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ON THE KINEMATICS OF THE TWO-PHOTON CHERENKOV EFFECT

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1 Introduction

The possibility of the two-photon Cherenkov effect was predicted by Frank and Tamm in [1]:

> We note in passing that for $v < c$ the conservation laws prohibit the emission of one particular photon as well as the simultaneous emission of a group of photons. However, for the superluminal velocity such higher order processes are possible although for them the radiation condition (2.4) is not necessary.

(Under this condition Tamm and Frank meant the one-photon radiation condition $\cos\theta = c/vn$). In this case, the conservation of energy and momenta does not prohibit the process in which a moving charge emits simultaneously two photons. There is no experimental confirmation of this effect.

The calculations of the two-photon radiation intensity are known [2-6], but they were performed without paying enough consideration to the exact kinematical relations. The goal of this treatment is to point out that the two photon Cherenkov effect will be strongly pronounced for special orientations of photons and the recoil charge. This makes easier the experimental search for the 2-photon Cherenkov effect.

The plan of our exposition is as follows. In section 2, for the pedagogical purposes, we consider the one-photon emission from a charge moving uniformly in medium. It turns out that the kinematics allows not only emission at the Cherenkov angle [7], but also in the forward direction, in accordance with the claim made in [8]. In section 3, devoted to the two-photon emission, the inequalities are obtained for the emission angles of two photons. In specific cases these inequalities reduce to equalities. For these particular cases a possible setup of experiments aiming to observe the two-photon emission is discussed. In the same section the relation of the solutions of the classical Maxwell equations to the quantum two-photon Cherenkov effect is discussed. A short discussion of the results obtained is given in section 4.

2 Pedagogical example: one-photon Cherenkov effect

Let a point-like charge e having the rest mass m_0 move in medium of the refractive index n. It emits the photon with the frequency ω. The conservation of energy and momentum gives

$$m_0c^2\gamma_0 = m_0c^2\gamma + \hbar\omega, \quad m_0\vec{v}_0\gamma_0 = m_0\vec{v}\gamma + \frac{\hbar\omega n}{c}\vec{e}_\gamma. \tag{2.1}$$

Here $\hbar$ is the Plank constant, $\vec{v}_0$ and $\vec{v}$ are the charge velocities before and after emitting the γ quanta, $\gamma = 1/\sqrt{1-\beta^2}$, $\gamma_0 = 1/\sqrt{1-\beta_0^2}$; $\vec{e}_\gamma$ and ω are the unit vector in the direction of emitted γ quanta and its frequency; n is the medium refractive index taken at the frequency ω. We rewrite (2.1) in the dimensionless form

$$\gamma_0 = \gamma + \epsilon, \quad \vec{\beta}_0\gamma_0 = \vec{\beta}\gamma + \epsilon n \vec{e}_\gamma. \tag{2.2}$$

Here $\vec{\beta} = \vec{v}/c$, $\vec{\beta}_0 = \vec{v}_0/c$, $\epsilon = \hbar\omega/m_0c^2$. Let $\vec{v}_0$ be directed along the z axis. We project all vectors on this axis and two others perpendicular to it

$$\vec{\beta}_0 = \beta_0\vec{e}_z, \quad \vec{\beta} = \beta[\vec{e}_z\cos\theta + \sin\theta(\vec{e}_x\cos\phi + \vec{e}_y\sin\phi)],$$

$$\vec{e}_\gamma = \vec{e}_z\cos\theta_\gamma + \sin\theta_\gamma(\vec{e}_x\cos\phi_\gamma + \vec{e}_y\sin\phi_\gamma)]. \tag{2.3}$$

Substituting (2.3) into (2.2), one obtains

$$\gamma_0 = \gamma + \epsilon, \quad \beta_0\gamma_0 = \beta\gamma\cos\theta + n\epsilon\cos\theta_\gamma,$$

$$\beta\gamma\sin\theta\cos\phi + n\epsilon\sin\theta_\gamma\cos\phi_\gamma = 0, \quad \beta\gamma\sin\theta\sin\phi + n\epsilon\sin\theta_\gamma\sin\phi_\gamma = 0. \tag{2.4}$$

From two last equations one finds

$$\sin\theta\sin(\phi - \phi_\gamma) = 0, \quad \sin\theta_\gamma\sin(\phi - \phi_\gamma) = 0. \tag{2.5}$$

For $\sin(\phi - \phi_\gamma) \neq 0$ it follows that $\theta = \theta_\gamma = 0$ and Eqs. (2.4) reduce to

$$\gamma_0 = \gamma + \epsilon, \quad \beta_0\gamma_0 = \beta\gamma + n\epsilon. \tag{2.6}$$

From this one easily obtains:

$$\beta = \frac{2n - \beta_0(n^2+1)}{n^2 + 1 - 2n\beta_0}, \quad \epsilon = \frac{2\gamma_0(\beta_0 n - 1)}{n^2 - 1}. \tag{2.7}$$

The conditions $\epsilon > 0$ and $0 < \beta < \beta_0$ give

$$\frac{1}{n} < \beta_0 < \frac{2n}{1+n^2}$$

for $n > 1$ and

$$n < \beta_0 < \frac{2n}{1+n^2} \tag{2.8}$$

for $n < 1$.

