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Теоретико-полевое описание ограниченных связями процессов диссипации энергии

Дается теоретико-полевое описание процессов диссипации, ограниченных высокой группой симметрии. Формализм излагается на примере процессов множественного рождения адронов, в которых переход к термодинамическому равновесию происходит в результате диссипации кинетической энергии сталкивающихся частиц в массы адронов. Динамика этих процессов ограничена необходимостью учитывать связи, ответственные за «невылетание» цветового заряда. Развита более общая S-матричная формулировка термодинамики неравновесных диссипативных процессов, найдено необходимое и достаточное условие корректности такого описания, сходное с условием ослабления корреляций, которое, по Боголюбову, должно иметь место при приближении системы к равновесию. Физически такая ситуация должна возникать в процессах с очень большой множественностью, по крайней мере в случае, если масса адронов отлична от нуля. Излагается также новая схема теории возмущений сильной связи, удобная для учета симметрийных ограничений на динамику процессов диссипации. Приводится обзор литературы, посвященной обсуждаемой проблеме.

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Manjavidze J. D., Sissakian A. N. E2-2002-93
A Field Theory Description of Constrained Energy-Dissipation Processes

We give a field theory description of dissipation processes constrained by a high-symmetry group. The formalism is presented in the example of the multiple-hadron production processes, where the transition to the thermodynamic equilibrium results from the kinetic energy of colliding particles dissipating into hadron masses. The dynamics of these processes are restricted because the constraints responsible for the color charge confinement must be taken into account. We develop a more general S-matrix formulation of the thermodynamics of nonequilibrium dissipative processes and find a necessary and sufficient condition for the validity of this description; this condition is similar to the correlation relaxation condition, which, according to Bogoliubov, must apply as the system approaches equilibrium. This situation must physically occur in processes with an extremely high multiplicity, at least if the hadron mass is nonzero. We also describe a new strong-coupling perturbation scheme, which is useful for taking symmetry restrictions on the dynamics of dissipation processes into account. We review the literature devoted to this problem.

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Dedicated to the memory of Aleksandr Mikhailovich Baldin,
an outstanding scientist and personality

1. Introduction

In this review, we attempt to describe nonequilibrium processes constrained by a high-symmetry group. The example of this type that is closest our interests is provided by multiple-hadron production processes, although these do not seem to limit the variety of physical applications of the formalism presented here.

We attempt to formulate the perturbation theory for a nonequilibrium problem, namely, the relativistic thermodynamics with constraints associated with the high symmetry of the problem. Specifically, we wish to describe the dissipation of the kinetic energy of colliding particles into hadron masses. We view this process as one of the forms of the initial-state thermalization, assuming that the incident energy goes into the hadron color constituent masses and into the color charge binding energy.

We note that the problem of defining the notion of equilibrium in dissipative systems is itself very important. For example, interest in this problem is often aroused because the transition to equilibrium in such systems is understood as the tendency to establish a certain "order" [1].

Within this setup of our problem, we are interested in the S-matrix formulation, where we can arbitrarily fix the initial and the final states at infinitely remote hypersurfaces σ_∞. We assume that symmetries acting via the corresponding conservation laws, including hidden ones (of the "polynomial type"), can constrain the dissipation process, thereby affecting the probability of the realization of specific asymptotic states on σ_∞. In what follows, we address the problem of how and to what extent this can occur.

1.1. Energy dissipation processes and multiple-particle production. Strictly speaking, we do not have the possibility to control the time of the multiple-production process in inelastic scattering experiments with relativistic particles (see [2], [3], where this problem was investigated within the formalism of Wigner functions [4], and also see [5], where it was discussed from general perspectives). Therefore, we can only indirectly estimate the proper time of the process by controlling just its result. For example, introducing the inelastic coefficient $\kappa = 1 - \varepsilon_{\max}/E$, where $\varepsilon_{\max}$ is the energy of the fastest particle in the given reference frame[1] and E is the total energy, we can control the dissipation degree by choosing specific values of κ. We can also consider the mean kinetic energy $1/\beta_c$ of the produced particles if fluctuations in the vicinity of β_c are not very large. The simplest parameter (albeit not necessarily from the experimentalist's standpoint) is the multiplicity n of the produced particles. It is obvious that dissipation is considerably large if $\kappa \to 1$ or if $n \to n_{\max} = E/m_h$ (where $m_h \simeq 0.2\,\text{GeV}$ is the characteristic hadron mass). Under these conditions, $\beta_c \to \infty$.

It is assumed that in hadron physics, the constraints that follow from the non-Abelian gauge symmetry lead to the confinement of the color charge inside colorless hadrons. In addition, and apparently more importantly, the same constraints prevent the complete thermalization of a very "hot" initial state (the one at high energies of colliding particles). Indeed, under complete thermalization, the mean multiplicity of the produced particles $\bar{n}(E)$ must be $\sim E$ [6]. But the experiment shows that the mean multiplicity is proportional to only $\log^2(E/m_h)$, although rare fluctuations can occur with the multiplicity $n \gg \bar{n}(E)$.

