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DYNAMICAL GROUP OF THE SCALAR SU(3) SYMMETRIC STRONG COUPLING THEORY

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I. Introduction

The existence of unitary particle multiplets [1] - the baryon octet (N , Ξ , Σ , Λ) with spin 1/2 and parity +1, the decuplet of baryon resonances (N^* , ...) with spin 3/2 and parity +1, the octet of baryon resonances with spin 1/2 and parity -1 - is now well established [2]. It leads to an effort at an explanation of these multiplets as a consequence of the dynamics of strongly interacting particles.

For the first time, such a problem was solved in a simplified form in the scalar and the pseudoscalar models of the static strong-coupling theory by Wentzel [3] and Pauli and Dancoff [4] in the case of the isotopic SU (2) internal symmetry group. For pseudoscalar meson they found an infinite series of the possible states characterized by isospin I and spin J for which $I = J$ and assumes the values 1/2, 3/2, 5/2, ... In a recent paper by Dothan and Ne'eman [5/x] it was shown that there is a unitary irreducible representation (UIR) of the symmetry group $[SU(2) \times SU(2)] \cdot T_9$ of the interaction Hamiltonian of the Pauli-Dancoff theory containing all the states with $I = J = 1/2, 3/2, \ldots$ which represent the solution of the pseudoscalar model in the static strong-coupling approximation. Therefore the symmetry group $[SU(2) \times SU(2)] \cdot T_9$ of the interaction Hamiltonian can be regarded as a dynamical or spectrum generating group of the Pauli-Dancoff strong-coupling theory.

The static strong-coupling theory for the case of the SU(3) internal symmetry group with scalar mesons (without the coupling of spin) was partly solved in a paper by Dullemond [8] who found the six lowest SU(3) rotational states and their energies.

x/ For the approach through the dispersion relations see [6] and [7].

The purpose of this work is the following: to use the symmetry group $SU(3) \cdot T_8$ of the interaction Hamiltonian of the SU(3)-invariant scalar strong-coupling theory as the spectrum generating group of the model and to determine the UIR of this non-compact group representing the complete infinite set of possible SU(3)-multiplets which are the solution for this model. The band of the isobar states we obtained agrees with the results of ref./8/ for the lowest levels.

2. The Hamiltonian and its Symmetry Properties

The interacting system of a static baryon octet source and an octet of scalar mesons in the static strong-coupling approximation is described by the Hamiltonian/8/*

$$H = \sum_{i=1}^{8} (\tfrac{1}{2} p_i^2 + \tfrac{1}{2}\mu^2 q_i^2) + H',$$

where μ is the average mass of the meson octet ,

$$H' = g \sum_{i=1}^{8} (\alpha F_i + (1-\alpha) D_i) q_i$$

is the interaction part of the Hamiltonian, g is the coupling constant and α the mixing parameter, q_i are the components of the meson octet $\vec{q} = (q_1 ... q_8)$ in the Cartesian basis, p_i are canonically conjugate to them and satisfy ordinary commutation relations $[q_i , p_j] = i\delta_{ij}$, $[q_i , q_j] = [p_i , p_j] = 0$. F_i and D_i are the 8x8 matrices with their elements defined in terms of Gell-Mann's/1/ f_{ijk} and d_{ijk}: $(F_i)_{jk} = -if_{ijk}$, $(D_i)_{jk} = d_{ijk}$. Their commutation relations are

$$[F_i , F_j] = if_{ijk} F_k , \quad [F_i , D_j] = if_{ijk} D_k .$$

* $\hbar = c = 1$

Hamiltonian H represents on the one hand an operator in the meson variables q_i and on the other hand an 8x8 matrix in the octet representation space of the bare baryon states. It is invariant under the group SU(3) generated by the operators

$$\hat{F}_i = F_i - i \sum_{j,k=1}^{8} f_{ijk} q_j \frac{\partial}{q_k} = F_i + \sum_{j,k=1}^{8} f_{ijk} q_j p_k$$

satisfying

$$[\hat{F}_i , \hat{F}_j] = if_{ijk} \hat{F}_k .$$

These eight operators are easily shown to commute with the Hamiltonian H.

