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QUANTUM NUMBERS IN THE LITTLE GROUPS OF THE POINCARE GROUP

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I. Introduction

We shall investigate the separation of variables in the equation

$$\Delta\psi = \lambda\psi , \tag{1}$$

where Δ is the Laplace operator in two-dimensional spaces with constant positive, zero or negative curvature, and demonstrate how the choice of the coordinate system is connected with the choice of an operator, diagonalized simultaneously with the Laplace operator, i.e. with the choice of certain integrals of motion and quantum numbers. (We call an operator L diagonal, if the eigenfunctions of (1) satisfy the equation $L\psi = \mu\psi$).

Such a problem is interesting from the physical point of view for several reasons.

Firstly, the obtained results can be applied to construct representations of the Poincaré group [1,2] since the groups of motions of the considered spaces, i.e. the three-dimensional rotation group $O(3)$, the group of motions of the euclidean plane E_2 and the three-dimensional Lorentz group $O(2,1)$ are little groups of the Poincaré group, corresponding to a time-like, isotropic or space-like [3] vector [x]. It follows that we are actually considering possible parametrizations of the relativistic spin in the physical or unphysical regions (or on the light cone). Such parametrizations should prove useful e.g. in the the analytical continuation of wave functions and amplitudes for particles with spin.

Secondly, the representation theory of compact and specially noncompact groups finds many applications in physics, e.g. in investigations concerning relativistic generalizations of the SU (6) symmetry, or in the obtaining of complete sets of functions with definite transformation properties, in terms of which physical quantities, like reaction amplitudes, can be expanded. The choice of the

[x] The little group, corresponding to a fixed null-vector is simply the homogeneous Lorentz group. The separation of variables on the corresponding three-dimensional hyperboloid $x_0^2 - x_1^2 - x_2^2 - x_3^2 = I$ has been considered in [10] and its connection with diagonal operators, quadratical in the generators of the Lorentz group in [6,7].

diagonal operator L determines the choice of the basis of the corresponding group representation. An analogous problem was first considered in connection with the invariant expansions of relativistic amplitudes in terms of the basis functions of the unitary representations of the Lorentz group[4]. It was shown in[5,6], that the diagonalization of operators, which are invariants of the subgroups of the Lorentz group, leads to coordinate systems with one centre. The diagonalization of other operators, quadratical in the generators of the Lorentz group[6,7] leads to coordinate systems of the elliptical or parabolical type.

Thirdly, in the investigation of higher symmetries in quantum mechanics, specifically in a search for all potentials, having a symmetry group larger than the evident geometrical symmetry group, it also proved necessary to find all operators, commuting with the Laplace operator and leading to the separation of variables[8,9].

We shall prove the following assertion:

<u>Theorem.</u> A linear self-adjoint operator L_k, being a homogeneous quadratical polynomial in the generators of the group of motions of the considered two dimensional space with constant curvature, corresponds to each system of coordinates in which the variables in equation (1) separate. The operator L_k is determined by the condition $L_K \psi = \mu \psi$, where ψ are solutions of (1), separated in K. As K runs through all types of coordinate systems allowing the separation of variables, L_k runs through all types of non-equivalent operators in the considered class. The operators L and $\tilde{L}$ are considered equivalent, if the relation

$$\tilde{L} = \alpha L' + \beta \Delta \tag{2}$$

holds, where α, β are real constants, and a motion of the space exists, transforming L into L'.

An analogous theorem for the euclidean space E_2 has been proved in[6]. Here we shall consider the cases $k>0$, $k=0$ and $k<0$ separately.

2. Integrals of Motion on a Two-Dimensional Sphere R_2 (Case of a Time-Like Vector)

It has been shown by Olevsky[10] that two coordinate systems exist in R_2 in which the variables in (1) separate. It can be shown that the operator

$$\boxed{L_S = L_3^2} \tag{3}$$

is diagonal in the spherical system

$$x = \sin\rho_1 \sin\rho_2 \qquad y = \sin\rho_1 \cos\rho_2 \qquad z = \cos\rho_1$$

and the operator

$$\boxed{L_E = L_3^2 + \sin^2 f L_2^2} \tag{4}$$

is diagonal in the elliptical system

$$x^2 = \frac{(\rho_1 - a)(\rho_2 - a)}{(c-a)(b-a)} \qquad y^2 = \frac{(\rho_1 - b)(\rho_2 - b)}{(c-b)(a-b)} \qquad z^2 = \frac{(\rho_1 - c)(\rho_2 - c)}{(a-c)(b-c)} \quad ,$$

$$c < \rho_2 < b < \rho_1 < a$$

Here $\sin^2 f = \frac{a-b}{a-c}$, $2f$ is the distance between the focusses; L_i, $i = 1,2,3$ are the generators of the group $O(3)$ and the Laplace operator is $\Delta = L_1^2 + L_2^2 + L_3^2$.