It can therefore be assumed that the constraints considerably affect the formation of the multiple-hadron production dynamics. But these constraints do not play a decisive role here, in contrast to the case of completely integrable problems [7], [8], because a certain fraction of energy still dissipates ($\bar{n}(E) \sim \log^2(E/m_h) \gg 1$). We therefore assume that the constraints only restrict the dynamics in some way (see

[1] We consider only the center-of-mass frame.

Sec. 1.3 for details). The problem considered here thus involves processes that are intermediate between completely integrable and completely thermalizable ones and is therefore sufficiently complicated.

Indeed, many facts have been accumulated in the physics of multiple production during the almost three-quarters of a century of its history (the pioneering works pertaining to multiple production are referred to in [9]), but there are not many rigorous results. We must first note the results based on causality and unitarity, for example, the proof of dispersion relations for the two-particle amplitudes (see [10] and the references therein and also [11]). There is also a principally important extension of this approach to inclusive processes [12]. Regarding asymptotic estimates, we must note the Froissaire and Pomeranchuk theorems for the total cross sections (see, e.g., [13]). This practically exhausts the results concerning formal foundations of the theory of strong interactions describing multiple-production processes.[2]

Over the decades of the development of multiple-production physics, many ideas based on heuristic assumptions have been tested. We mention some of them here. First is the idea based on the experimental observation that the mean transverse momentum of produced hadrons is constrained and is practically independent of the energy E and the multiplicity n (at least for "moderate" E and n). This observation leads to the multiperipheral approach [14] and the Regge model related to it [15]. Even without a comprehensive theoretical justification, these schemes remain the main instruments for describing multiple production qualitatively.

Experimental data were considerably systematized using the notion of scale invariance at short distances [16]. Next, the notions of duality (which follows from crossing symmetry and the sum rules for the final energy [17]) seem fundamental [18]. But we lack a consistent scheme based on these ideas that would not contradict the unitarity condition and could produce experimentally verifiable predictions [19].

From the methodological standpoint, it is important to keep in mind that the dissipation of the incident energy E into hadron masses is a multicomponent process and each of its components has its own space–time scale and apparently different multiple-production mechanism. Grasping this idea has been attempted from the phenomenological [20] as well as from the purely formal standpoint [21] using decomposition into correlation functions. The latter approach is similar to the Mayer group decomposition [22].

Despite all these efforts, we still lack a detailed quantitative theory of inelastic hadron interactions. At the same time, inelastic interactions are responsible for the main share of contributions to the total cross section of hadrons (see, e.g., [23]). Discrepancies in the qualitative estimates of the role of specific inelastic processes are therefore an essential obstruction to further experimental investigations (in modern problems, these investigations are very sensitive to the background conditions).

We note that the physics of multiple production is also interesting for its own sake, although it may not presently be the "mainstream" of physics as compared, for example, to the standard model and related problems (the discovery of the Higgs boson, the mass hierarchy problem, etc.). In this sense, the production processes of a very large number of hadrons may be especially interesting because the maximum number of the degrees of freedom must be excited at a very high multiplicity [7].

The remaining uncertainty in estimating the contributions that are dominant at high energies [24] and the fact that the Regni fractal dimension is nontrivial and has an essential dependence on the energy and type of the interacting particles [25] certainly abate the hope that the qualitative theory of multiple-hadron production can ever be completely created (for example, this phenomenon seems comparable to turbulence in complexity). The observable decline in the number of publications on multiple production during the last two decades is a direct consequence. In this connection, we note that precisely the asymptotic regime with respect to multiplicity can be the simplest (see Sec. 6), because we expect that the *statistical description* [7] is applicable in this domain of multiplicities (which means that the process details must not be crucially important under these conditions). But we must emphasize here that in contrast to the

[2] We do not discuss purely axiomatic constructions including the Wightman functions or the Haag theorem.

hydrodynamic and statistical multiple-production models developed previously [6], [26], we expect the onset of the hydrodynamic stage [27] of the thermalization process only in the domain of very high multiplicities (in other words, it can be realized only as a sufficiently rare fluctuation in the dissipation process).

We attempt to construct a theory here that could describe the dynamics of strong interactions both at short distances, where the effect of symmetry constraints seems insignificant, and at long distances, where taking the constraints into account is principally important. We see in what follows that despite its formal compactness, the final expression (see Eq. (5.38) below) for the generating functional of the observables (the scattering cross section, correlation functions, etc.) is in fact extremely cumbersome; therefore, more than likely, only numerical methods can be effective to any extent (see Sec. 6). We are presently working in this direction, but the description of experimental predictions is beyond the scope of this review [28].

1.2. Microcanonical formalism. Strictly speaking, the amplitudes of the production of n particles depend on $3n-4$ variables. This number is too large,[3] and hoping to construct an exact scheme for their description with all these variables being essential would be naive (see above and also [25]). Therefore, to formulate the problem qualitatively, we first attempt to find conditions under which the system can be described by a limited number of parameters. Obviously, statistical physics methods must be adapted for this purpose.

