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SELF-DUAL YANG–MILLS EQUATION AND DEFORMATION OF SURFACES

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1 Introduction

Several nonlinear phenomena in physics, modelled by the nonlinear differential equations, can describe also the evolution of surfaces in time. The interaction between differential geometry of surfaces and nonlinear differential equations has been studied since the 19th century. This relationship is based on the fact that most of the local properties of surfaces are expressed in terms of nonlinear differential equations. Since the famous sine-Gordon and Liouville equations, the interrelation between nonlinear differential equations of the classical differential geometry of surfaces and modern soliton equations has been studied from various points of view in numerous papers. In particular, the relationship between deformations of surfaces and integrable systems in 2+1 dimensions was studied by several authors [1-14, 27-28, 30].

The Self-Dual Yang - Mills equation (SDYME) is a famous example of nonlinear differential equations in four dimensions integrable by the inverse scattering method [16]-[17]. Ward conjectured that all integrable (1+1)-dimensional nonlinear differential equations may be obtained from SDYME by reduction [18] (see the book [19] and references therein). More recently, many soliton equations in 2+1 dimensions have been found as reductions of the SDYME [20]-[23].

In this paper we study the deformation of surfaces in the context of its connection with integrable systems in 2+1 and 3+1 dimensions. We show that many integrable (2+1)-dimensional nonlinear differential equations can be obtained from the deformed or (2+1)-dimensional Gauss - Mainardi - Codazzi equation (GMCE) describing the deformation (motion) of the surface, as exact particular cases. At the same time, integrable isotropic spin systems in 2+1 dimensions are exact reductions of the (2+1)-dimensional or, in other words, deformed Gauss - Weingarten equation (GWE). Also we show that the deformed GMCE is the exact reduction of two famous multidimensional integrable system, namely, the Yang - Mills - Higgs - Bogomolny equation and the SDYME.

2 Fundamental facts on the theory of surfaces

Let us consider a smooth surface in R^3 with local coordinates x and t, where $\mathbf{r}(x,t)$ is a position vector. The first and second fundamental forms of this surface are given by

$$I = d\mathbf{r}^2 = Edx^2 + 2Fdxdt + Gdt^2, \tag{1a}$$

$$II = d\mathbf{r} \cdot \mathbf{n} = Ldx^2 + 2Mdxdt + Ndt^2, \tag{1b}$$

where by definition

$$E = \mathbf{r}_x^2, \quad F = \mathbf{r}_x \cdot \mathbf{r}_t, \quad G = \mathbf{r}_t^2,$$

$$L = \mathbf{r}_{xx} \cdot \mathbf{n}, \quad M = \mathbf{r}_{tx} \cdot \mathbf{n}, \quad N = \mathbf{r}_{tt} \cdot \mathbf{n}.$$

The unit normal vector $\mathbf{n}$ to the surface is given by

$$\mathbf{n} = \frac{\mathbf{r}_x \wedge \mathbf{r}_t}{|\mathbf{r}_x \wedge \mathbf{r}_t|}.$$

There exists the third fundamental form

$$III = d\mathbf{n} \cdot d\mathbf{n} = edx^2 + 2fdxdt + gdt^2. \tag{2}$$

This form, in contrast to II, does not depend on the choice of $\mathbf{n}$ and contains no new information, since it is expressible in terms of I and II as

$$III = 2H \cdot II - K \cdot I, \tag{3}$$

where K, H are the gaussian and mean curvatures, respectively. As is well known in surface theory, the GWE for surface can be written as

$$\mathbf{r}_{xx} = \Gamma^1_{11}\mathbf{r}_x + \Gamma^2_{11}\mathbf{r}_t + L\mathbf{n}, \tag{4a}$$

$$\mathbf{r}_{xt} = \Gamma^1_{12}\mathbf{r}_x + \Gamma^2_{12}\mathbf{r}_t + M\mathbf{n}, \tag{4b}$$

$$\mathbf{r}_{tt} = \Gamma^1_{22}\mathbf{r}_x + \Gamma^2_{22}\mathbf{r}_t + N\mathbf{n}, \tag{4c}$$

$$\mathbf{n}_x = P^1_1\mathbf{r}_x + P^2_1\mathbf{r}_t, \tag{4d}$$

$$\mathbf{n}_t = P^1_2\mathbf{r}_x + P^2_2\mathbf{r}_t, \tag{4e}$$

where

$$\Gamma^1_{11} = \frac{GE_x - 2FF_x + FE_t}{2g}, \quad \Gamma^2_{11} = \frac{2EF_x - EE_t - FE_x}{2g},$$