In the past, the possibility of the one-photon radiation in the forward direction by a charge moving in medium was suggested by Tyapkin on purely intuitive grounds [8]. Equations (2.6)-(2.8) tell us that this assumption is not in conflict with the kinematics.

Let now $\sin(\phi - \phi_\gamma) = 0$. There are no physical solutions of (2.4) if $\phi = \phi_\gamma$. It remains only $\phi = \phi_\gamma + \pi$. Then,

$$\gamma_0 = \gamma + \epsilon, \quad \beta_0\gamma_0 = \beta\gamma\cos\theta + n\epsilon\cos\theta_\gamma, \quad \beta\gamma\sin\theta = n\epsilon\sin\theta_\gamma. \tag{2.9}$$

These equations have the well-known solution found by Ginsburg [7]

$$\cos\theta_\gamma = \frac{1}{\beta_0 n}[1 + \frac{\epsilon(n^2-1)}{2\gamma_0}], \quad \cos\theta = \frac{\beta^2\gamma^2 + \beta_0^2\gamma_0^2 - n^2(\gamma_0-\gamma)^2}{2\beta\gamma\beta_0\gamma_0}. \tag{2.10}$$

The conditions that the r.h.s. of these equations should be smaller than 1 and greater than -1, lead to the following conditions:

$$\frac{|2n - \beta_0(n^2+1)|}{n^2+1-2n\beta_0} < \beta < \beta_0, \quad \epsilon < \frac{2\gamma_0(\beta_0 n - 1)}{n^2-1}. \tag{2.11}$$

Eqs. (2.9)-(2.11) can be realized only for $n > 1, \beta_0 > 1/n$.

3 Two-photon Cherenkov effect

3.1 General formulae

The energy-momentum conservation gives

$$\gamma_0 = \gamma + \epsilon_1 + \epsilon_2, \quad \gamma_0\vec{\beta}_0 = \gamma\vec{\beta} + \epsilon_1 n_1 \vec{e}_1 + \epsilon_2 n_2 \vec{e}_2. \tag{3.1}$$

Here

$$\epsilon_1 = \frac{\hbar\omega_1}{m_0c^2}, \quad \epsilon_2 = \frac{\hbar\omega_2}{m_0c^2}, \quad n_1 = n(\omega_1), \quad n_2 = n(\omega_2),$$

ω_1 and ω_2 are the frequencies of the γ quanta 1 and 2, and $\vec{e}_1$ and $\vec{e}_2$ are the unit vectors along the directions of their propagation. Projecting (3.1) on the same axes as above one gets

$$\gamma_0 = \gamma + \epsilon_1 + \epsilon_2, \quad \gamma_0\beta_0 = \gamma\beta\cos\theta + \epsilon_1 n_1 \cos\theta_1 + \epsilon_2 n_2 \cos\theta_2,$$

$$\beta\gamma\sin\theta\cos\phi + \epsilon_1 n_1 \sin\theta_1 \cos\phi_1 + \epsilon_2 n_2 \sin\theta_2 \cos\phi_2 = 0,$$

$$\beta\gamma\sin\theta\sin\phi + \epsilon_1 n_1 \sin\theta_1 \sin\phi_1 + \epsilon_2 n_2 \sin\theta_2 \sin\phi_2 = 0. \tag{3.2}$$

From the last two equations one finds

$$\cos(\phi_1 - \phi) = \frac{\epsilon_2^2 n_2^2 \sin^2\theta_2 - \epsilon_1^2 n_1^2 \sin^2\theta_1 - \beta^2\gamma^2\sin^2\theta}{2\beta\gamma\epsilon_1 n_1 \sin\theta\sin\theta_1},$$

$$\cos(\phi_2 - \phi) = \frac{\epsilon_1^2 n_1^2 \sin^2\theta_1 - \epsilon_2^2 n_2^2 \sin^2\theta_2 - \beta^2\gamma^2\sin^2\theta}{2\beta\gamma\epsilon_2 n_2 \sin\theta\sin\theta_2}. \tag{3.3}$$

For the given β_0 (initial charge velocity), β, θ, ϕ (the final charge velocity and its direction), ϵ_1, θ_1 (the frequency and the inclination angle towards the motion axis for the first photon) the first and second of Eqs. (3.2) define the frequency and the inclination angle towards the motion axis for the second photon) while Eqs.