Defining a potential energy operator $V(\vec{q})$

$$V(\vec{q}) = \tfrac{1}{2}\mu^2 \sum_{i=1}^{8} q_i^2 + H'$$

the Hamiltonian H is expressed as a sum of two parts, the first one being "kinetic" and the other one "potential"

$$H = \tfrac{1}{2} \sum_{i=1}^{8} p_i^2 + V(\vec{q}) ,$$

From the form of $V(\vec{q})$ it is seen that it commutes with the q_i and F_i's , i = 1, ... , 8. Since

$$[\hat{F}_i , q_j] = if_{ijk} q_k$$

it follows that the symmetry group of $V(\vec{q})$ is the group $G = SU(3) \cdot T_8$, a semidirect product of the symmetry group SU(3) of the whole Hamiltonian H and the abelian group of translations T_8 generated by q_i's, i = 1, ... 8 (T_8 is an invariant subgroup of G). Lie algebra of G has the following commutation relations:

$$[\hat{F}_i , \hat{F}_j] = if_{ijk} \hat{F}_k , \quad [\hat{F}_i , q_j] = if_{ijk} q_k , \quad [q_i , q_j] = 0$$

As

$$[\frac{1}{2}\sum_{i=1}^{8} p_i^2 , q_j] = -ip_j \neq 0$$

the symmetry group G of $V(\vec{q})$ is broken by the "kinetic" term to the group SU(3). The basic assumption of the strong-coupling theory is that the coupling constant g is sufficiently great ($g \gg \mu^{3/2}$) so that the kinetic term $\frac{1}{2}\sum_{i=1}^{8} p_i^2$ can be considered as a perturbation. With this assumption one can consider the non-compact symmetry group of $V(\vec{q})$ as a spectrum generating group of the model under consideration. There is a UIR of G that will contain all the admissible states of the strongly coupled system investigated here. It is necessary to determine this UIR from some physically reasonable requirements. They will be discussed in the next paragraph.

3. Unitary Irreducible Representations of the Group SU(3)

UIR's of the non-compact groups given by a semidirect product of a semi-simple Lie group and an abelian invariant subgroup were investigated in the works of Wigner[9], Mackey[10], Hermann[11], Cook and Sakita and others by the method of so called induced representations.

As follows from these papers a UIR of the group SU(3) · T_8 is unambiguously determined by fixing the variables characterizing the basic position $\overset{0}{\vec{q}}$ of a vector $\vec{q}$ in the octet space and by choosing the UIR of the little group of this vector $\overset{0}{\vec{q}}$.

Every vector $\vec{q} = (q_1, \ldots, q_8)$ in the octet space which transforms according to the adoint representation of the SU(3) group may be transformed into the basic position

$$\overset{0}{\vec{q}} = (0,0,\overset{0}{q}_3, 0,0,0,0,\overset{0}{q}_8).$$

The basic position $\overset{0}{\vec{q}}$ of $\vec{q}$ is unambiguously characterized by the two independent radial variables $\overset{0}{q}_3$ and $\overset{0}{q}_8$ which can be parametrized[8] :

$$\overset{0}{q}_3 = f\sin\phi , \quad \overset{0}{q}_8 = f\cos\phi .$$

We assume $\phi \neq 0$. Then the little group of $\overset{0}{\vec{q}}$ that leaves this vector invariant is the subgroup $U(1)_{I_3} \otimes U(1)_Y$ of SU(3) generated by the two commuting operators F_3 and F_8.

An induced representation of SU(3) · T_8 is thus determined by giving the values of the two radial variables f and ϕ, and of the two quantum numbers I_3 and Y, characterizing the UIR of the little group $U(1)_{I_3} \otimes U(1)_Y$.

The values f and ϕ determining the physical UIR of G we are looking for are chosen in such a way that an eigenvalue of the potential energy operator $V(\vec{q})$ (call it $E = E(\vec{q}) = E(f,\phi)$) would assume its absolute minimum $E_0 = E_0(f_0, \phi_0)$ for this choice of $f = f_0$ and $\phi = \phi_0$.[x] Further, there is an eigenvector of $V(\vec{q})$ corresponding to this smallest eigenvalue E_0. It is characterized by the quantum numbers I_{30} and Y_0. Now I_{30} and Y_0 determine the UIR of the little group. The characteristics f_0, ϕ_0, I_{30} and Y_0 fix then the induced physical UIR of the group SU(3) · T_8. The ground state of the interacting system is the lowest representations of SU(3) containing the vector with quantum numbers I_{30} and Y_0 in its representation space. The ground state characterized by f_0, ϕ_0, I_{30} and Y_0 represents the lowest state that appears in the spectrum of the system.

It is possible to establish the composition of this UIR of G with respect to the maximal compact subgroup SU(3) by means of the explicit relation for the matrix elements of translation operators q_i between different representations of SU(3) contained in the representation space of the UIR of G. The basis of this space is formed by the representations of the maximal compact subgroup SU(3).