$$\Gamma^1_{12} = \frac{GE_t - FG_x}{2g}, \quad \Gamma^2_{12} = \frac{EG_x - FE_t}{2g},$$

$$\Gamma^1_{22} = \frac{2GF_t - GG_x - FG_t}{2g}, \quad \Gamma^2_{22} = \frac{EG_t - 2FF_t + FG_x}{2g}, \tag{5}$$

$$P_1^1 = \frac{MF - LG}{g}, \quad P_1^2 = \frac{LF - ME}{g},$$
$$P_2^1 = \frac{NF - MG}{g}, \quad P_2^2 = \frac{MF - NE}{g}$$

Here

$$g = EG - F^2$$

Now we introduce the orthogonal basis as

$$\mathbf{e}_1 = \frac{\mathbf{r}_x}{\sqrt{E}},$$
$$\mathbf{e}_2 = \mathbf{n},$$
$$\mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2.$$

Hence

$$\mathbf{r}_t = \frac{F}{\sqrt{E}}\mathbf{e}_1 - \sqrt{\frac{g}{E}}\mathbf{e}_3.$$

Then the GWE takes the form

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_x = A \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{6a}$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_t = C \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{6b}$$

where

$$A = \begin{pmatrix} 0 & \kappa & -\sigma \\ -\kappa & 0 & \tau \\ \sigma & -\tau & 0 \end{pmatrix}, \tag{7a}$$

$$C = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix} \tag{7b}$$

and

$$\kappa_g = \kappa = \frac{L}{\sqrt{E}},$$
$$\tau_g = \tau = -\sqrt{\frac{g}{E}}P_1^2,$$
$$\kappa_n = \sigma = \frac{\sqrt{g}}{E}\Gamma_{11}^2$$

and

$$\omega_1 = -\sqrt{\frac{g}{E}}P_2^2,$$

$$\omega_2 = \frac{\sqrt{g}}{E}\Gamma^2_{12},$$
$$\omega_3 = \frac{M}{\sqrt{E}}.$$

Here $\kappa_n, \kappa_g, \tau_g$ are called the normal curvature, geodesic curvature and geodesic torsion, respectively. In the case

$$\sigma = 0$$

the first equation of this GWE coincides in fact with the Frenet equation for the curves. So, all that we are doing in the next sections is true for the motion (deformation) of curves when $\sigma = 0$.

The compatibility condition for the GWE (6) gives the GMCE as

$$A_t - C_x + [A, C] = 0, \tag{8}$$

or in elements

$$\kappa_t = \omega_{3x} + \tau\omega_2 - \sigma\omega_1, \tag{9a}$$

$$\tau_t = \omega_{1x} + \sigma\omega_3 - \kappa\omega_2, \tag{9b}$$

$$\sigma_t = \omega_{2x} + \kappa\omega_1 - \tau\omega_3. \tag{9c}$$

We can reformulate the linear system (6) in 2×2 matrix form as

$$\psi_x = U\psi,$$

$$\psi_t = V\psi,$$

where

$$U = \frac{1}{2i}\begin{pmatrix} \tau & \kappa + i\sigma \\ \kappa - i\sigma & -\tau \end{pmatrix},$$

$$V = \frac{1}{2i}\begin{pmatrix} \omega_1 & \omega_3 + i\omega_2 \\ \omega_3 - i\omega_2 & -\omega_1 \end{pmatrix}.$$

3 Deformation of surfaces

Now we would like to consider the deformation of the surface with respect to y. We postulate that such deformation or motion of the surface is governed by the system