(3.3) define the azimuthal angles for the 1 and 2 photons. These angles are not independent:

$$\cos(\phi_2 - \phi_1) = \frac{\beta^2\gamma^2 \sin^2\theta - \epsilon_1^2 n_1^2 \sin^2\theta_1 - \epsilon_2^2 n_2^2 \sin^2\theta_2}{2\epsilon_1 n_1 \epsilon_2 n_2 \sin\theta_1 \sin\theta_2}. \tag{3.4}$$

The conditions

$$-1 < \cos(\phi_1 - \phi) < 1, \quad -1 < \cos(\phi_2 - \phi) < 1, \quad -1 < \cos(\phi_2 - \phi_1) < 1$$

lead to the following restrictions on θ, θ_1 and θ_2:

$$\frac{|n_1\epsilon_1 \sin\theta_1 - n_2\epsilon_2 \sin\theta_2|}{\beta\gamma} \le \sin\theta \le \frac{n_1\epsilon_1 \sin\theta_1 + n_2\epsilon_2 \sin\theta_2}{\beta\gamma},$$

$$\frac{|\beta\gamma \sin\theta - n_2\epsilon_2 \sin\theta_2|}{n_1\epsilon_1} \le \sin\theta_1 \le \frac{\beta\gamma \sin\theta + n_2\epsilon_2 \sin\theta_2}{n_1\epsilon_1},$$

$$\frac{|\beta\gamma \sin\theta - n_1\epsilon_1 \sin\theta_1|}{n_2\epsilon_2} \le \sin\theta_2 \le \frac{\beta\gamma \sin\theta + n_1\epsilon_1 \sin\theta_1}{n_2\epsilon_2}. \tag{3.5}$$

The energy of the recoil charge enters only through the $\beta\gamma\sin\theta$ term. It can be excluded using the relations

$$\beta\gamma = \sqrt{(\gamma_0 - \epsilon_1 - \epsilon_2)^2 - 1},$$

$$\beta\gamma \sin\theta = [\beta^2\gamma^2 - (\gamma_0\beta_0 - \epsilon_1 n_1 \cos\theta_1 - \epsilon_2 n_2 \cos\theta_2)^2]^{1/2}. \tag{3.6}$$

For the extremely relativistic charges ($\gamma_0 \gg \epsilon_1$, $\gamma_0 \gg \epsilon_2$)

$$\sin\theta = \sqrt{\frac{2}{\gamma_0}}[\epsilon_1(n_1\cos\theta_1 - 1) + \epsilon_2(n_2\cos\theta_2 - 1)]^{1/2},$$

that is, $\theta \to 0$ when $\beta_0 \to 1$. It follows from this that

$$\epsilon_1(n_1\cos\theta_1 - 1) + \epsilon_2(n_2\cos\theta_2 - 1) \ge 0.$$

This inequality cannot be satisfied if both n_1 and n_2 are smaller than 1. In the same relativistic limit

$$\beta\gamma \sin\theta = \sqrt{2\gamma_0}[\epsilon_1(n_1\cos\theta_1 - 1) + \epsilon_2(n_2\cos\theta_2 - 1)]^{1/2}$$

is finite despite the large $\sqrt{\gamma_0}$ factor. This becomes evident if we rewrite the first of equations (3.5) in the form

$$|n_1\epsilon_1 \sin\theta_1 - n_2\epsilon_2 \sin\theta_2| \le \beta\gamma\sin\theta \le n_1\epsilon_1 \sin\theta_1 + n_2\epsilon_2 \sin\theta_2$$

and note that θ enters into two last inequalities (3.5) through the same combination $\beta\gamma\sin\theta$.

3.2 Particular cases

Inequalities (3.5) reduce to equalities when the recoil charge moves in the same direction as the initial one ($\theta = 0$) or when one of the photons moves along the direction of motion of the initial charge. We consider these cases separately.

3.2.1 A charge does not change the direction of motion

Let $\theta = 0$, that is a charge does not change the motion direction. Then, from the two first equations (3.5) it follows that

$$n_1\epsilon_1 \sin\theta_1 = n_2\epsilon_2 \sin\theta_2. \tag{3.7}$$

It follows from (3.4) that

$$\cos(\phi_2 - \phi_1) = -1, \quad \phi_2 = \phi_1 + \pi, \tag{3.8}$$

that is, photons fly in the opposite azimuthal directions. Then, Eqs. (3.2) reduce to

$$\gamma_0 = \gamma + \epsilon_1 + \epsilon_2, \quad \gamma_0\beta_0 - \gamma\beta = \epsilon_1 n_1 \cos\theta_1 + \epsilon_2 n_2 \cos\theta_2,$$

$$n_1\epsilon_1 \sin\theta_1 = n_2\epsilon_2 \sin\theta_2. \tag{3.9}$$

From this one easily obtains $\cos\theta_1$ and $\cos\theta_2$

$$\cos\theta_1 = \frac{(\beta_0\gamma_0 - \beta\gamma)^2 + \epsilon_1^2 n_1^2 - \epsilon_2^2 n_2^2}{2(\beta_0\gamma_0 - \beta\gamma)\epsilon_1 n_1},$$

$$\cos\theta_2 = \frac{(\beta_0\gamma_0 - \beta\gamma)^2 - \epsilon_1^2 n_1^2 + \epsilon_2^2 n_2^2}{2(\beta_0\gamma_0 - \beta\gamma)\epsilon_2 n_2}. \tag{3.10}$$

The conditions $-1 < \cos\theta_1 < 1$ and $-1 < \cos\theta_2 < 1$ lead to the inequality which can be presented in the following two equivalent forms:

$$|\epsilon_1 n_1 - \epsilon_2 n_2| \le \beta_0\gamma_0 - \beta\gamma \le \epsilon_1 n_1 + \epsilon_2 n_2,$$

$$|\beta_0\gamma_0 - \beta\gamma - \epsilon_1 n_1| \le n_2(\gamma_0 - \gamma - \epsilon_1) \le \beta_0\gamma_0 - \beta\gamma + \epsilon_1 n_1. \tag{3.11}$$