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[x] $V(\vec{q})$ is invariant under G. Therefore, $V(\vec{q})$ in the space of a UIR of G is a multiple of the unit operator and its eigenvalues E are functions of the invariant variables f, ϕ, $E = E(f,\phi)$. The eigenvalues $E(f,\phi)$ of $V(\vec{q})$ represent the energy of the system in the strong coupling limit. This energy is the same for all states of the UIR of G. Thus it is natural to find the lowest eigenvalue of $E(f,\phi)$ as a function of f and ϕ and it will play the role of the ground-state energy of the system in the strong limit.

Matrix elements of the translation operators q_ρ in this basis are given by the equation (for derivation see the Appendix)

$$<(\lambda'),\nu',I'|q_\rho|(\lambda),\nu,I_0>_{(f_0,\phi_0,I_{30}Y_0)}=\sqrt{\frac{N_\lambda}{N_{\lambda'}}}\{f\sin\phi\sum_\gamma\begin{pmatrix}\lambda & 8 & \lambda'_\gamma\\ I_0 I_{30} Y_0 & 100 & I'_0 I_{30} Y_0\end{pmatrix} \tag{1}$$

$$\begin{pmatrix}\lambda & 8 & \lambda'_\gamma\\ \nu & \rho & \nu'\end{pmatrix}+f\cos\phi\sum_\gamma\begin{pmatrix}\lambda & 8 & \lambda'\\ I_0 I_{30} Y_0 & 000 & I'_0 I_{30}\ Y_0\end{pmatrix}\begin{pmatrix}\lambda & 8 & \lambda'_\gamma\\ \nu & \rho & \nu'\end{pmatrix},$$

where q_ρ are the spherical components of $\vec{q}$ the numbers $(\lambda),(\lambda')$ determine the representation of SU(3) and N_λ, $N_{\lambda'}$ are their dimensions ν, ν' are the states in the representations (λ), (λ'), respectively, and $(f_0,\phi_0,(I_{30},\phi Y_0)$ characterize the UIR of the non-compact groups $SU(3)\cdot T_8$. The unsymmetrized Clebsch-Gordan coefficients of SU(3) are denoted according to /12/. A state $|(\lambda),\nu,I_0>$ in the UIR of G is characterized by (λ), ν and the index I_0 that runs through all the possible values of isotopic spin characterizing the isotopic spin projection I_{30}.

4. The Minimum Condition for the Potential Energy Operator $V(\vec{q})$

The potential energy operator $V(\vec{q})$ is a matrix in the octet space of baryon states:

$$V(\vec{q}) = g\sum_{i=1}^{8}(\alpha F_i + (1-\alpha)D_i)q_i + \tfrac{1}{2}\mu^2\sum_{i=1}^{8}q_i^2\cdot\hat{1},$$

where $\hat{1}$ is the 8×8 unit matrix.

For every octet vector $\vec{q}=(q_1,\ldots,q_8)$ there exists an SU(3) transformation u by means of which it can be transformed into the basic position

$$\overset{0}{\vec{q}} = u\vec{q}. \tag{2}$$

To the transformation u there corresponds x/ an SU(3) transformation U in the space of baryon states by means of which the operator $V(\vec{q})$ goes over to the form

$$U^+V(\vec{q})U = g\sum_{i=3,8}(\alpha F_i + (1-\alpha)D_i)\overset{0}{q}_i + \tfrac{1}{2}\mu^2\sum_{i=3,8}\overset{0}{q}{}_i^2\cdot\hat{1}. \tag{3}$$

In order to find the eigenvalues of $V(\vec{q})$ explicitly we have to pass from/the Cartesian basis in the octet space of baryons used so far to a spherical one. There the matrices F_3, F_8, D_8 become diagonal and D_3 may be trivially diagonalized afterwards.

The vectors $\vec{e}_i$ of the Cartesian basis in octet space have the components

$$(\vec{e}_i)_j = \delta_{ij} \quad i,j=1,\ldots 8.$$

The vectors $\vec{e}_\rho$ of the spherical basis in octet space are defined by

$$I_3\vec{e}_\rho = I_3\vec{e}_\rho \quad \text{where} \quad F_3 = I_3,$$

$$Y\vec{e}_\rho = Y\vec{e}_\rho \quad \text{where} \quad F_8 = \frac{\sqrt{3}}{2}Y, \quad \rho=(I,I_3,Y).$$

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x/ Such a correspondence follows from the invariance of $V(\vec{q})$ under SU(3)

$$e^{-i\sum_{i=1}^{8}\alpha_i\hat{F}_i}\,V(\vec{q})\,e^{i\sum_{i=1}^{8}\alpha_i\hat{F}_i} = V(\vec{q})$$

