gamma^{k}_{ji}); \quad \Omega^{k}_{ij} = -\Omega^{k}_{ji} \tag{2.12}$$

which is a tensor and according to Cartan[29] (see also[30-35]) defines the torsion of the space[x].

It would be more natural, therefore, to lay down in the ground of our speculations the stronger demand: the metric tensor in the point x should be obtained from the one in the other point y by parallel displacement i.e.[xx]

$$g_{ij|\ell} = g_{ij,\ell} - \Gamma^{s}_{i\ell}\, g_{sj} - \Gamma^{s}_{j\ell}\, g_{is}. \tag{2.13}$$

It is easy to see, that the general conditions of consistency of metrics with the affine connexion follow from this demand, but the inverse assertion is, generally speaking, not true. Thus, Eq.(2.13) generally imposes essential restrictions on the geometry of the space. However, it is easy to verify, that for the space with the absolute parallelism the condition (2.13) is automatically satisfied. Actually, the metric tensor g_{ij} is defined in terms of the ennuple $h^{i}_{(a)}$ by the relation (2.9). The ennuples in different points transform into another ones by parallel displacement i.e.

$$h^{i}_{(a)|\ell} = 0; \quad h_{(a)i|\ell} = 0. \tag{2.14}$$

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[x] The geometric meaning of the tensor Ω is described e.g. in[30, 32-34].

[xx] We denote by the symbol $|a$ the covariant derivative.

This condition may be obtained in a formal way, using Eqs. (2.4), (2.5) and the definition of the covariant derivative of the vector

$$A^i_{|\ell} \equiv A^i_{,\ell} + \Gamma^i_{j\ell} A^j ; A_i|_\ell \equiv A_{i,\ell} - \Gamma^j_{i\ell} A_j . \qquad (2.15)$$

Thus, we see that in any space with the absolute parallelism metrics agrees with the affine connexion and, moreover, the change of a metric tensor by transition from one point to another may be obtained with the aid of the parallel displacement.

3. Pseudoeuclidean space with the torsion and the free electromagnetic field

Consider now the simplest space with the absolute parallelism, namely the space, in which we retain the usual pseudoeuclidean metrics

$$g_{ij} = \delta_{ij} , \; g^{ij} = \delta^{ij} . \qquad (3.1)$$

We shall call such spaces pseudoeuclidean spaces with torsion. The metric relations in these spaces are the same as in the usual Minkowsky geometry, but the parallel displacement is essentially different because of the torsion. We shall not rewrite all the formulae of the preceding section, and keep in mind, that one should always set δ^{ij} , δ_{ij} instead of g^{ij} , g_{ij} .

Let us write down several significant relations

$$\Gamma_{ijk} = \delta_{i\ell} \Gamma^{\ell}_{jk} , \; \Omega_{ijk} = \Omega^{\ell}_{ij} \delta_{\ell k} . \qquad (3.2)$$

Eqs. (2.4) and (2.8) lead to the condition that the matrix $h^i_{(a)}$ is pseudoorthogonal:

$$h^{(a)}_i h_{(a)j} = h_{(a)i} \delta^{ab} h_{(b)j} = \delta_{ij} . \qquad (3.3)$$

From Eq. (2.5) we may now find, that

$$\Gamma_{ijk} = -\Gamma_{jik} . \tag{3.4}$$

This symmetry condition may be obtained also from Eq. (2.13) keeping in mind that $\delta_{ij,\ell} = 0$.

From the definition (2.12) of the torsion tensor Ω and from the condition (3.4) we may find the useful relation

$$\Gamma_{ijk} = -\Omega_{ijk} + \Omega_{jki} + \Omega_{kij} \tag{3.5}$$

from which, in particular, follows the tensor character of the affine connexion Γ_{ijk} with respect to the transformations, preserving the metrics (3.1).