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_x = A\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{10a}$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_y = B \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{10b}$$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}_t = C \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{10c}$$

where

$$A = \begin{pmatrix} 0 & \kappa & -\sigma \\ -\kappa & 0 & \tau \\ \sigma & -\tau & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & \gamma_3 & -\gamma_2 \\ -\gamma_3 & 0 & \gamma_1 \\ \gamma_2 & -\gamma_1 & 0 \end{pmatrix}, \tag{11}$$

$$C = \begin{pmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{pmatrix}$$

and γ_j are real functions. The system (10) will be called the deformed or (2+1)-dimensional GWE. We remark that first and third equations of the system (10) are the equations (6) and A, C coincide with formulas (7). The compatibility conditions of the deformed GWE (10) gives the deformed or (2+1)-dimensional GMCE of the form

$$A_t - C_x + [A, C] = 0, \tag{12a}$$

$$A_y - B_x + [A, B] = 0, \tag{12b}$$

$$B_t - C_y + [B, C] = 0. \tag{12c}$$

As we see, equation (12a) is in fact the GMCE (8). This fact explains why we call (12) the deformed GMCE. The linear problem (Lax representation) associated with the system (12) can be written as

$$\Psi_z = \lambda^2 \Psi_{\bar{z}} + (F^- - \lambda^2 F^+)\Psi, \tag{13a}$$

$$\Psi_t = -i\lambda \Psi_{\bar{z}} + (C + i\lambda F^+)\Psi, \tag{13b}$$

where

$$F^{\pm} = A \pm iB,$$

$$z = \frac{1}{2}(x + iy),$$

$$\bar{z} = \frac{1}{2}(x - iy).$$

So we can confirm that the deformed GMCE (12) is a candidate to be integrable in the sense that for it there exists the Lax representation with the spectral parameter (13). Higher hierarchy of the deformed GMCE (12) can be obtained as the compatibility condition of the linear system

$$\Psi_z = \lambda^2 \Psi_{\bar{z}} + (F^- - \lambda^2 F^+)\Psi, \tag{14a}$$

$$\Psi_t = -i\lambda^n \Psi_{\bar{z}} + \sum_{j=0}^{m} \lambda^j F_j \Psi. \tag{14b}$$

4 Deformation of surfaces induced by (2+1)-dimensional integrable systems

In this section we would like to attract your attention to some aspects of the relation between the deformation of surfaces and integrable systems in 2+1 dimensions.

4.1 Integrable systems in 2+1 dimensions and the deformed GMCE

Our first observation is that some important integrable systems in 2+1 dimensions are particular reductions of equations (12). In fact, the well-known (2+1)-dimensional integrable systems such as the Kadomtsev - Petviashvili equation, the Davey - Stewartson equation and so on, can be obtained from the deformed GMCE (12) as some reductions. We support this statement by presenting some examples.

Example 1. *The Davey - Stewartson II equation.* Let the matrices A, B, C in the equations (12) have the following form

$$A = \sqrt{2i}\lambda\sigma_3 + \frac{1}{\sqrt{2}}\bar{q}\sigma^+ + \frac{1}{\sqrt{2}}q\sigma^-, \tag{15a}$$

$$B = -\frac{i\lambda}{\sqrt{2}}\sigma_3 + \frac{1}{\sqrt{2}}\bar{q}\sigma^+ + \frac{1}{\sqrt{2}}q\sigma^-, \tag{15b}$$

$$C = -\frac{i}{2}(|q|^2 + \phi_y + 3\lambda^2)\sigma_3 - 3\lambda\bar{q}\sigma^+ - 3\lambda q\sigma^-, \tag{15c}$$

where we used the isomorphism $so(3) \cong su(2)$ to write the matrices A, B, C in 2×2 form,

$$\sigma^{\pm} = \sigma_1 \pm i\sigma_2.$$

Substituting (15) into the system (12) after some algebra we get the Davey - Stewartson II equation [24]

$$iq_t + \frac{1}{2}(q_{xx} + q_{yy}) - (|q|^2 + \phi_y)q = 0, \tag{16a}$$

$$\phi_{xx} + \phi_{yy} + 2(|q|^2)_y = 0. \tag{16b}$$

Example 2. *The Kadomtsev - Petviashvili equation.* Now we consider the case when

$$\tau = \sigma = \omega_1 = \omega_2 = \gamma_1 = \gamma_2 = 0.$$

Then the matrices A, B, C take the form

$$A = \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix}, \tag{17a}$$

$$B = \begin{pmatrix} 0 & \gamma_3 \\ -\gamma_3 & 0 \end{pmatrix}, \tag{17b}$$

$$C = \begin{pmatrix} 0 & \omega_3 \\ -\omega_3 & 0 \end{pmatrix}. \tag{17c}$$

Substituting these expressions into (12) we get

$$\kappa_y = \gamma_{3x}, \tag{18a}$$

$$\gamma_{3t} = \omega_{3y}, \tag{18b}$$

$$\kappa_t = \omega_{3x}. \tag{18c}$$

Now we assume that the function ω_3 has the form

$$\omega_3 = -\kappa_{xx} - 3\kappa^2 - 3\alpha^2\partial_x^{-1}\gamma_{3y}. \tag{19}$$