These inequalities can be easily resolved (see Appendix). For definiteness, we suggest that $n_2 > n_1$. There are the following possibilities depending on n_1, n_2, β_0 and β.

$$1) \quad n_2 > 1 > n_1.$$

In this case inequality (3.11) has the solution

$$\beta_2 < \beta < \beta_0 \quad \text{for} \quad \frac{1}{n_2} < \beta_0 < \frac{2n_2}{1+n_2^2} \tag{3.12}$$

and

$$0 < \beta < \beta_0 \quad \text{for} \quad \beta_0 > \frac{2n_2}{1+n_2^2}. \tag{3.13}$$

Here

$$\beta_1 = \frac{2n_1 - \beta_0(1+n_1^2)}{1+n_1^2-2n_1\beta_0}, \quad \beta_2 = \frac{2n_2 - \beta_0(1+n_2^2)}{1+n_2^2-2n_2\beta_0}.$$

When the conditions (3.12) and (3.13) are satisfied, the dimensionless energy of the first photon belongs to the interval

$$\frac{n_2(\gamma_0-\gamma)-(\beta_0\gamma_0-\beta\gamma)}{n_1+n_2} \le \epsilon_1 \le \frac{n_2(\gamma_0-\gamma)-(\beta_0\gamma_0-\beta\gamma)}{n_2-n_1}. \tag{3.14}$$

The energy of the second photon is positive if $\epsilon_2 = \gamma_0 - \gamma - \epsilon_1 > 0$. Since the inequality

$$\epsilon_1 < \frac{n_2(\gamma_0-\gamma)-(\beta_0\gamma_0-\beta\gamma)}{n_2-n_1} < \gamma_0 - \gamma \tag{3.15}$$

holds when inequalities (3.12) and (3.13) are satisfied, the positivity of ϵ_2 is guaranteed.

2) $n_2 > n_1 > 1$.

For $n_1 < (1+n_2^2)/2n_2$ (this corresponds to the following chain of inequalities $1/n_2 < 2n_2/(1+n_2^2) < 1/n_1 < 2n_1/(1+n_1^2)$) one obtains:

$$\beta_2 < \beta < \beta_0 \quad \text{for} \quad \frac{1}{n_2} < \beta_0 < \frac{2n_2}{1+n_2^2},$$

$$0 < \beta < \beta_0 \quad \text{for} \quad \frac{2n_2}{1+n_2^2} < \beta_0 < \frac{1}{n_1}$$

and

$$0 < \beta < \beta_1 \quad \text{for} \quad \frac{1}{n_1} < \beta_0 < \frac{2n_1}{1+n_1^2}. \tag{3.16}$$

For $n_1 > (1+n_2^2)/2n_2$ (this corresponds to the chain of inequalities $1/n_2 < 1/n_1 < 2n_2/(1+n_2^2) < 2n_1/(1+n_1^2)$) one obtains:

$$\beta_2 < \beta < \beta_0 \quad \text{for} \quad \frac{1}{n_2} < \beta_0 < \frac{1}{n_1},$$

$$\beta_2 < \beta < \beta_1 \quad \text{for} \quad \frac{1}{n_1} < \beta_0 < \frac{2n_2}{1+n_2^2}$$

and

$$0 < \beta < \beta_1 \quad \text{for} \quad \frac{2n_2}{1+n_2^2} < \beta_0 < \frac{2n_1}{1+n_1^2}. \tag{3.17}$$

When β and β_0 lie inside the intervals defined by (3.16) and (3.17), ϵ_1 satisfies the same inequality (3.14).

On the other hand, the inequality

$$\frac{n_2(\gamma_0-\gamma)-(\beta_0\gamma_0-\beta\gamma)}{n_1+n_2} \le \epsilon_1 \le \frac{n_2(\gamma_0-\gamma)+(\beta_0\gamma_0-\beta\gamma)}{n_2+n_1} \tag{3.18}$$

holds when

$$\beta_1 < \beta < \beta_0 \quad \text{for} \quad \frac{1}{n_1} < \beta_0 < \frac{2n_1}{1+n_1^2}$$

and

$$0 < \beta < \beta_0 \quad \text{for} \quad \beta_0 > \frac{2n_1}{1+n_1^2}. \tag{3.19}$$

There are no solutions of (3.11) when both n_1 and n_2 are smaller than 1.

A further analysis of (3.10) and (3.11) requires the knowledge of the dispersion law $n(\omega)$. These equations are convenient when the charge energy moving along the z axis can be measured.

As a result, we obtain the following procedure for measurement of the two-photon Cherenkov radiation. Put the charge particle detector on the axis of motion. It should be tuned in such a way as to detect a particular charge velocity in the intervals (3.12), (3.13),(3.13) or (3.17). Correspondingly, the energy of one of the photons should be chosen in the intervals (3.14) or (3.18). The energy of the other photon is found from the first of Eqs. (3.1). Put the photon detectors under the polar angles given by (3.10) and, in accordance with (3.8), under opposite azimuthal angles. Since θ_1 and θ_2 are uniquely determined by β_0, β and ϵ_1, the corresponding radiation intensities should have sharp maxima at these angles.
If the measurement of the recoil charge is not possible, one can place photon detectors tuned into the coincidence at the angles given by (3.10) from which the $\beta\gamma\sin\theta$ term should be excluded using the relations (3.6). This is especially clear for the relativistic case considered below.