To prove the theorem stated in the introduction we must show that any operator of the type

$$L = A_{ik} L_i L_k \qquad A_{ik} = A_{ki} \tag{5}$$

(summation from 1 to 3 over repeated indices) is equivalent to L_S or L_E (or to the zero operator). However, this is obvious since under a motion of R_3 the generators L_i transform like a vector

$$L'_i = a_{ik} L_k ,$$

where a_{ik} are elements of a real orthogonal matrix. The polynomial L is transformed into

$$L' = A'_{ik} L_i L_k ,$$

where $A'_{ik} = a_{\ell i} A_{\ell m} a_{mk}$ i.e. in matrix form we have

$$A' = a^T A a \qquad a^T a = 1 \tag{6}$$

(the letter T means transposition).

It is well known that any real symmetrical matrix A_{ik} can be diagonalized by an orthogonal transformation (6), so that we have to consider only three possibilities.

1) All three eigenvalues λ_i of A are different (they are naturally always real). Transformation (6) can be chosen so that

$$L' = \lambda_1 L_1^2 + \lambda_2 L_2^2 + \lambda_3 L_3^2 . \tag{7}$$

Let us consider the case $\lambda_1 > \lambda_2 > \lambda_3$ (any other inequality simply corresponds to an interchange of the axes). We then have

$$\tilde{L} = \frac{1}{\lambda_3 - \lambda_1}(L' - \lambda_1 \Delta) = L_E .$$

2) Two of the eigenvalues coincide $(\lambda_1 = \lambda_2 \neq \lambda_3)$: the operator L' can be written as

$$L' = \lambda_1 (L_1^2 + L_2^2) + \lambda_3 L_3^2 \tag{8}$$

and

$$\tilde{L} = \frac{1}{\lambda_3 - \lambda_1}(L' - \lambda_1 \Delta) = L_s .$$

3) All three eigenvalues coincide $(\lambda_1 = \lambda_2 = \lambda_3)$.

Thus

$$L' = \lambda_1 (L_1^2 + L_2^2 + L_3^2) \tag{9}$$

and

$$\tilde{L} = L' - \lambda_1 \Delta = 0$$

This completes the proof of the theorem for R_3 .

3. Integrals of Motion on the Euclidean Plane E_2 (Case of an Isotropical Vector)

The variables in equation (1) can be separated in four coordinate systems in the space E_2 /11/. Let us denote the generators of the group of motions P_1 , P_2 (infinitesimal translations) and M (an infinitesimal rotation). The Laplace operator is $\Delta = P_1^2 + P_2^2$. The separable coordinate systems and corresponding diagonal operators are:

1) Cartesian system x , y with the operator

$$\boxed{L_D = P_1^2 - P_2^2} , \tag{10}$$

2) The polar system $x = r\cos\phi$, $y = r\sin\phi$ with the operator

$$\boxed{L_S = M^2} , \tag{11}$$

3) The elliptical system $x = \ell\xi\eta$, $y = \ell\sqrt{(\xi^2 - 1)(1 - \eta^2)}$ ($\ell > 0$ is the focal distance) with the operator

$$\boxed{L_E = A^2 + \frac{\ell^2}{2}(P_1^2 - P_2^2) ,} \tag{12}$$

4) The parabolical system $x = \frac{1}{2}(\xi^2 - \eta^2), y = \xi\eta$ with the operator

$$\boxed{L_p = MP_2 + P_2 M} . \tag{13}$$

It has been proved in [6] that the most general operator of the considered type

$$L = aM^2 + b_1(MP_1 + P_1M) + b_2(MP_2 + P_2M) + c_1P_1^2 + 2c_2P_1P_2 + c_3P_2^2 \qquad (14)$$

is equivalent (in the sense of equation (2)) to one of these four types.

4. Integrals of Motion on the Hyperboloid L_2 (Case of a Space-Like Vector)

Nine types of separable coordinate systems exist in the two-dimensional space L_2 [10]. Let us denote the generators of the group $O(2,1)$ – K_1, K_2 (infinitesimal hyperbolic rotations) and M_3 (an infinitesimal space rotation), the Laplace operator is $\Delta = K_1^2 + K_2^2 - M_3^2$.