This equation can be written in the form

$$e^{-i\sum_{i=1}^{8}\alpha_i F_i}\,V(\vec{q})\,e^{i\sum_{i=1}^{8}\alpha_i F_i} = e^{i\sum_{ijk=1}^{8}\alpha_i f_{ijk}q_j p_k}\,V(\vec{q})\,e^{-i\sum_{ijk=1}^{8}\alpha_i f_{ijk}q_j p_k}$$

The parameters α_i, $i = 1,\ldots 8$ are chosen in such a way that

$$e^{i\sum_{ijk=1}^{8}\alpha_i f_{ijk}q_j p_k}\,q_r\,e^{-i\sum_{ijk=1}^{8}\alpha_i f_{ijk}q_j p_k} = (u\vec{q})_r = \overset{0}{q}_r$$

then

$$U = e^{i\sum_{i=1}^{8}\alpha_i F_i}$$

The transition from the Cartesian basis to the spherical one is provided by a unitary 8x8 matrix

$$\begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \\ \vec{e}_4 \\ \vec{e}_5 \\ \vec{e}_6 \\ \vec{e}_7 \\ \vec{e}_8 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & +\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_{110} \\ \vec{e}_{1-10} \\ \vec{e}_{100} \\ \vec{e}_{\frac{1}{2}\frac{1}{2}1} \\ \vec{e}_{\frac{1}{2}-\frac{1}{2}1} \\ \vec{e}_{\frac{1}{2}\frac{1}{2}-1} \\ \vec{e}_{\frac{1}{2}-\frac{1}{2}-1} \\ \vec{e}_{000} \end{pmatrix}$$

Then the matrices F_i and D_i are transformed to the spherical basis according to the relations

$$F_i^S = Z^+ F_i Z \qquad D_i^S = Z^+ D_i Z$$

We give F_3^S, F_8^S, D_3^S, D_8^S explicitly:

$$F_3^S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad F_8^S = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$D_3^S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad D_8^S = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Instead of the parameters g, α let us define the new parameters /8/ g', β:

$$g'\cos\beta = \frac{g(1-\alpha)}{\sqrt{3}} \; ; \quad g'\sin\beta = g\alpha \; ; \quad \mathrm{tg}\beta = \sqrt{3}\,\frac{\alpha}{1-\alpha} \; ; \quad g' = g\left(\frac{(1-\alpha)^2}{3} + \alpha^2\right)^{\frac{1}{2}} .$$

With these new parameters and the parameters f and ϕ of $\vec{q}^{\,0}$ the transformed potential energy operator $V(\vec{q})$ can be written in the form

$$U^+ V(\vec{q})U = g'f(\sqrt{3}\cos\beta\sin\phi\, D_3^S + \sqrt{3}\cos\phi\cos\beta\, D_8^S +$$

$$+\sin\beta\sin\phi\, F_3^S + \sin\beta\cos\phi\, F_8^S) + \tfrac{1}{2}\mu^2 f^2 \cdot \hat{1} .$$

In the matrix form

$$U^{+}V(\vec{q})U = g'f\begin{pmatrix} \cos(\beta-\phi) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos(\beta+\phi) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\beta\cos\phi & 0 & 0 & 0 & 0 & \cos\beta\sin\phi \\ 0 & 0 & 0 & \cos(\beta+\phi-\frac{2\pi}{3}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos(\beta-\phi-\frac{2\pi}{3}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos(\beta+\phi+\frac{2\pi}{3}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos(\beta-\phi+\frac{2\pi}{3}) & 0 \\ 0 & 0 & \cos\beta\sin\phi & 0 & 0 & 0 & 0 & -\cos\beta\cos\phi \end{pmatrix} + \tfrac{1}{2}\mu^2 f^2 \cdot \hat{1}$$