Extremely relativistic case. The above equations are simplified if both the initial and recoil charges are extremely relativistic ($\beta_0 \approx 1, \quad \beta \approx 1$). Then one has

$$\cos\theta_1 = \frac{(\epsilon_1+\epsilon_2)^2 + \epsilon_1^2 n_1^2 - \epsilon_2^2 n_2^2}{2(\epsilon_1+\epsilon_2)\epsilon_1 n_1}, \quad \cos\theta_2 = \frac{(\epsilon_1+\epsilon_2)^2 - \epsilon_1^2 n_1^2 + \epsilon_2^2 n_2^2}{2(\epsilon_1+\epsilon_2)\epsilon_2 n_2} \tag{3.20}$$

instead of (3.10). Inequality (3.11) reduces to

$$\frac{1-n_1}{n_2-1}\epsilon_1 < \epsilon_2 < \frac{n_1+1}{n_2-1}\epsilon_1$$

for $n_2 > 1 > n_1$ and to

$$\frac{n_1-1}{n_2+1}\epsilon_1 < \epsilon_2 < \frac{n_1+1}{n_2-1}\epsilon_1 \tag{3.21}$$

for $n_2 > n_1 > 1$.

Non-dispersive medium. Also , the simplification of (3.10) and (3.11) takes place for the non-dispersive medium. It turns out that ϵ_1 satisfies the inequality

$$\frac{n(\gamma_0-\gamma) - (\beta_0\gamma_0 - \beta\gamma)}{2n} \le \epsilon_1 \le \frac{n(\gamma_0-\gamma) + (\beta_0\gamma_0 - \beta\gamma)}{2n} \tag{3.22}$$

which is valid under the condition

$$n(\gamma_0 - \gamma) > \beta_0\gamma_0 - \beta\gamma. \tag{3.23}$$

In a manifest form, this equation for $n > 1$ looks like

$$\frac{2n - \beta_0(n^2+1)}{1 + n^2 - 2n\beta_0} \le \beta \le \beta_0, \quad \text{for} \quad \frac{1}{n} < \beta_0 < \frac{2n}{1+n^2}$$

and

$$0 < \beta < \beta_0 \quad \text{for} \quad \beta_0 > \frac{2n}{1+n^2}. \tag{3.24}$$

There are no solutions of (3.22) for $n < 1$.
If in addition both charges are extremely relativistic, one gets for the non-dispersive medium $(n > 1)$

$$\cos\theta_1 = \frac{(\epsilon_1+\epsilon_2)^2 + n^2(\epsilon_1^2 - \epsilon_2^2)}{2(\epsilon_1+\epsilon_2)\epsilon_1 n}, \quad \cos\theta_2 = \frac{(\epsilon_1+\epsilon_2)^2 - n^2(\epsilon_1^2 - \epsilon_2^2)}{2(\epsilon_1+\epsilon_2)\epsilon_2 n},$$

$$\frac{n-1}{n+1}\epsilon_1 < \epsilon_2 < \frac{n+1}{n-1}\epsilon_1, \quad n > 1.$$

3.2.2 One of the photons moves along the direction of motion of the initial charge

For definiteness, let this photon be the second one ($\theta_2 = 0$). Then, it follows from (3.5) that $\beta\gamma\sin\theta = n_1\epsilon_1\sin\theta_1$. Substituting this into (3.3) one finds $\cos(\phi_1 - \phi) = -1$, $\phi_1 = \phi - \pi$, that is, the recoil charge and photon fly in the opposite azimuthal directions. As a result, one gets the following equations:

$$\gamma_0 = \gamma + \epsilon_1 + \epsilon_2,$$

$$\beta_0\gamma_0 = \epsilon_2 n_2 + \epsilon_1 n_1 \cos\theta_1 + \beta\gamma\cos\theta,$$

$$\beta\gamma\sin\theta = n_1\epsilon_1\sin\theta_1.$$

From this one easily finds θ_1 and θ:

$$\cos\theta = \frac{(\gamma_0\beta_0 - \epsilon_2 n_2)^2 + \gamma^2\beta^2 - \epsilon_1^2 n_1^2}{2\gamma\beta(\gamma_0\beta_0 - \epsilon_2 n_2)},$$

$$\cos\theta_1 = \frac{(\gamma_0\beta_0 - \epsilon_2 n_2)^2 - \gamma^2\beta^2 + \epsilon_1^2 n_1^2}{2\epsilon_1 n_1(\gamma_0\beta_0 - \epsilon_2 n_2)}. \tag{3.25}$$

The conditions that the r.h.s. of these equations be smaller than 1 and greater than -1, give the following inequality

$$|\gamma_0\beta_0 - \epsilon_1 n_1 - \epsilon_2 n_2| < \beta\gamma < |\gamma_0\beta_0 + \epsilon_1 n_1 - \epsilon_2 n_2|. \tag{3.26}$$