Let us first enumerate these coordinate systems and the corresponding diagonal operators. As usual we denote the Weierstrass coordinates x, y, t (for simplicity we set $k = -1$).

1) Horocyclic system

$$x = \tfrac{1}{2}[e^{\rho_1} + e^{-\rho_1}(\rho_2^2 - 1)] \quad y = \rho_2 e^{-\rho_1} \quad t = \tfrac{1}{2}[e^{\rho_1} + e^{-\rho_1}(\rho_2^2 + 1)]$$

$$\boxed{L_0 = (K_1 + M_3)^2 = K_1^2 + M_3^2 + K_1M_3 + M_3K_1}, \qquad (15)$$

2) Equidistant system

$$x = \operatorname{ch}\rho_1 \operatorname{sh}\rho_2 \quad y = \operatorname{sh}\rho_1 \quad t = \operatorname{ch}\rho_1 \operatorname{ch}\rho_2$$

$$\boxed{L_{Eq} = K_2^2}, \qquad (16)$$

3) Spherical system

$$x = \operatorname{ch}\rho_1 \cos\rho_2 \quad y = \operatorname{sh}\rho_1 \sin\rho_2 \quad t = \operatorname{ch}\rho_1$$

$$\boxed{L_S = M_3^2}, \qquad (17)$$

4) Elliptical system

$$x^2 = \frac{(\rho_1 - b)(\rho_2 - b)}{(a-b)(b-c)} \quad y^2 = \frac{(\rho - a)(a - \rho_2)}{(a-b)(a-c)} \quad t^2 = \frac{(\rho_1 - c)(a - \rho_2)}{(a-c)(b-c)}$$

$$c < b < \rho_2 < a < \rho_1$$

$$\boxed{L_E = M_3^2 + \operatorname{sh}^2 f\, K_2^2}, \qquad (18)$$

where $\operatorname{sh}^2 f = \frac{a-b}{b-c}$ and $2f$ is the focal distance

5) Hyperbolic system

$$x = \frac{(\rho_1 - c)(c-\rho_2)}{(a-c)(b-c)} \quad y^2 = \frac{(\rho_1 - a)(a-\rho_2)}{(a-b)(a-c)} \quad t^2 = \frac{(\rho_1 - b)(b-\rho_2)}{(a-b)(b-c)}$$

$$\rho_2 < c < b < a < \rho_1$$

$$\boxed{L_H = K_2^2 - \sin^2 \alpha L_3^2}, \qquad (19)$$

where $\sin^2\alpha = \frac{b-c}{a-c}$ and 2α is the angle between two focal lines.

6) Semihyperbolic system

$$x^2 = -\frac{(\rho_1 - a)(a-\rho_2)}{2[(a-\gamma)^2+\delta^2]} - \tfrac{1}{2} + \frac{1}{2\delta}\sqrt{\frac{[(\rho_1-\gamma)^2+\delta^2][(\rho_2-\gamma)^2+\delta^2]}{(a-\gamma)^2+\delta^2}}$$

$$y^2 = \frac{(\rho_1 - a)(a-\rho_2)}{(a-\gamma)^2+\delta^2} \qquad \rho_2 < a < \rho_1$$

$$t^2 = \frac{(\rho_1 - a)(a-\rho_2)}{2[(a-\gamma)^2+\delta^2]} + \tfrac{1}{2} + \frac{1}{2\delta}\sqrt{\frac{[(\rho_1-\gamma)^2+\delta^2][(\rho_2-\gamma)^2+\delta^2]}{(a-\gamma)^2+\delta^2}}$$

$$\boxed{L_{SH} = M_3 K_1 + K_1 M_3 + \mathrm{sh}\,2f K_2^2}, \qquad (20)$$

where $\mathrm{sh}\,2f = \frac{a-\gamma}{\delta}$ and $2f$ is the distance between the focus of the semi-hyperbolas and the basis line of their equidistant curve.