Let us count the number of the independent functions, with define the geometry of the pseudoeuclidean space with torsion. In virtue of the orthogonality conditions (3.3), the matrix $h_{(a)i}$ has only six independent elements, in terms of which one can express all the geometric quantities. One would take the antisymmetric part of the matrix $h_{(a)i}$ as the independent functions, but in our case we may proceed in another way. Consider any Lorentz frame of reference and set

$$h_{(a)i}(x) = \ell_{(a)i} + F_{ij}(x)\ell^{j}_{(a)} \tag{3.6}$$

where $\ell_{(a)i}$ is the arbitrary constant pseudoorthogonal matric. The matrix F_{ij} is evidently a tensor under any Lorentz transformations of coordinates and does not change under transformation of the ennuple (2.10). In the limit of the extreemely small torsion the ennuple h does not depend on x, so we may assume, that tensor $h_{(a)i}$ becames also very small. The ennuple $h_{(a)i}$ will coincide with the uniquelly determined one (according to section 2), if we set $\ell_{(a)i} = \delta_{ai}$.

From Eq. (2.4) and using the arbitrariness of $\ell_{(a)i}$ we find the condition on the tensor F_{ij}

$$F_{ij} + F_{ji} + F_{ik}\,\delta^{k\ell}\,F_{j\ell} = 0 . \tag{3.7}$$

Thus, the matrix $\delta_{ij} + F_{ij}$ is pseudoorthogonal. The symmetric part of this matrix

$$\delta_{ij} + \bar{F}_{ij} = \delta_{ij} + \frac{1}{2}(F_{ij} + F_{ji}) \tag{3.8}$$

may be expressed in terms of the antisymmetric part

$$f_{ij} = \frac{1}{2}(F_{ij} - F_{ji}).$$

Namely (see Appendix)

$$\bar{F}_{ij} = \frac{1}{4}[\bar{s} - \frac{2s}{s+4}]\delta_{ij} + \frac{2}{s+4}(f_{ik}\delta^{k\ell}f_{\ell j}), \bar{s} = \bar{F}_k{}^k, \ s = f^{ij} f_{ji} . \tag{3.9}$$

We may also express $\bar{s}$ in terms of s and $d = \det|f_{ij}|$ (see Appendix) but the corresponding relations are rather involved and we make here no use of them. For the small torsion we have evidently

$$F_i^{\cdot k} = \frac{1}{2} f_i^{\cdot \ell} \ f_\ell^{\cdot k} + o(f^2) \tag{3.10}$$

It is useful to write down the explicit expression of Γ_{ijk} in terms of F_{ij}

$$\begin{aligned} \Gamma_{ijk} &= F_{ji,k} + F_i^{\cdot s} F_{js,k} = \\ &= -f_{ij,k} + \frac{1}{2}(F_i^{\cdot s} F_{js,k} - F_j^{\cdot s} F_{is,k}) . \end{aligned} \tag{3.11}$$

In the last expression the validity of the antisymmetry property (3.4) is evident. Inserting Eq. (3.9) in Eq. (3.11) we may express the connexion Γ_{ijk} , and, consequently, the torsion Ω_{ijk} in terms of six independent and still arbitrary functions f_{ij}. Therefore the geometry of the space-time under consideration is still quite arbitrary.

To get rid of this ambiguity and restrict in some manner the choice of the spaces, we shall proceed along the way, which has proved extremely successful in the Einstein's general theory of relativity[42]. The way of reasoning of Einstein is in plane words the following. Let us find all the irreducible tensors, which may be constructed with the aid of the tensor, defining the geometry (in the Einstein theory it is the Riemannian curvature tensor $R^{i}_{jk\ell}$. Take the simplest irreducible tensor and put it to be equal to zero. The simplest nontrivial equation, really restricting the geometry, is the Einstein equation, which for the free case reads $R_{ij} - \frac{1}{2} g_{ij} R = 0$.