We do not further elaborate Eq.(3.26) by presenting it in a manifest form similarly as it was done for (3.11).
These equations are useful when one is able to measure only the photons energies. In fact, substituting (3.6) into (3.25) one gets the polar angles of recoil charge and the 1-st photon. Making the same substitution in (3.26), one finds the set of available ϵ_1 and ϵ_2:

$$|\gamma_0\beta_0 - \epsilon_1 n_1 - \epsilon_2 n_2| < [(\gamma_0 - \epsilon_1 - \epsilon_2)^2 - 1]^{1/2} < |\gamma_0\beta_0 + \epsilon_1 n_1 - \epsilon_2 n_2|. \tag{3.27}$$

The measurement procedure reduces to the following one. Choose the photon energies ϵ_1 and ϵ_2. Check whether they satisfy (3.27). Put the photon counters at the initial direction of the charge motion, and at the angle θ_1 defined in (3.25). The counters tuned into the coincidence will detect photons arising from the two-photon Cherenkov effect. Since θ_1 is uniquely defined by kinematics, the radiation intensity should have maximum at this angle for the photon with the energy ϵ_1.

3.3 Back to the general two-photon Cherenkov effect

The situation is more complicated for the general two-photon Cherenkov radiation described by Eqs. (3.2)-(3.5). It is easy to check that only one of inequalities (3.5) is independent. It is convenient to choose the first of them rewriting it in the form

$$(n_1\epsilon_1 \sin\theta_1 - n_2\epsilon_2 \sin\theta_2)^2 \le \beta^2\gamma^2 \sin^2\theta \le (n_1\epsilon_1 \sin\theta_1 + n_2\epsilon_2 \sin\theta_2)^2. \tag{3.28}$$

This inequality is satisfied trivially for particular cases $\theta = 0$ and $\theta_1 = 0$ considered above. However, there are other solutions of (3.28) for which θ_1, θ_2 and θ are uniquely defined.

3.3.1 Another particular case

To find this case we substitute $\beta\gamma \sin\theta$ from (3.6) to (3.28) thus obtaining the following inequality

$$\cos\theta_2^{(1)} < \cos\theta_2 < \cos\theta_2^{(2)}, \tag{3.29}$$

where

$$\cos\theta_2^{(1)} = A - R, \quad \cos\theta_2^{(2)} = A + R, \tag{3.30}$$

$$A = \frac{c_1}{2n_2\epsilon_2}\frac{\beta_0\gamma_0 - \epsilon_1 n_1 \cos\theta_1}{Z^2},$$

$$R = \frac{\beta_0\gamma_0\epsilon_1^2 n_1^2 \sin\theta_1}{\epsilon_2 n_2 Z^2}[(\cos\theta_1 - \cos\theta_1^{(1)})(\cos\theta_1^{(2)} - \cos\theta_1)]^{1/2},$$

$$\cos\theta_1^{(1)} = \frac{\epsilon_1^2 n_1^2 + \beta_0^2\gamma_0^2 - (\epsilon_2 n_2 + \beta\gamma)^2}{2\beta_0\gamma_0\epsilon_1 n_1},$$

$$\cos\theta_1^{(2)} = \frac{\epsilon_1^2 n_1^2 + \beta_0^2\gamma_0^2 - (\epsilon_2 n_2 - \beta\gamma)^2}{2\beta_0\gamma_0\epsilon_1 n_1}, \tag{3.31}$$

$$c_1 = \epsilon_1^2 n_1^2 + \epsilon_2^2 n_2^2 + (\epsilon_1 + \epsilon_2)(2\gamma_0 - \epsilon_1 - \epsilon_2) - 2\beta_0\gamma_0\epsilon_1 n_1 \cos\theta_1,$$

$$Z^2 = \epsilon_1^2 n_1^2 + \beta_0^2\gamma_0^2 - 2\beta_0\gamma_0\epsilon_1 n_1 \cos\theta_1.$$

We see that for each $\cos\theta_1$ from the interval

$$\cos\theta_1^{(1)} < \cos\theta_1 < \cos\theta_1^{(2)}$$

there is a continuum of $\cos\theta_2$ values given by (3.29). The corresponding angular radiation intensities are rather broad.

The notable exceptions (in addition to the trivial cases $\theta = 0$ and $\theta_1 = 0$ considered above) are $\cos\theta_1 = \cos\theta_1^{(1)}$ and $\cos\theta_1 = \cos\theta_1^{(2)}$ when $R = 0$ and $\cos\theta_2^{(1)} = \cos\theta_2^{(2)}$.