Hence and from (18c) we obtain

$$\kappa_t + 6\kappa\kappa_x + \kappa_{xxx} + 3\alpha^2\gamma_{3y} = 0, \tag{20a}$$

$$\gamma_{3x} = \kappa_y, \tag{20b}$$

or

$$(\kappa_t + 6\kappa\kappa_x + \kappa_{xxx})_x + 3\alpha^2\kappa_{yy} = 0. \tag{21}$$

It is the famous Kadomtsev - Petviashvili equation.

Example 3. *The Lame equation.* Now we consider the case when the matrices A, B, C take the form (see, e. g. the Ref. [26])

$$A = \begin{pmatrix} 0 & -\beta_{21} & -\beta_{31} \\ \beta_{21} & 0 & 0 \\ \beta_{31} & 0 & 0 \end{pmatrix}, \tag{22a}$$

$$B = \begin{pmatrix} 0 & \beta_{12} & 0 \\ -\beta_{12} & 0 & -\beta_{32} \\ 0 & \beta_{32} & 0 \end{pmatrix}, \tag{22b}$$

$$C = \begin{pmatrix} 0 & 0 & \beta_{13} \\ 0 & 0 & \beta_{23} \\ -\beta_{13} & -\beta_{23} & 0 \end{pmatrix}. \tag{22c}$$

In this case, the (2+1)-dimensional GMCE (12) takes the form

$$\frac{\partial \beta_{ij}}{\partial u^k} = \beta_{ik}\beta_{kj}, \tag{23a}$$

$$\frac{\partial \beta_{ij}}{\partial u^i} + \frac{\partial \beta_{ji}}{\partial u^j} + \sum_{m\neq i,j} \beta_{mi}\beta_{mj} = 0, \tag{23b}$$

where

$$u^1 = x, \quad u^2 = y, \quad u^3 = t.$$

It is the Lame equation, integrability of which was proved by V. E. Zakharov in [2].

So we have presented three examples of integrable equations in 2+1 dimensions which are the particular reductions of the deformed GMCE (12).

4.2 Integrable spin systems in 2+1 dimensions and the deformed GWE

Now let us consider the relation between integrable spin systems in 2+1 dimensions and the deformed GWE (10). Our second observation is that many integrable isotropic spin systems in 2+1 dimensions are particular reductions of equations (10). We support this statement by citing two examples.

Example 4. *The M-I equation.* First, we consider the M-I equation which reads as

$$\mathbf{S}_t = (\mathbf{S} \wedge \mathbf{S}_y + u\mathbf{S})_x, \tag{24a}$$

$$u_x = -\mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y) \tag{24b}$$

and which is integrable (see, e. g. Refs. [9, 13]). In this case we take the identification

$$\mathbf{e}_1 = \mathbf{S}, \tag{25}$$

where $\mathbf{S}$ is the solution of the M-I equation (24) and

$$\mathbf{e}_1^2 = \kappa^2 + \sigma^2 = \mathbf{S}_x^2.$$

Then the M-I equation (24) becomes

$$\mathbf{e}_{1t} = (\mathbf{e}_1 \wedge \mathbf{e}_{1y} + u\mathbf{e}_1)_x, \tag{26a}$$

$$u_x = -\mathbf{e}_1 \cdot (\mathbf{e}_{1x} \wedge \mathbf{e}_{1y}). \tag{26b}$$

Now let us assume

$$\begin{aligned} \tau &= f_x, \\ \gamma_1 &= f_y + u, \\ \omega_1 &= f_t + \partial_x^{-1}(\sigma\omega_3 - \kappa\omega_2), \\ \omega_2 &= -\gamma_{3x} - \gamma_2\tau + u\sigma, \\ \omega_3 &= \gamma_{2x} - \gamma_3\tau + u\kappa, \end{aligned} \tag{27}$$

where $f(x, y, t, \lambda)$ is a real function. Taking into account the formulas (27) and after eliminating the vectors $\mathbf{e}_2$ and $\mathbf{e}_3$, the system (10) takes the form (26). This means that the M-I equation (24) is the particular exact reduction of the (2+1)-dimensional GWE (10) with the choice (27).