7) Elliptic-parabolical system

$$2x = \frac{(\rho_1 - a)(a-\rho_2)}{(a-b)^{3/2}[(\rho_1-b)(\rho_2-b)]^{1/2}} + \left[\frac{a-b}{(\rho_1-b)(\rho_2-b)}\right]^{1/2} - \left[\frac{(\rho_1-b)(\rho_2-b)}{a-b}\right]^{1/2}$$

$$y^2 = \frac{(\rho_1 - a)(a-\rho_2)}{(a-b)^2} \qquad b<\rho_2<a<\rho_1$$

$$2t = \frac{(\rho_1 - a)(a-\rho_2)}{(a-b)^{3/2}[(\rho_1-b)(\rho_2-b)]^{1/2}} + \left[\frac{a-b}{(\rho_1-b)(\rho_2-b)}\right]^{1/2} + \left[\frac{(\rho_1-b)(\rho_2-b)}{a-b}\right]^{1/2}$$

$$\boxed{L_{EP} = (a-b)K_2^2 + K_1^2 + M_3^2 + K_1 M_3 + M_3 K_1}, \qquad (21)$$

8) Hyperbolic-parabolical system

$$2x = \frac{(\rho_1 - a)(a - \rho_2)}{(a-b)^{3/2}[(\rho_1 - b)(b - \rho_2)]^{1/2}} + \left[\frac{a-b}{(\rho_1 - b)(b-\rho_2)}\right]^{1/2} - \left[\frac{(\rho_1 - b)(b - \rho_2)}{a-b}\right]^{1/2}$$

$$y^2 = \frac{(\rho_1 - a)(a - \rho_2)}{(a-b)^2} \qquad \rho_2 < b < a < \rho_1$$

$$2t = \frac{(\rho_1 - a)(a - \rho_2)}{(a-b)^{3/2}[(\rho_1 - b)(b - \rho_2)]^{1/2}} + \left[\frac{a-b}{(\rho_1 - b)(b-\rho_2)}\right]^{1/2} + \left[\frac{(\rho_1 - b)(b - \rho_2)}{a-b}\right]^{1/2}$$

$$\boxed{L_{HP} = -(a-b)K_2^2 + K_1^2 + M_3^2 + (K_1 M_3 + M_3 K_1)}\,, \tag{22}$$

9) Semicircular-parabolical system

$$x = \frac{(\rho_1 - \rho_2)^2}{8[(\rho_1 - a)(a - \rho_2)]^{3/2}} - \tfrac{1}{2}[(\rho_1 - a)(a - \rho_2)]^{1/2}$$

$$2y = \left[\frac{\rho_1 - a}{a - \rho_2}\right]^{1/2} - \left[\frac{a - \rho_2}{\rho_1 - a}\right]^{1/2} \qquad \rho_2 < a < \rho_1$$

$$t = \frac{(\rho_1 - \rho_2)^2}{8[(\rho_1 - a)(a - \rho_2)]^{3/2}} + \tfrac{1}{2}[(\rho_1 - a)(a - \rho_2)]^{1/2}$$

$$\boxed{L_{CP} = K_1 K_2 + K_2 K_1 + K_2 M_3 + M_3 K_2}\,. \tag{23}$$

We have introduced names for the coordinate systems, corresponding to the form of the coordinate lines. The definitions of the corresponding curves can be found e.g. in /12/.

As in the cases considered above we must prove that any operator of the type

$$L = aK_1^2 + b(K_1 K_2 + K_2 K_1) + cK_2^2 + d(K_1 M_3 + M_3 K_1) + e(K_2 M_3 + M_3 K_2) + fM_3^2 \tag{24}$$

is equivalent (cf. equation (2)) to one of the operators (15)-(23).

P r o o f.

Let us consider the symmetrical matrix

$$X = \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} \qquad (25)$$

and the characteristic equation

$$\det X(\mu) = \begin{pmatrix} a-\mu & b & d \\ b & c-\mu & e \\ d & e & f+\mu \end{pmatrix} = 0 \qquad (26)$$

Under a motion of the space L_2 (hyperbolic motion) the operator L transforms into L', determined by the matrix X', where

$$X' = a^T X a \qquad (27)$$

and a is the matrix of a hyperbolic motion i.e.

$$a^T I a = I \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \qquad (28)$$

Hyperbolic motions are investigated e.g. in /12,13/ in connection with the classification of second order curves (conics) on the Lobachevsky plane. It is shown that the hyperbolic motions leave the following set of quantities invariant:

1) Roots of the characteristic equation (26): μ_1, μ_2, μ_3 ; or the quantities

$$S = a + c - f = \mu_1 + \mu_2 + \mu_3$$
$$T = A + C - F = -(\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1)$$
$$R = \det X = -\mu_1\mu_2\mu_3 ,$$

where A, C and F are the minors of X corresponding to the elements a, c and f.