The $\cos\theta_2^{(1)}$ and $\cos\theta_2^{(2)}$ corresponding to $\cos\theta_1^{(1)}$ and $\cos\theta_1^{(2)}$, respectively, are obtained by substitution $\cos\theta_1 = \cos\theta_1^{(1)}$ and $\cos\theta_1 = \cos\theta_1^{(2)}$ into A and are given by

$$\cos\theta_2^{(1)} = \frac{\beta_0^2\gamma_0^2 - \epsilon_1^2 n_1^2 + (\epsilon_2 n_2 + \beta\gamma)^2}{2\beta_0\gamma_0(\epsilon_2 n_2 + \beta\gamma)}$$

for $\cos\theta_1 = \cos\theta_1^{(1)}$ and

$$\cos\theta_2^{(2)} = \frac{\beta_0^2\gamma_0^2 - \epsilon_1^2 n_1^2 + (\epsilon_2 n_2 - \beta\gamma)^2}{2\beta_0\gamma_0(\epsilon_2 n_2 - \beta\gamma)} \tag{3.32}$$

for $\cos\theta_1 = \cos\theta_1^{(2)}$.

Obviously, the r.h.s. of equations (3.31) and (3.32) defining $\cos\theta_1^{(i)}$ and $\cos\theta_2^{(i)}$ should be smaller than 1 and greater than -1. This defines the interval of ϵ_1 and ϵ_2 for which the solution discussed exists.

The polar angle of the recoil charge is found from the relation

$$\beta\gamma\cos\theta_i = \beta_0\gamma_0 - \epsilon_1 n_1 \cos\theta_1^{(i)} - \epsilon_2 n_2 \cos\theta_2^{(i)}, \tag{3.33}$$

where $\cos\theta_1^{(i)}$ and $\cos\theta_2^{(i)}$ are the same as in (3.31) and (3.32).

Since θ, θ_1 and θ_2 are now fixed and are no longer connected by inequalities, the corresponding angular radiation intensities should have sharp maxima at θ_1 and θ_2 (similarly to the single-photon Cherenkov effect).

In general, to each angle θ_1 there corresponds the interval of θ_2 defined by (3.29). The maxima of corresponding radiation intensities are rather diffused.

Only for special cases:
1) when the recoil charge moves in the same direction as the initial charge (see section 3.2.1);
2) when one of the photons moves along the direction of the initial charge (see section 3.2.2) and
3) for the orientations of the photons and recoil charge defined by (3.31)- (3.33),
the directions of the recoil charge and photons are uniquely defined similarly to the single-photon Cherenkov effect. The corresponding radiation intensities should have sharp maxima for such orientations.

3.3.2 Relativistic case

In the relativistic limit the inequality (3.28) reduces to

$$\epsilon_1 + \epsilon_2 \le n_1\epsilon_1 \cos\theta_1 + n_2\epsilon_2 \cos\theta_2$$

$$\epsilon_1 + \epsilon_2 \ge n_1\epsilon_1 \cos\theta_1 + n_2\epsilon_2 \cos\theta_2,$$

which are compatible only if

$$\epsilon_1(n_1 \cos\theta_1 - 1) + \epsilon_2(n_2 \cos\theta_2 - 1) = 0. \tag{3.34}$$

This equation has no solutions if both n_1 and n_2 are smaller than 1. We extract $\cos\theta_2$:

$$\cos\theta_2 = \frac{1}{n_2} + \frac{\epsilon_1(1 - n_1 \cos\theta_1)}{n_2\epsilon_2}. \tag{3.35}$$

For definiteness we choose $n_2 > n_1$ and $n_2 > 1$. The right hand side of this equation should be smaller than 1 and greater than -1. This leads to the following inequality for $\cos\theta_1$:

$$\frac{1}{n_1} - \frac{\epsilon_2(n_2 - 1)}{\epsilon_1 n_1} \le \cos\theta_1 \le \frac{1}{n_1} + \frac{\epsilon_2(n_2 + 1)}{\epsilon_1 n_1}.$$

It is convenient to rewrite this equation in a manifest form.

Let $n_2 > n_1 > 1$.

Then, available θ_1 lie in the following intervals

$$-1 < \cos\theta_1 < 1 \quad \text{for} \quad \epsilon_2 > \epsilon_1 \frac{1 + n_1}{n_2 - 1},$$

$$\frac{1}{n_1} - \frac{\epsilon_2(n_2 - 1)}{\epsilon_1 n_1} < \cos\theta_1 < 1 \quad \text{for} \quad \epsilon_1 \frac{n_1 - 1}{n_2 + 1} < \epsilon_2 < \epsilon_1 \frac{n_1 + 1}{n_2 - 1}$$

and

$$\frac{1}{n_1} - \frac{\epsilon_2(n_2 - 1)}{\epsilon_1 n_1} \le \cos\theta_1 \le \frac{1}{n_1} + \frac{\epsilon_2(n_2 + 1)}{\epsilon_1 n_1} \quad \text{for} \quad 0 < \epsilon_2 < \epsilon_1 \frac{n_1 - 1}{n_2 + 1}.$$

Let $n_2 > 1$, $n_1 < 1$.

Then, available values of θ_1 belong to the intervals

$$-1 < \cos\theta_1 < 1 \quad \text{for} \quad \epsilon_2 > \epsilon_1 \frac{1 + n_1}{n_2 - 1},$$

and

$$\frac{1}{n_1} - \frac{\epsilon_2(n_2 - 1)}{\epsilon_1 n_1} < \cos\theta_1 < 1 \quad \text{for} \quad \epsilon_1 \frac{1 - n_1}{n_2 - 1} < \epsilon_2 < \epsilon_1 \frac{n_1 + 1}{n_2 - 1}.$$

It follows from these equations that there is a continuum of pairs θ_1,θ_2 connected by (3.34). This means that in a general case, rather broad distributions of radiation intensities should be observed. The kinematical consideration is not sufficient now and concrete calculations are needed.