In our case the geometry is thoroughly defined by the tensor Ω_{ijk}. Let us split it into the irreducible tensors[x]. We can do it easly using the operations of symmetrization, alternation, contraction and multiplication by metric tensor δ_{ij} and Levi-Chivita tensor density. In this way we evidently cannot construct any irreducible scalars or second rank tensors, but we easily find irreducible vector V_i and pseudovector A_i

$$V_i = \Omega^{\cdot\,\cdot\,k}_{ik} \; ; \; A_i = \epsilon_{ijk\ell}\, \Omega^{jk\ell} . \tag{3.12}$$

In the absence of the matter (free case) we have not any other vector or pseudovector and it is natural to assume the following equations for the torsion

$$V_i = 0; \; A_i = 0 . \tag{3.13}$$

In the following we shall see, that these equations are the nonlinear equations for the tensor f_{ij} which generalize the Maxwell equations and coincide with the latter for the small values of $|f_{ij}|$. This solves the problem of consistency at least for the case of the small torsion[xx].

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x The general device for constructing the irreducible tensors is developed by Cartan[29]. Here we may use simple considerations.

xx For the full solution of this problem it would be necessary to deduce this equation from a Lagrangian.

From the second equation (3.13) and Eq. (3.5) it follows, that

$$\Omega_{ijk} = -\frac{1}{2}\Gamma_{ijk} . \tag{3.14}$$

Therefore, using Eq. (3.11) we obtain

$$\Omega_{ijk} = \frac{1}{2} f_{ij,k} - \frac{1}{4}[F_i^{\cdot s} F_{jsk} - F_j^{\cdot s} F_{is,k}] \tag{315}$$

and we may rewite Eqs. (3.13) in the form

$$f^{ij}_{\cdot\cdot,j} = \frac{1}{2}[F^{is} F^{j}_{\cdot s,j} - F^{js} F^{i}_{\cdot s,j} \quad] \tag{3.16a}$$

$$\begin{aligned} f_{ij,k} + f_{jk,i} + f_{ki,j} = \frac{1}{2}[& F_i^{\cdot s} F_{js,k} + F_j^{\cdot s} F_{ks,i} + F_k^{\cdot s} F_{is,j} - \\ & - F_j^{\cdot s} F_{is,k} - F_k^{\cdot s} F_{js,i} - F_i^{\cdot s} F_{ks,j}] . \end{aligned} \tag{3.16b}$$

In the approximation of the weak torsion nonlinear terms in Eqs. (3.16a) and (3.16b) may be neglected and we find, that the tensor f_{ij} satisfies Maxwell equations

$$f^{ij}_{\cdot\cdot,j} = 0, \; f_{ij,k} + f_{jk,i} + f_{ki,j} = 0 . \tag{3.17}$$

This leads us to a conjecture, that the tensor f_{ij} is proportional to a tensor of electromagnetic field H_{ij}

To come to the idea concerning the coefficient of proportionality, we note that the electromagnetic tensor has the dimension that of square mass (in units $h = c = 1$) whereas tensor f_{ij} is dimensionless. We have explained in the Introduction, that we think to be natural to seek for the unified theory of the weak and electromagnetic interactions. Following this line, we may use the universal weak interaction constant

$$G = \frac{10^{-5}}{m_p^2} \tag{3.18}$$

to obtain the magnitude of the constant, which connect the tensors f and H. From the dimensionality consideration we set

$$f_{ij} = \lambda\,(Ge)\,H_{ij} \tag{3.19}$$

where λ is a dimensionless number and the factor e is written explicitly to stress, that the effects of the space torsion (or, as we shall see, of the CP-nonconservation) are displayed only in weak electromagnetic interactions (see[20-22]). Of course, we may hope to obtain the exact form of the relations (3.18) i.e. the value of the constant only in a more complete theory, consistently taking into account both the curvature and the torsion of the space.

The smallness of the constant G allows us to justify the neglection of the nonlinear terms in Eqs. (3,16). Actually, for this neglection be valid, the following condition should be fulfilled

$$|f_{ij}| = \lambda\, Ge\, |H_{ij}| \ll 1; \;\; \lambda\, Ge\, |E| \ll 1, \;\; \lambda\, Ge\, |H| \ll 1$$

where E and H are correspondingly the vectors of the electric and magnetic fields. In other words these conditions read

$$|E| \ll 5\cdot 10^{27}\ \frac{\text{volt}}{\text{cm}}, \quad \epsilon \ll \lambda^2\, 10^{55}\ \frac{m_p}{\text{cm}^3}$$

where ϵ is the density of electromagnetic energy. Thus, it is evident, that in all the usual cases we can neglect the nonlinear terms in Eqs. (3.16), having a numerical estimate for the possibility of this neglection.