Example 5. *The Ishimori equation.* Now we assume that the functions γ_i, ω_i are given by the formulas

$$\begin{aligned} \tau &= f_x + \frac{1}{2}u_y, \\ \gamma_1 &= f_y + \frac{1}{2\alpha^2}u_x, \\ \omega_1 &= f_t + \partial_x^{-1}(\frac{1}{2}u_{yt} + \sigma\omega_3 - \kappa\omega_2), \end{aligned} \tag{28}$$

where as in the previous case, $f(x, y, t, \lambda)$ is a real function. Using these expressions and doing as for the M-I equation, i. e., eliminating the vectors $\mathbf{e}_2$, $\mathbf{e}_3$ from the system (10), we obtain the following equation for the unit vector $\mathbf{e}_1$

$$\mathbf{e}_{1t} = \mathbf{e}_1 \wedge (\mathbf{e}_{1xx} + \alpha^2\mathbf{e}_{1yy}) + u_x\mathbf{e}_{1y} + u_y\mathbf{e}_{1x}, \tag{29a}$$

$$\alpha^2 u_{yy} - u_{xx} = 2\alpha^2\mathbf{e}_1 \cdot (\mathbf{e}_{1x} \wedge \mathbf{e}_{1y}). \tag{29b}$$

After the identification (25), the equation (29) takes the form

$$\mathbf{S}_t = \mathbf{S} \wedge (\mathbf{S}_{xx} + \alpha^2\mathbf{S}_{yy}) + u_x\mathbf{S}_y + u_y\mathbf{S}_x, \tag{30a}$$

$$\alpha^2 u_{yy} - u_{xx} = 2\alpha^2 \mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y), \tag{30b}$$

that is, the Ishimori equation [25]. So we have shown that the M-I and Ishimori equations are the particular reductions of the deformed GWE (10). Similarly, we can show that the other isotropic spin systems in 2+1 dimensions are the exact reductions of the system (10) at least for existing known integrable isotropic (2+1)-dimensional spin systems.

4.3 Integrable spin systems in 2+1 dimensions as exact reductions of the M-0 equation

Now let us consider the (2+1)-dimensional isotropic M-0 equation (about our notations, see. e.g., [13-15], [31-32])

$$\mathbf{e}_{1t} = \omega_3 \mathbf{e}_2 - \omega_2 \mathbf{e}_3, \tag{31a}$$

$$\tau_y - \gamma_{1x} = \mathbf{e}_1 \cdot (\mathbf{e}_{1x} \wedge \mathbf{e}_{1y}), \tag{31b}$$

which sometimes we write in terms of $\mathbf{S}$ as

$$\mathbf{S}_t = \theta_1 \mathbf{S}_x + \theta_2 \mathbf{S}_y, \tag{32a}$$

$$\tau_y - \gamma_{1x} = \mathbf{S} \cdot (\mathbf{S}_x \wedge \mathbf{S}_y), \tag{32b}$$

where θ_j are some real functions. We note that many integrable isotropic spin systems in 2+1 dimensions are particular reductions of the (2+1)-dimensional isotropic M-0 equation (31), (32). For example, the M-I equation (24) is the particular case of (32) as

$$\theta_1 = \frac{\omega_3\gamma_2 - \omega_2\gamma_3}{\kappa\gamma_2 - \sigma\gamma_3}, \tag{33a}$$

$$\theta_2 = \frac{\omega_2\kappa - \omega_3\sigma}{\kappa\gamma_2 - \sigma\gamma_3}. \tag{33b}$$

5 The (2+1)-dimensional GMCE as exact reduction of the Yang - Mills - Higgs - Bogomolny equation

One of the most interesting and important integrable equations in 2+1 dimensions is the following Yang - Mills - Higgs - Bogomolny equation [19]

$$\Phi_y + [\Phi, B] + C_x - A_t + [C, A] = 0, \tag{34a}$$

$$\Phi_t + [\Phi, C] + A_y - B_x + [A, B] = 0, \tag{34b}$$

$$\Phi_x + [\Phi, A] + B_t - C_y + [B, C] = 0. \tag{34c}$$

The third important observation is that the deformed GMCE (12) is the particular case of the equation (34). In fact, if in the Yang - Mills - Higgs - Bogomolny equation we put

$$\Phi = 0$$

then it becomes the deformed GMCE (12). So we can suggest that the deformed GMCE is a candidate to be integrable as the exact reduction of the integrable equation (34).