Hence it is seen that the eigenvectors of $V(\vec{q})$ and the eigenvalues corresponding to them are given by the equations

$$V(\vec{q})U\vec{e}_{110} = \{g'f\cos(\beta-\phi) + \tfrac{1}{2}\mu^2 f^2\}\, U\vec{e}_{110}$$

$$V(\vec{q})U\vec{e}_{1-10} = \{g'f\cos(\beta+\phi) + \tfrac{1}{2}\mu^2 f^2\}\, U\vec{e}_{1-10}$$

$$V(\vec{q})U\vec{e}_{\frac{1}{2}\frac{1}{2}1} = \{g'f\cos(\beta+\phi-\tfrac{2\pi}{3}) + \tfrac{1}{2}\mu^2 f^2\}U\vec{e}_{\frac{1}{2}\frac{1}{2}1}$$

$$V(\vec{q})U\vec{e}_{\frac{1}{2}-\frac{1}{2}1} = \{g'f\cos(\beta-\beta-\tfrac{2\pi}{3}) + \tfrac{1}{2}\mu^2 f^2\}U\vec{e}_{\frac{1}{2}-\frac{1}{2}1}$$

$$V(\vec{q})U\vec{e}_{\frac{1}{2}+\frac{1}{2}-1} = \{g'f\cos(\beta+\phi+\tfrac{2\pi}{3}) + \tfrac{1}{2}\mu^2 f^2\}U\vec{e}_{\frac{1}{2}+\frac{1}{2}-1}$$

$$V(\vec{q})U\vec{e}_{\frac{1}{2}-\frac{1}{2}-1} = \{g'f\cos(\beta-\phi+\tfrac{2\pi}{3}) + \tfrac{1}{2}\mu^2 f^2\}U\vec{e}_{\frac{1}{2}-\frac{1}{2}-1}$$

$$V(\vec{q})U\vec{e}_2 = \{-g'f\cos\beta + \tfrac{1}{2}\mu^2 f^2\}U\vec{e}_2$$

$$V(\vec{q})U\vec{e}_8 = \{g'f\cos\beta + \tfrac{1}{2}\mu^2 f^2\}U\vec{e}_8\,,$$

where

$$\vec{e}_8 = \cos\frac{\phi}{2}\,\vec{e}_{000} - \sin\frac{\phi}{2}\,\vec{e}_{100}$$

$$e_8 = \sin\frac{\phi}{2}\,\vec{e}_{000} + \cos\frac{\phi}{2}\,\vec{e}_{100}.$$

The eigenvalues depend only on the invariant octet variables and the parameters $+g', \beta$. Assuming $\beta \neq 0$ the smallest eigenvalue of $V(\vec{q})$ may be obtained in 6 different ways, i.e. 6 possible choices of f and ϕ: $f = g'/\mu^2$ in all cases, the eigenvector corresponding to the minimal eigenvalue $-g'^2/2\mu^2$ is:

$\phi_1 = \beta \pm \pi$	$U\vec{e}_{110}$
$\phi_2 = -\beta \pm \pi$	$U\vec{e}_{1-10}$
$\phi_3 = -\beta - \frac{\pi}{3}$	$U\vec{e}_{\frac{1}{2}\frac{1}{2}1}$
$\phi_4 = \beta + \frac{\pi}{3}$	$U\vec{e}_{\frac{1}{2}-\frac{1}{2}1}$
$\phi_5 = -\beta + \frac{\pi}{3}$	$U\vec{e}_{\frac{1}{2}\frac{1}{2}-1}$
$\phi_6 = \mp\beta - \frac{\pi}{3}$	$U\vec{e}_{\frac{1}{2}-\frac{1}{2}-1}$.

From these 6 possibilities we choose for further investigation case 3, i.e. we fix the variables $f = g'/\mu^2$, $\phi_3 = -\beta - \frac{\pi}{3}$ and the quantum numbers of the third eigenvector are $I_3 = ½$, $Y = 1$. Let us denote this set of numbers $(f_0, \phi_0, I_{30}, Y_0)$. They fix the UIR of $SU(3) \cdot T_8$ we have been looking for.

5. Structure of the UIR $(f_0, \phi_0, I_{30}, Y_0)$

The UIR of $SU(3) \cdot T_8$ can be realized in an infinite-dimensional space spanned by the basis vectors of the finite-dimensional UIR's of the maximal compact subgroup SU(3). Explicit knowledge of the matrix elements of the abelian generators q_ρ in such a basis permits us to identify the representations of SU(3) from which the UIR $(f_0, \phi_0, I_{30}, Y_0)$ of G is built up. This allows us to determine all the physically admissible states of the system.

From (1) and the properties of the Clebsch-Gordan coefficients of the SU(3) group[13] it is easy to show that the UIR $(f_0, \phi_0, I_{30}, Y_0)$ contains all representations $(\lambda) = (p,q)$ of SU(3) satisfying the condition

$$p = q + 3m, \quad q = 0,1\dots, \quad m = 0, \pm 1, \pm 2, \dots, p \geq 0$$

with the exception of the SU(3) singlet (0,0).