2) The quantities

r_i = rank of $X(\mu_i)$

$\epsilon_i = \mathrm{sgn}[A(\mu_i) + C(\mu_i) + F(\mu_i)]$

$K_i = \epsilon_i [A(\mu_i) + C(\mu_i) - F(\mu_i)]$.

3) For $r_i \le 1$ an additional invariant exists

$$\eta_i = \mathrm{sgn}[\mu_i (a + c + f - \mu_i)] .$$

It is proved in /12/ that the values of these invariants completely determine the class, to which a considered conic belongs. Here we shall briefly recapitulate this classification and establish a correspondence with the classification of operators (15)-(23).

Let us first enumerate all "canonical" forms to which the operator (24) can be reduced by hyperbolic motions in dependence on the invariants of the matrix X.

1. All characteristic numbers μ_1, μ_2, μ_3 are different and real:

$$L_1 = \mu_1 K_1^2 + \mu_2 K_2^2 - \mu_3 K_3^2 . \tag{29}$$

2. One real characteristic number μ_1 and two complex ones $\mu_{2,3} = \alpha \pm i\beta$:

$$L_2 = \mu_1 (K_1^2 + K_2^2) - (\mu_1 + 2q) M_3^2 + p(K_1 M_3 + M_3 K_1) \qquad \alpha = \mu_1 + q \quad \beta^2 = p^2 - q^2 > 0 . \tag{30}$$

3. A simple characteristic number μ_1 and a double one $\mu_2 = \mu_3$:

a) $\epsilon_1 = -1, \quad \epsilon_2 = 1$

$$L_3 = \mu_1 (K_1^2 + K_2^2) - (2\mu_2 - \mu_1) M_3^2 + (\mu_2 - \mu_1)(K_1 M_3 + M_3 K_1) , \tag{31}$$

b) $\epsilon_1 = -1, \quad \epsilon_2 = -1$

$$L_4 = (2\mu_2 - \mu_1) K_1^2 + \mu_1 (K_2^2 - M_3^2) + (\mu_1 - \mu_2)(K_1 M_3 + M_3 K_1) , \tag{32}$$

c) $\epsilon_1 = 1, \quad \epsilon_2 = 0$

$$L_5 = \mu_2 (K_1^2 + K_2^2) - \mu_1 M_3^2 , \tag{33}$$

d) $\epsilon_1 = -1, \quad \epsilon_2 = 0$

$$L_6 = \mu_2 (K_1^2 - M_3^2) + \mu_1 K_2^2 . \tag{34}$$

4. A triple characteristic number $\mu_1 = \mu_2 = \mu_3$

a) $\epsilon_1 = -1,$

$$L_7 = K_1^2 + K_2^2 - M_3^2 + K_1 K_2 + K_2 K_1 - K_2 M_3 - M_3 K_2 , \tag{35}$$

b) $\epsilon_1 = 0, \quad \eta_1 = 1$

$$L_8 = 2K_1^2 + K_2^2 - K_1 M_3 - M_3 K_1 , \tag{36}$$

c) $\epsilon_1 = 0$, $\eta_1 = -1$

$$L_9 = K_2^2 - 2M_3^2 + K_1 M_3 + M_3 K_1 , \qquad (37)$$

d) $\epsilon_1 = \eta_1 = 0$.

$$L_{10} = K_1^2 + K_2^2 - M_3^2 . \qquad (38)$$

Let us discuss the forms (29)-(38).

Geometrically five different types of curves correspond to (29), depending on the possible values of μ_1. However, with the help of the transformation

$$\tilde{L} = \alpha L + \beta \Delta \qquad (39)$$

and an interchange of the coordinate axes they can be reduced to the cases

1. $\mu_1 > \mu_2 > \mu_3$ — L_1 reduces to L_E
2. $\mu_1 > \mu_3 > \mu_2$ — L_1 reduces to L_H
3. A single curve corresponds to (30). It is easy to see that all the invariants corresponding to $\tilde{L}_2 = \frac{1}{\sqrt{p^2-q^2}}[-L_2+(\mu_1+q)\Delta]$ and L_{SH} coincide, so that L_2 can be reduced to L_{SH}.
4. Two curves correspond to (31) (different for $\frac{\mu_1}{\mu_2}>1$ and for $\frac{\mu_1}{\mu_2}<1$), however, the corresponding operators are connected by (39). The operators L_3 and $\tilde{L}_{EP} = \frac{\mu_1-\mu_2}{a-b} L_{EP} + \mu_2 \Delta$ have the same invariants, so that L_3 reduces to L_{EP}.
5. Three curves correspond to (32) but in all cases L_4 can be reduced to L_{HP} in complete analogy with the previous case.
6. Formula (33) describes three types of curves, but can always be reduced to L_S.
7. Formula (34) describes two types of curves, but can always be reduced to L_{Eq}.
8. Formula (35) describes a single curve. The operators L_7 and $\tilde{L}_{CP} = L_{CP} + \Delta$ have the same invariants, so that L_7 can be reduced to L_{CP}.
9. Formulae (36) and (37) describe different curves, but L_8 and L_9 can both be transformed using (39) into $(K_1 - M_3)^2$ connected with L_0 by a hyperbolic motion, so that they both reduce to L_0.
10. Formula (38) describes a single curve and L_{10} is equivalent to the zero operator.