6 The (2+1)-dimensional GCME as exact reduction of the SDYME

Now we study the relationship between the deformed GMCE (12) and the SDYME. The SDYME reads as [29]

$$F_{\mu\nu} = ^{\star} F_{\mu\nu}, \tag{35}$$

where $\star$ is the Hodge star operator and the Yang - Mills field defined as

$$F_{\mu\nu} = \frac{\partial A_\nu}{x_\mu} - \frac{\partial A_\mu}{x_\nu} - [A_\mu, A_\nu].$$

Let

$$x_\alpha = z + it,$$

$$x_{\bar{\alpha}} = z - it,$$

$$x_\beta = x + iy,$$

$$x_{\bar{\beta}} = x - iy$$

be the null-coordinates in the Euclidean space for which the metric has the form

$$ds^2 = dx_\alpha dx_{\bar{\alpha}} + dx_\beta dx_{\bar{\beta}}.$$

Now the SDYME takes the form [16-17,19]

$$F_{\alpha\beta} = 0, \tag{36a}$$

$$F_{\bar{\alpha}\bar{\beta}} = 0, \tag{36b}$$

$$F_{\alpha\bar{\alpha}} + F_{\beta\bar{\beta}} = 0, \tag{36c}$$

where

$$A_\alpha = A_z + iA_t,$$
$$A_{\bar{\alpha}} = A_z - iA_t,$$
$$A_\beta = A_x + iA_y,$$
$$A_{\bar{\beta}} = A_x - iA_y.$$

The associated linear system is [19]

$$(\partial_\alpha + \lambda\partial_{\bar{\beta}})\Psi = (A_\alpha + \lambda A_{\bar{\beta}})\Psi, \tag{37a}$$

$$(\partial_\beta - \lambda\partial_{\bar{\alpha}})\Psi = (A_\beta - \lambda A_{\bar{\alpha}})\Psi, \tag{37b}$$

where λ is the spectral parameter and

$$\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial z} - i\frac{\partial}{\partial t},$$
$$\frac{\partial}{\partial x_{\bar{\alpha}}} = \frac{\partial}{\partial z} + i\frac{\partial}{\partial t},$$
$$\frac{\partial}{\partial x_\beta} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y},$$
$$\frac{\partial}{\partial x_{\bar{\beta}}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}.$$

Our fourth observation: the deformed GMCE (12) is the particular reduction of the SDYME (36). In fact, we consider the following reduction of the SDYME

$$A_\alpha = -iC, \tag{38a}$$

$$A_{\bar{\alpha}} = iC, \tag{38b}$$

$$A_\beta = A - iB, \tag{38c}$$

$$A_{\bar{\beta}} = A + iB, \tag{38d}$$

and assume that A, B, $C \in so(3)$ and independent of z. In this case, from the SDYME (36) we obtain the (2+1)-dimensional GMCE (12) in the Euclidean coordinates.

7 Conclusion

In this paper, we have considered some deformations or, in other terminology, motions of surfaces. We have shown that the corresponding deformed GMCE is integrable in the sense that the associated linear problem (Lax representation) exists with the spectral parameter. We demonstrated that several important integrable systems in 2+1 dimensions, such as the Davey - Stewartson II, Kadomstev - Petviashvili and Lame equations are some exact reductions of the deformed GMCE. Although, all known integrable (2+1)-dimensional isotropic spin systems can be obtained from the deformed GMCE, we have showed that such spin systems can be obtained also from the deformed GWE as exact reductions. Finally, we proved that the deformed GMCE is the particular case of two famous integrable systems, namely, the Yang - Mills - Higgs - Bogomolny equation and SDYME. It goes in favour of integrability of the deformed GMCE.

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References

[1] *Konopelchenko B.G.* Phys. Lett. A. V.183. P.153. 1993.

[2] *Zakharov V.E.* Duke Math. J. V.94. P.1. 1998

[3] *Taimanov I.A.* Amer. Math. Soc. Transl. Ser.2. V.179. P.131. 1997.

[4] *Myrzakulov R., Lakshmanan M.* On the geometrical and gauge equivalence of certain (2+1)-dimensional integrable spin model and nonlinear Schrodinger equation. Preprint HEPI, Alma-Ata, 1996.