It is useful to discuss the problem of the uniqueness of our choice of equations. Inasmuch as in the theory of the pseudoeuclidean space with the torsion, $\Gamma_{ij,k}$ is also a tensor, it would seem, that we may obtain another set of equations instead of (3.13), if we change Ω by Γ. However, one may be easly convinced, that

$$\Gamma_{ijk} + \Gamma_{jki} + \Gamma_{kij} = \Omega_{ijk} + \Omega_{jki} + \Omega_{kij} \tag{3.19}$$

and so both the second equations coincide. From the second equation Eq.(3.14) follows immediately and we obtain

$$\Gamma^{ik}_{\cdot\cdot k} = -2\Omega^{ik}_{\cdot\cdot k} \tag{3.20}$$

and the first equations are the same also. In the more complicated spaces only Ω is tensor and this question does not arise at all.

4. Spinor field in pseudoeuclidean space with torsion. CP-violation in interaction of spinor particles with the electromagnetic field

In the space with the absolute parallelism the equations for spinor particles may be introduced quite naturally. The most simple way to achieve the goal is to use the described above formalism of the absolutely parallel ennuples. While constructing the spinor equations we shall follow V.Fock[38,39], who gave elaborated in detail ennuple method for Riemannian spaces without torsion. As we shall see, the case of pseudoeuclidean spaces with torsion provides an additional simplification and the introduction of spinors goes without trouble.

Let us define the set of the usual Dirac matrices satisfying the anticommutation relation

$$\{\gamma^{(a)}, \gamma^{(b)}\} \equiv \gamma^{(a)}\gamma^{(b)} + \gamma^{(b)}\gamma^{(a)} = 2\delta^{ab} . \tag{4.1}$$

In the space under consideration these matrices are not the objects of the vector nature, because the parallel displacement of $\gamma_{(\alpha)}$ to another point give another set of matrices $\gamma'_{(\alpha)}$. However, we can easily construct from the vector objects, using the ennuple coefficients $h^{i}_{(\alpha)}$. Let us define the following matrices β_i which depend on the point

$$\beta_i = \gamma^{(\alpha)} h_{(\alpha) i} \; . \tag{4.2}$$

From Eq. (3.3) it follows, that these matrices satisfy the anticommutation relations

$$\{\beta_i \, \beta_j\} = 2\delta_{ij} \tag{4.3}$$

and their vector character is evident.

Consider now bilinear spinor combinations $\bar{\psi} B \psi$ where B is a matrix from the algebra of matrices β_i and $\bar{\psi} = \overset{+}{\psi} \beta_0$. To define the transformation of the spinors ψ and $\bar{\psi}$ in the process of parallel displacement we demand (compare[39]) the quantity $\bar{\psi}\psi$ to be scalar and $\bar{\psi} \beta_i \psi$ to be a vector. Then, if we make the parallel displacement from the point x^k to the point $x^k + \delta x^k$ these bilinear combinations should aquire the following increments

$$\delta \bar{\psi}(x) \psi(x) = 0; \tag{4.4a}$$

$$\delta \bar{\psi}(x) \beta_i \psi(x) = \Gamma^{i}_{jk} (\bar{\psi} \beta_i \psi) \delta x^k \tag{4.4b}$$

The increment $\delta\psi$ of the spinor is, by definition, equal to

$$\delta\psi(x) = C_k(x) \psi(x) \delta x^k \tag{4.5}$$

From Eq. (4.4a) we find

$$\delta \bar{\psi}(x) = -\bar{\psi}(x) C_k \delta x^k \tag{4.6}$$

and from Eq. (4.4b) we obtain the relation

$$\bar{\psi}[\beta_i C_k]\psi + \bar{\psi}\delta\beta_i \psi = \Gamma^i_{jk} (\psi \beta_i \psi) \delta x^k \tag{4.7}$$

where (see Eq. (2.3))

$$\delta \beta_i = \gamma^{(a)} h_{(a)i} = \Gamma^i_{jk} \beta_i \delta x^k \tag{4.8}$$