Thus although there are 20 types of nondegenerate curves (det $X \neq 0$)

in the Lobachevsky plane (12 real, 2 imaginary, 5 ideal ones and the absolute), we have shown that they correspond to our 9 classes of operators. The degenerate case $\det X = 0$ need not be considered, since degenerate curves can always be transformed into nondegenerate ones by the transformation (39). This completes the proof.

5. Conclusions

We have investigated the group theoretical origin of coordinate systems, allowing the separation of variables in equation (1) for the spaces R_2, E_2 and L_2.

Let us recapitulate the main results.

1. The diagonalization of a homogeneous quadratical polynomial in the generators of the group of motions corresponds to each separating coordinate system.

2. All different types of coordinate systems can be obtained by classifying all operators of the considered type into classes of equivalency, where two operators L and $\tilde{L}$ belong to the same class, if they are related by transformation (2). A coordinate system of a certain type corresponds to each class of operators and systems of the same type, but shifted or rotated with respect to each other, correspond to different operators in the same class.

3. Operators that are invariants of subgroups of the group of motions, correspond to coordinate systems with one centre (i.e. systems that are not elliptical or parabolical). Different types of coordinate systems correspond to different and unequivalent subgroups (spherical coordinates in R_2, cartesian and polar coordinates in E_2, horocyclic, equidistant and spherical coordinates in L_2).

According to our opinion, the fact that operators, commuting with the Laplace operators of a given group can be distributed into a finite and quite definite number of equivalency classes, is of great physical interest. Mathematically it limits the choice of the basis for the group representations. In non-relativistic quantum mechanics this fact severely limits the number of potentials, allowing a dynamical invariance group (this has been considered for the two-dimensional case in /8,9/ and an investigation of three-dimensional potentials is now under way).

From the point of view of relativistic spin theory our results mean that it is only possible to measure quite definite quantities, characterizing the "projection" of the spin, if the total spin is fixed. For physical particles with non-zero mass it is either possible to measure the usual component (e.g. L_3) or the eigenvalue of the operator $L_E = L_3^2 + \sin^2 f L^2$, but no other independent quantity. The situation is similar, but more sophisticated in the case of zero mass particles, or unphysical particles with $m^2 < 0$.

The group 0(2,1) describes particles with imaginary mass, but is physically important in the study of scattering problems, in which the symmetry properties of amplitudes, not of single-particle wave functions, are considered. The group 0(2,1) is the symmetry group related to a reaction in a given channel, considered from the cross-channel (it describes the amplitude in the non-physical region).

It would be interesting to consider possible experiments determining the quantity $L_3^2 + \sin^2 f L_2^2$ for physical particles with $m^2 > 0$, to clarify the role of the focus distance f in the corresponding experimental device and to consider the connection of the found operators for $m^2 = 0$ with the Stokes polarization parameters. A discussion of these problems, as well as an investigation of the properties of cross-channel amplitudes connected with the "components" enumerated in § 4 will be presented separately. Although the representation theory of the groups 0(3), 0(2,1) and E_2 is well known, explicit representations have only been constructed in the polar coordinate system, thus corresponding to the diagonalization of a space rotation generator. It would be of interest to construct representations, corresponding to the diagonalization of all other independent operators, found in this paper, i.e. to give expressions for the basis functions, matrix elements of infinitesimal and finite transformations etc. and to find the operators, connecting these quantities in various systems. Such an investigation, specifically concerning the relation between the results of this paper and the representation theory of the group 0(2,1), developped by Bargmann, will be published elsewhere.

It should be noted that the results of this paper can be generalized to spaces with higher dimensions, however in such cases it is necessary to classify sets of commuting operators, quadratical and symmetrical in the generators of the corresponding group, instead of a single operator, as in the case of the groups of rank I, considered in the present paper.

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