[5] *Myrzakulov R.* Spin systems and soliton geometry. Alma-Ata: FTI, 2001.

[6] *Tenenblat K.* Transformations of manifolds and applications to differential equations. Longman, 1998.

[7] *Schief W.K.* Proc. R. Soc. Lond. A. V.453. P.1671. 1997.

[8] *Myrzakulov R., Vijayalakshmi S., Nugmanova G.N., Lakshmanan M.* Phys. Lett. A. V.233. P.391. 1997.

[9] *Myrzakulov R., Nugmanova G.N., Syzdykova R.N.* J.Phys. A: Math.Gen. V.31. P.9535. 1998.

[10] *Myrzakulov R., Vijayalakshmi S., Syzdykova R.N., Lakshmanan M.* J. Math. Phys. V.39. P.2122. 1998.

[11] *Martina L., Myrzakul Kur., Myrzakulov R., Soliani G.* J. Math. Phys. V.42. P.1397. 2001.

[12] *Myrzakulov R., Danlybaeva A.K., Nugmanova G.N.* Theor. Math. Phys. V.118. P.347. 1999.

[13] *Lakshmanan M., Myrzakulov R., Vijayalakshmi S., Danlybaeva A.K.* J. Math. Phys. V.39. P.3765. 1998.

[14] *Chou K.S., Qu C.Z.* J. Phys. Soc. Jpn. V.71. P.1039. 2002.

[15] *Estevez P.G., Hernaez G.A.* Lax pair, Darboux Transformations and solitonic solutions for a (2+1)-dimensional nonlinear Schrodinger equation. E-preprint: solv-int/9910005.

[16] *Ward R.S.* Phys. Lett. A. V.61. P.81. 1977.

[17] *Belavin A.A., Zakharov V.E.* Phys. Lett. B. V.73. P.53. 1978.

[18] *Ward R.S.* Philos. Trans. R. Soc. London Ser. A, V.315. P.451. 1985.

[19] *Ablowitz M.J., Clarkson P.A.* Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge: Cambridge University Press, 1992.

[20] *Ablowitz M.J., Chakravarty S., Takhtajan L.* Commun. Math. Phys. V.158. P.289. 1993.

[21] *Chakravarty S., Kent A.L., Newmann E.T.* J. Math. Phys. V.36. P.763. 1995.

[22] *Ivanova T.A., Popov A.D.* Phys. Lett. A. V.205. P.158. 1995.

[23] *Legare M.* J. Nonlin. Phys. V.3. P.266. 1996.

[24] *Brunelli L.C., Das A.* Mod. Phys. Lett. A. V.9. P.1267. 1994.

[25] *Ishimori Y.* Prog. Theor. Phys. V.72. P.33. 1984.

[26] *Schief W.K.* Proc. R. Soc. Lond. A. V.453. P.1671. 1997.

[27] *Kozhamkulov T.A., Serikbaev N.S., Koshkinbaev A.D., Myrzakul Kur., Caiymbetova S.K., Rahimov F.K., Myrzakulov R.* On some nonlinear models of magnets. Preprint JINR P17-2003-171, Dubna, Russia, 2003.

[28] *Myrzakulov R., Rahimov F.K.* Soliton Theory of Magnets and Differential Geometry. Almaty: FTI, 2003.

[29] *Ablowitz M.J., Chakravarty S., Halburd R.G.* J. Math. Phys. V.44. P.3147. 2003.

[30] *Martina L., Kozhamkulov T.A., Myrzakul Kur., Myrzakulov R.* Integrable Heisenberg ferromagnets and soliton geometry of curves and surfaces. In: Nonlinear Physics: Theory and Experiment. II. Eds. M.J.Ablowitz, M.Boiti, F.Pempinelli, B.Prinari. Singapore: World Scientific, 2003.

[31] *Gutshabash E.Sh.* Some note on Ishimori's magnet model. In: Problems of Quantum Field Theory and Statistical Physics. Part 17. "Zapiski Nauchnykh Seminarov POMI". V.291. P.155. 2002. E-preprint: nlin.SI/0302002. 2003.

[32] *M.Lakshmanan.* Geometrical interpretation of (2+1)-dimensional integrable nonlinear evolution equations and localized solutions. Mathematisches Forschungsinstitut Oberwolfach. Report/40/97.ps, P.9. 1997.

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