Inserting Eq. (4.8) into Eq. (4.7), we find

$$[\beta_i, C_k] = 0 \tag{4.9}$$

and come to the following spinor connexion C_k [x]

$$C_k = ieIA_k \tag{4.10}$$

where I is the unit matrix and A_k is an arbitrary real vector, which has been often identified with the electromagnetic potential. We also may admit this interpretation of the vector A_k to obtain the usual electromagnetic interaction, but we note, that this interpretation is not obligatory (see [18,19]). If we do not take into account this ambiguity, the spinor do not change in the process of parallel displacement. The simplicity of this result is due to the fact, that we

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[x] In fact any four-dimensional matrix C_k, which commutes with all the matrices β_j satisfying the conditions (4.3) is proportional to unit matrix (see e.g. [43]).

really consider the most simple deviation from the pseudoeuclidean case.

Now, the covariant derivative of the spinor reads

$$\psi_{|k} = \psi_{,k} - C_k \psi = (\partial_k - i e A_k)\psi \tag{4.11}$$

$$\bar{\psi}_{|k} = \bar{\psi}_{,k} + \bar{\psi} C_k = (\partial_k + i e A_k)\bar{\psi}. \tag{4.12}$$

If one neglect the arbitrary term $i e A_k$ the covariant derivative coincides with the usual one. The Dirac equation in our case may be represented in the form

$$i \beta^k \psi_{|k} - m\psi = 0; \quad i\bar{\psi}_{|k} \beta^k + m\bar{\psi} = 0. \tag{4.13}$$

Eq.(4.13) and the condition $\beta_{i|j} = 0$ guarantee the conservation of the current $\bar{\psi}\beta^i\psi$ of the spinor particles

$$(\bar{\psi}\beta^i\psi)_{|i} = 0. \tag{4.14}$$

From the relations (4.11) (4.12) (4.9) it follows also that

$$(\bar{\psi}\beta^i\psi)_{|i} = (\bar{\psi}\beta^i_{,i}\psi) = h^i_{(a),i}(\bar{\psi}\gamma^{(a)}\psi). \tag{4.15}$$

Taking into account the Maxwell equations (3.17) we obtain that in our (free) case

$$(\bar{\psi}\beta^i\psi)_{,i} = 0. \tag{4.16}$$

The problem of the current conservation in the next approximation needs the account of the current in the righthand side of the Maxwell equations and goes beyond the framework of the present approach.

Let us take now the limit of the small torsion in Eq. (4.13). Keeping only the first order terms, we obtain

$$i\gamma^k(\partial_k - ieA_k)\psi - m\psi + if^{k\ell}(\partial_k - ieA_k)\gamma_\ell\psi = 0 \tag{4.17}$$

$$i(\partial_k + ieA_k)\bar{\psi}\gamma^k + m\bar{\psi} + if^{k\ell}(\partial_k + ieA_k)\bar{\psi}\gamma_\ell = 0 \tag{4.18}$$

where

$$\gamma^k = \ell^k_{(a)}\gamma^{(a)}.$$

The last terms in these equations correspond to the interaction Lagrangian

$$\mathcal{L} = \frac{1}{2}\lambda\, Ge\,[\bar{\psi}(\partial_k - ieA_k)\gamma_\ell\psi - (\partial_k + ieA_k)\bar{\psi}\gamma_\ell\psi] \tag{4.19}$$

which we have constructed earlier [20-22] on the basis of intuitive considerations on the connection between the electromagnetic field and the torsion of the space-time. This Lagrangian is CP-odd (or T-odd) and C-odd.

This simple Lagrangian cannot however explain the observed CP-violation. Some generalization should be done for the case of interaction of several spinor particles. Further, to introduce the terms which do not conserve parity P, one would demand the interaction (4.19) to be γ_5 - invariant. We have done this earlier[20-22] but the hypothesis, which are necessary in doing this evidently are beyond the contents of the present simple model and are situated in the field of the conjectural unified theory of weak and electromagnetic interactions.