Multiplicity of a UIR (λ) of SU(3) in the UIR $(f_0, \phi_0, I_{30}, Y_0)$ is given by the number of the isotopic multiplets (with hypercharge $Y_0 = 1$ and containing a state with the projection $I_{30} = 1/2$) which appear in the given UIR (λ) of SU(3). This multiplicity can be determined by applying a theorem describing the structure of a UIR of SU(3) with respect to its subgroup of isotopic spin (e.g. 13):

Let p, q be two numbers labelling the UIR (p, q) of SU(3). Then to each pair of integers κ, μ obeying the inequalities

$$p + q \geq \kappa \geq q \geq \mu \geq 0.$$

there corresponds an isotopic multiplet contained once in the UIR(p, q) and characterized by isospin and hypercharge

$$I = ½(\kappa - \mu) \qquad Y = \kappa + \mu - \frac{2p + 4q}{3},$$

respectively.

We are looking for all the isomultiplets with $Y = 1$ and $I_3 = 1/2$ which are contained in the UIR $(q + 3m, q)$, i.e. all the isomultiplets with $I = (2k+1)/2, k = 0,1\dots$ and $Y = 1$, contained in $(q + 3m, q)$.

From $I = 1/2(\kappa - \mu)$ we get $\kappa - \mu = 2k+1$ and from $Y = 1$ we get $\kappa + \mu = 2(q+m)+1$, hence $\kappa = q + m + k + 1$, $\mu = q + m - k$. From the first inequality $0 \leq \mu \leq q$ we find $m \leq k \leq q + m$; from $q \leq \kappa \leq p + q$ we obtain

$$q \leq q + m + k + 1 \leq 2q + 3m \quad \text{i.e.} \quad 0 \leq m + k + 1 \leq p.$$

The integer k has to satisfy both these inequalities simultaneously. Consequently for $m = 1,2,3 \dots$ there are $q + 1$ possible values of k; for $m = 0$ there are q possible values of k; and for $m = -1, -2, -3, \dots$ there are $p + 1$ possible values of k.

This means for $m \geq 1$ representation $(q + 3m, q)$ appears $(q+1)$ times in $(f_0, \phi_0, ½, 1)$; representation (q, q) appears q times and for $m \leq -1$, $q + 3m \geq 0$ representation $(q + 3m, q)$ appears $(q + 3m + 1)$ times there.

The six lowest representations of SU (3) that appear in the representation $(f_0, \phi_0, ½, 1)$ of $SU(3) \cdot T_8$ are placed in the table:

(p, q)	Dimension $N_{(p,q)}$	Multiplicity $M_{(p,q)}$	Possible values of I_0 for given UIR(p, q)
(1,1)	8	1	1/2
(3,0)	10	1	3/2
(0,3)	10^x	1	1/2
(2,2)	27	2	1/2, 3/2
(4,1)	35	2	3/2, 5/2
(1,4)	35	2	1/2, 3/2

In this manner we have found the set of all possible SU(3) - representations which appear in our representation $(f_0, \phi_0\ ½, 1)$ of the spectrum generating group SU(3). T_8. All these states represent the physically realizable states of the strongly coupled system with the baryon octet as the ground state.

6. Discussion

We obtained a band of SU(3) rotational excited states (isobar states) of the SU(3) invariant theory of scalar mesons strongly coupled to a static baryon.

This approach provides a simple explanation of the presence of SU(3) singlet in the case of pure D coupling[14]. In this case $\beta = 0$ or π. Hence the eigenvalue of the potential energy operator $V(\vec{q})$ reaches its minimum values in the states $U\vec{e}_3$ or $U\vec{e}_8$ with $I_{30} = Y_0 = 0$, $f_0 = g'/\mu^2$ and ϕ is arbitrary except some discrete values. As above the UIR $(f_0, \phi, I_{30} = 0, Y_0 = 0)$ of G so defined consists of all SU(3) representations (p, q), $p = q$ (mod 3) but including the SU(3) singlet (0,0) now.

The solution of the problem given here is not complete in the point that we have not a general formula for the energies of all the isobar states[x]. It would be necessary to find the dependence of the transformed kinetic energy operator $U^+ ½ \sum_{i=1}^{8} p_i^2 U$ on the invariant operators of SU(3) group (which is the symmetry group of H). However we do not know a convenient parametrization of the octet space that would allow to perform this calculation explicitly.

In conclusion we would like to express our thanks to Dr. V.G.Kadyshevsky for turning out attention to the problem and his constant interest and encouragement. We are also grateful to Dr. B.Sakita for sending his paper to us before publication.

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[x] The energies of the lowest SU(3) representations were calculated in[8].