In the specific case $\theta = 0$, $\cos\theta_1$ and $\cos\theta_2$ also satisfy (3.34) but their values are fixed by (3.20).

3.4 Relation to the classical Cherenkov effect

We discuss now how the classical electromagnetic field strengths (which are the solutions of the Maxwell equations with classical currents in their r.h.s.) are related to the quantum field strengths operators. In quantum electrodynamics [9,10] the classical electromagnetic field strengths are defined as eigenvalues of positive frequency parts of the quantum field strengths operators (taken in the Heisenberg representation) when they act on the so-called coherent states. The latter can be presented as an infinite sum over states with a fixed photon numbers. The coefficients of these states are related to the Fourier components of the classical currents. Therefore, classical solutions of the Maxwell equations involve contributions from states with arbitrary photon numbers. The afore-said is valid only for the current flowing in vacuum. If one suggests that the same reasoning can be applied to the charge motion in medium, the classical formulae describing Cherenkov radiation should contain contributions from the states with arbitrary photon numbers.

4 Discussion and Conclusion

Using the analogy with the Doppler effect for the scattering of light by a charge moving in medium, Frank [11, 12] obtained the following condition for the emission of two photons:

$$\epsilon_1(\beta n_1 \cos\theta_1 - 1) + \epsilon_2(\beta n_2 \cos\theta_2 - 1) = 0, \qquad (4.1)$$

where β is the initial charge velocity. In the relativistic limit ($\beta \approx 1$), (4.1) coincides with equation (3.34) following from the relativistic kinematics. However, for arbitrary β, (4.1) is not compatible with exact kinematical inequalities (3.28) and (3.29) for the two-photon emission and, therefore, the above analogy with the Doppler effect is not at least complete.

It turns out that highly relativistic charges are not convenient for the observation of the two-photon Cherenkov effect. As we have seen, in this case the recoil charge flies in the almost forward direction and it will be rather difficult to discriminate it from the recoil charge moving exactly in the forward direction (only for this particular kinematics the photon emission angles θ_1 and θ_2 are fixed (see (3.10)). It is desirable to choose the energy of the initial charge only slightly above the summary energy of two photons. Certainly, kinematics itself cannot tell us how frequently the recoil charge or one of the photons moves exactly in the forward direction. For this, concrete calculations are needed. The goal of this treatment is to point out that the two-photon Cherenkov effect will be pronounced for the special orientations of photons and the recoil charge. This makes easier the experimental search for the 2-photon Cherenkov effect.

Appendix

It is easy to check that inequality (3.11) leads to (3.14) when

$$\frac{\beta_0\gamma_0 - \beta\gamma}{n_2} \leq \gamma_0 - \gamma \leq \frac{\beta_0\gamma_0 - \beta\gamma}{n_1} \tag{A.1}$$

and to (3.18) for

$$\gamma_0 - \gamma \geq \frac{\beta_0\gamma_0 - \beta\gamma}{n_1} \ . \tag{A.2}$$

We need, therefore, to resolve the conditions

$$\gamma_0 - \gamma < \frac{1}{n}(\beta_0\gamma_0 - \beta\gamma) \tag{A.3}$$

and

$$\gamma_0 - \gamma > \frac{1}{n}(\beta_0\gamma_0 - \beta\gamma) \tag{A.4}$$

for various relations between n and β_0.

Inequality $(A.3)$

For $n > 1$ inequality (A.3) reduces to

$$\beta < \beta_0 \quad \text{for} \quad \beta_0 < \frac{1}{n} \tag{A.5}$$

and to

$$\beta < \beta_c \quad \text{for} \quad \frac{1}{n} < \beta_0 < \frac{2n}{1+n^2}. \tag{A.6}$$

Here we put

$$\beta_c = \frac{2n - \beta_0(1+n^2)}{1+n^2-2n\beta_0}.$$

Inequality (A.3) cannot be satisfied for $\beta_0 > 2n/(1+n^2)$.
For $n < 1$, inequality (A.3) holds for all β in the interval $0 < \beta < \beta_0$.

Inequality $(A.4)$

For $n > 1$, inequality (A.4) reduces to

$$\beta_c < \beta < \beta_0 \quad \text{for} \quad \frac{1}{n} < \beta_0 < \frac{2n}{1+n^2} \tag{A.7}$$

and

$$0 < \beta < \beta_0 \quad \text{for} \quad \beta_0 > \frac{2n}{1+n^2}. \tag{A.8}$$

For $n < 1$, there are no solutions of (A.4).
Putting $n = n_1 < 1$ in (A.3), $n = n_2 > 1$ in (A.4) and combining them, one gets

(3.12) and (3.13).
Putting $n = n_1 > 1$ in (A.3), $n = n_2 > 1$ in (A.4) and combining them, one gets (3.16) and (3.17).
Putting $n = n_1 > 1$ in (A.4), one gets (3.19).

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