5. Conclusion

The main results of this work is proof of a possibility to connect the electromagnetic field with the space-time torsion and to deduce the equations for the electromagnetic field on the basis of the simple geometric considerations. It is very important, that this geometric theory of the electromagnetic field does not contradict the usual Maxwell equations and, quite the reverse, gives a possibility to obtain these equations and the estimate for their applicability. The other essential result is the derivation of the CP-odd interaction of the spinor particles with the electromagnetic field, which arise in the geometric theory quite automatically, without any additional conjectures.

The main problem to be solved is the construction of the unified theory of the weak and electromagnetic interactions. It seems, that for the solution one should try to unite present deas with the geometric ideas conserning weak interactions[17-19].

There are several interesting problems, however, even in the framework of this work. The examination of the nonlinear equations, generalizing Maxwell equations seems to be quite interesting. It would be useful to give a Lagrangian formulation of the theory and to construct a vector-potential (or its substitute) in the general nonlinear case. The difficult task of the global structure of the space-time is of great interest, the problem of the possibility to construct the continuous spinor field being connected with this problem.

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Appendix

Let us deduce the relation between the symmetric part of the pseudoorthogonal matrix and the antisymmetric one. One may represent any pseudoorthogonal matrix L_{ik} with the aid of the pseudoorthogonal transformation SLS^{-1} (where S is pseudoorthogonal) in one of the two forms

$$L^{(1)j}_{i} = \begin{vmatrix} \operatorname{ch}\chi & \operatorname{sh}\chi & 0 & 0 \\ \operatorname{sh}\chi & \operatorname{ch}\chi & 0 & 0 \\ 0 & 0 & \cos\phi & \sin\phi \\ 0 & 0 & -\sin\phi & \cos\phi \end{vmatrix}; \quad L^{(2)j}_{i} = \begin{vmatrix} 1+\frac{t^2}{2} & -\frac{t^2}{2} & t & 0 \\ t^2/2 & 1-t^2/2 & t & 0 \\ t & -t & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{A.1}$$

Let us set

$$L^{(1,2)j}_{i} = \delta^{j}_{i} + G^{(1,2)j}_{i}. \tag{A.2}$$

For the matrix $G^{(1)}$ the relation is valid

$$\bar{G}^{(1)}_{ij} = \frac{1}{4}\left(\bar{s} - \frac{2s}{\bar{s}+4}\right)\delta_{ij} + \frac{2}{\bar{s}+4}\left(\hat{G}^{(1)}_{ik}\,\delta^{k\ell}\,\hat{G}^{(1)}_{\ell j}\right), \tag{A.3}$$

where

$$\bar{s} = \bar{G}^{\cdot i}_{i} = G^{\cdot i}_{i}\,; \quad s = \hat{G}^{\cdot k}_{i}\,\hat{G}^{\cdot i}_{k} \tag{A.4}$$

$$\bar{G}^{(1)}_{ij} = \frac{1}{2}\left(G^{(1)}_{ij} + G^{(1)}_{ji}\right); \quad \hat{G}^{(2)}_{ij} = \frac{1}{2}\left(G^{(1)}_{ij} - G^{(1)}_{ji}\right). \tag{A.5}$$

From the equations

$$d = -uv, \quad s = 2(u-v) \tag{A.6}$$

where

$$d = \det(f) = \epsilon_{ijk\ell}\, f^{i}_{1} f^{j}_{2} f^{k}_{3} f^{\ell}_{4} \tag{A.7}$$

one may find the relation connecting $\bar{s}$ with s and d.

It is easy to verify, the matrix $G^{(2)}$ satisfies the relation

$$\bar{G}^{(2)}_{ij} = \frac{1}{2}\hat{G}^{(2)k}_{i} G^{(2)}_{kj} \qquad (A.8)$$

which coincides with Eq. (A.3) if one takes into account, that

$$\bar{s} = s_2 = d_2 = 0. \qquad (A.9)$$

Inasmuch as the relation (A.3) is invariant under pseudoorthogonal transformations, we have the proof that the Eq. (A.3) is valid for any pseudoorthogonal matrix.

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