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APPENDIX

Derivation of the Matrix Elements of q_ρ in the Basis $|(\lambda)\nu I_0 \rangle$

Here, the notation of[15] and[12] is used. Let us take a vector

$$\vec{\overset{0}{q}} = f(0\ \ 0\ \ \sin\phi\ \ 0\ \ 0\ \ 0\ \ 0\ \ \cos\phi)$$

labelling a particular character of T_8. A little group of $\vec{\overset{0}{q}}$, $(\phi \neq 0)$ is the group $U(1)_{I_3} \times U(1)_Y$ of rotations (α, β) about the axes $\vec{e}_8$ and in $\vec{e}_8$ in octet space. Let K be a semi-direct product

$$K = (U(1)_I \otimes U(1)_Y) \cdot T_8$$

and

$$k \to L(k), \quad k \in K$$

be a particular UIR of K in a Hilbert space $H(L,K)$:

$$k = (\vec{a}, \alpha, \beta), \quad L(\vec{a}, \alpha, \beta) = e^{i\vec{q}\cdot\vec{a}}\, e^{iI_{30}\alpha}\, e^{iY_0\beta}.$$

We consider functions $\{f\}$ from G to $H(L,K)$ which are " L-covariant along left cosets of G modulo K"

$$f(gk) = f(g)L^*(k), \quad g \in G, \quad k \in K. \qquad \text{(A.1)}$$

This property allows us to consider the functions only on the cosets of G modulo K which can be labelled by characters of T_8 which are elements of an orbit of $\overset{0}{\vec{q}}$

$$\vec{q} = R\overset{0}{\vec{q}}$$

here R is an SU(3) transformation in octet space. The orbit of $\overset{0}{\vec{q}}$ is a 6-dimensional hypersurface $S_6(f, \phi)$ in $\vec{q}$-space defined by two SU(3)-invariant functions of $\vec{q}$ /8/

$$\sum_{i=1}^{8} q_i^2 = f^2, \quad \sqrt{3}\sum_{ijk=1}^{8} d_{ijk}\, q_i q_j q_k = -f^3 \cos 3\phi .$$

We obtain a set of representatives for G/K by taking in each class $\vec{q}$ a particular rotation $R_{\vec{q}}$ leading from $\overset{0}{\vec{q}}$ to $\vec{q}$, $\vec{q} = R_{\vec{q}}\overset{0}{\vec{q}}$. The functions $f(R_{\vec{q}})$ have a finite norm with respect to the left invariant scalar product

$$(f_2, f_1) = \int_{\text{orbit of } \overset{0}{\vec{q}}} d\vec{q}\, f_2(R_{\vec{q}}) f_1(R_{\vec{q}}).$$

Then the UIR (f, ϕ, I_{30}, Y_0) of G induced by the UIR $L(k)$ of K can be written

$$[U(\vec{a},R)f](R_{\vec{q}}) = e^{i\vec{q}\vec{a}}\, e^{iI_{30}\alpha}\, e^{iY_0\beta}\, f(R_{R^{-1}\vec{q}}),$$

where

$$(\alpha, \beta) = R^{-1}_{\vec{q}}\, R R_{R^{-1}\vec{q}}.$$

The functions $f(R_{\vec{q}})$ can be formally expanded in terms of the irreducible representations of SU(3)

$$f(R_{\vec{q}}) = \sum_{\lambda\nu\nu'} f^{(\lambda)}_{\nu'\nu}(1) D^{(\lambda)}_{\nu'\nu}(R_{\vec{q}}) = \sum_{\lambda\nu\nu'} f^{(\lambda)}_{\nu'\nu}(1) D^{(\lambda)*}_{\nu\nu'}(R^{-1}_{\vec{q}}), \qquad \text{(A.2)}$$

where $D^{(\lambda)}_{\nu'\nu}(R)$ is a matrix element of a finite SU(3) transformation R in a UIR (λ) of SU(3), as defined in /12/, $\nu = (I, I_3, Y)$. It can be derived from (A.1) (see /15/) that we must take fixed values of I_3 and Y in (A.2);

$$I_3 = I_{30}, \quad Y = Y_0 .$$

Hence we should expand f in term of a complete set of functions $D^{(\lambda)*}_{(II_{30}Y_0),\nu'}(R^{-1}_{\vec{q}})$ with the additive quantum numbers I_{30} and Y_0 fixed:

$$f(R_{\vec{q}}) = \sum_{\lambda I\nu'} f^{(\lambda)}_{I\nu'}(1) D^{(\lambda)*}_{(II_{30}Y_0),\nu'}(R^{-1}_{\vec{q}}). \qquad \text{(A.3)}$$

From (A.3) it follows immediately what UIR's of SU(3) are contained in the UIR $(f; \phi\, I_{30}, Y_0)$ of G; it consists of all SU(3) representations which contain a vector with $I_3 = I_{30}, Y = Y_0$ in their representation space.

We introduce a non-normalizable basis $|\vec{q}\rangle$ and states

$$|\Phi\rangle = \int_{\text{orbit of } \overset{0}{\vec{q}}} d\vec{q}\, f(R_{\vec{q}})\, |\vec{q}\rangle$$

with

$$\langle \vec{q}' | \vec{q} \rangle = \delta_{\overset{0}{\vec{q}}} (\vec{q}, \vec{q}'), \quad f(R_{\vec{q}}) = \langle \vec{q} | \Phi \rangle$$

where $\delta_{\overset{0}{\vec{q}}}(\vec{q}, \vec{q}')$ is the invariant δ function on $S_8(f, \phi)$

$$\int_{\text{orbit of } \overset{0}{\vec{q}}} \delta_{\overset{0}{\vec{q}}} (\vec{q}, \vec{q}')\, f(\vec{q}')\, d\vec{q}' = f(\vec{q}) .$$

We pass to a new orthogonormal basis

$$|(\lambda), \nu, I_0 \rangle_{(f\phi I_{30} Y_0)} = \sqrt{N_\lambda} \int_{\text{orbit of } \overset{0}{\vec{q}}} d\vec{q}\; D^{(\lambda)*}_{(I_0 I_{30} Y_0), \nu} (R^{-1}_{\vec{q}}) |\vec{q}\rangle$$

or conversely

$$|\vec{q}\rangle = \sum_{\lambda' \nu' I_0'} \sqrt{N_{\lambda'}}\, D^{(\lambda')}_{(I_0' I_{30} Y_0), \nu'} (R^{-1}_{\vec{q}}) |(\lambda'), \nu', I_0' \rangle_{(f\phi I_{30} Y_0)}$$

with

$${}_{(f\phi I_{30} Y_0)}\langle (\lambda'), \nu', I_0' | (\lambda), \nu, I_0 \rangle_{(f\phi I_{30} Y_0)} = \delta_{\lambda\lambda'}\, \delta_{\nu\nu'}\, \delta_{I_0 I_0'} .$$

To calculate a matrix element

$$M = \langle (\lambda'), \nu', I_0' | q_\rho | (\lambda), \nu, I_0 \rangle_{(f\phi I_{30} Y_0)} ,$$

where q_ρ are the sperical components of generators of T_8 we substitute for the states from (A.4)):

$$M = \sqrt{N_{\lambda'} N_{\lambda}} \int d\vec{q}' \int_{\text{orbit of } \overset{0}{\vec{q}}} d\vec{q}\; D^{(\lambda')}_{(I'\, I\ \ Y), \nu'} (R^{-1}_{\vec{q}'}) D^{(\lambda)*}_{(I_0 I_{30} Y_0), \nu} (R^{-1}_{\vec{q}}) \langle \vec{q}' | q_\rho | \vec{q} \rangle .$$

Using

$$\langle \vec{q}' | q_\rho | \vec{q} \rangle = q_\rho\, \delta_{\overset{0}{\vec{q}}} (\vec{q}, \vec{q}')$$

we get

$$M = \sqrt{N_\lambda N_{\lambda'}} \int_{\text{orbit of } \overset{0}{\vec{q}}} d\vec{q}\; D^{(\lambda')}_{(I_0' I_{30} Y_0), \nu'} (R^{-1}_{\vec{q}}) D^{(\lambda)*}_{(I_0 I_{30} Y_0), \nu} (R^{-1}_{\vec{q}})\, q_\rho . \qquad \text{(A.5)}$$

It remains to express $\vec{q}$ as an SU(3) transformed $\overset{0}{\vec{q}}$:

$$q_\rho = q_{(I I_3 Y)} = \sum_\sigma \overset{0}{q}_\sigma\, D_{\sigma, (I I_3 Y)} (R^{-1}_{\vec{q}}) =$$

$$= f\, [\, D^{(8)*}_{(100), (I I_3 Y)} (R^{-1}_{\vec{q}}) \sin\phi + D^{(8)*}_{(000), (I I_3 Y)} (R^{-1}_{\vec{q}}) \cos\phi\,] .$$

After the substitution of this in (A.5) and using relations (13.8) and (13.6) of [12] we get the resulting formula (1)[x].

[x] This formula was given in [7] for the first time.