It is remarkable that under the conditions resembling the Bogoliubov correlation relaxation principle [29] [4] (see Sec. 2.1, where the Bogoliubov conditions are derived), the system arising as a result of particle production must attain *equilibrium*. Namely, the generating functional of the inelastic scattering cross sections $\rho(\alpha, z)$ evaluated using the S-matrix under the above conditions precisely coincides with the partition function of the equilibrium thermodynamics in the Schwinger–Keldysh formulation [30] with the corresponding Kubo–Martin–Schwinger periodic boundary conditions [31]. Here we must note the formal statement that thermodynamic theories constructed on the base of the Kubo–Martin–Schwinger periodic boundary conditions can describe only the systems that are in equilibrium in the canonical meaning of this term [32].

We show that the above relaxation of correlations is the necessary and sufficient condition for the validity of our thermodynamics based on the S-matrix formalism. Describing the kinetic stage of the process was attempted in [3] based on the *local equilibrium* hypothesis [33]. But if the correlations do not vanish, we are left with the standard S-matrix description operating with $3n-4$ independent variables. We reconsider this problem in Sec. 2.1.

It follows from the above that, generally speaking, the correlation relaxation conditions must not be satisfied in hadron processes, because symmetry constraints are involved. But the asymptotic regime with respect to n can be considered. In that case, if the conservation laws corresponding to the symmetry of the problem only constrain the dynamics, then selecting very high multiplicities results in suppressing the effect of those constraints associated with a given symmetry. This must obviously simplify the theory. (The introduction of the asymptotic regime with respect to n is also convenient because a small parameter $\sim \bar{n}(E)/n$ then arises in the theory. We can then use the fact that the momenta of produced particles must be relatively small.) The principle importance of the physics of very high multiplicities was discussed in detail in [7], and we do not consider this problem in what follows. We note here that the *thermalization*, i.e., the correlation relaxation effect, seems to be attainable at lower values of $n/\bar{n}(E)$ in ion–ion collisions [34]. We note that n determines only the number of momenta in the argument of the multiple-production amplitude $a_n(q_1, q_2, \dots, q_n; E)$. Therefore, whenever we are interested in the asymptotic behavior with respect to n, for example, we must evaluate the modulus $\left|a_n(q_1, q_2, \dots, q_n; E)\right|^2$ integrated over the entire phase volume

[3] With modern accelerator energies, the mean multiplicity of produced particles is up to one hundred.

[4] The importance of the Bogoliubov correlation relaxation principle for multiple-hadron production processes has been emphasized many times by A. M. Baldin in discussions with one of us (A. N. S.).

because n appears as a parameter in our formulas only in that case.

All this naturally leads to the idea that instead of multiparticle amplitudes, we must consider precisely the generating functions (or functionals) $\rho(\alpha, z)$ expressed through the integrals of $|a_n|^2$ weighted by the corresponding parameters α and z. In the simplest version of the theory, to preserve the possibility of "tuning" the final state of the dissipation process of the incident energy at will, we therefore introduce the dependence on the four-vector $\alpha = (-i\beta, \vec{\alpha})$, which is conjugate to the momentum of the produced particles, and on the parameter z, which is conjugate to the number of particles (see the definition in Eq. (2.1)).

We first show how to introduce a thermodynamic formalism ("rough," using the definition proposed in [27]) that is "economical" (because it uses a restricted set of parameters—the temperature $\sim 1/\beta$, the chemical potential $\sim \log z$, etc.) and at the same time is capable of describing the system. We find the necessary and sufficient conditions for this description (see Sec. 2.1). Details and an additional list of references can be found in [3], [7].

1.3. Constrained quantization. Because the Lagrangians of modern field theories possess a high-symmetry group [35], [36] while the computational scheme must operate with only independent degrees of freedom, the problem of how to select these latter arises. In the canonical formalism, the corresponding constraint equations are used for this [37]. But this procedure is sufficiently complicated and unclear in many respects due to its unwieldiness.

At an early stage of constructing the theory of strong interactions based on the Yang–Mills gauge theory, it was natural to use the conventional computational scheme that practically repeated the quantum electrodynamics with its excellent reputation. It is then essential that with the Slavnov–Taylor identity [38] applied in this formulation, the Yang–Mills theory becomes renormalizable [39]. The Faddeev–Popov method [40] used for this helps separate the dynamic degrees of freedom from the purely gauge ones. But the price for this is that the effective action of non-Abelian gauge theories is non-Hermitian. This considerably complicates the derivation of gauge-invariant results because gauge invariance is restored only in summing different diagram contributions. In addition, the procedure of selecting gauge degrees of freedom is itself ambiguous in strong gauge fields [41], [42]. In relation to this, reformulating the perturbation theory in terms of gauge-invariant fields was proposed in a number of works (see, e.g., [43]). In what follows, we show how this can be achieved (see Sec. 5.4).

These problems do not play an essential role if the interactions and, correspondingly, the fields are weak in the applications under consideration. As a result, the *asymptotic freedom* was predicted in this *weak-coupling regime*, based on the phenomenon of the antiscreening of a color charge [44]: the "running" expansion parameter is

$$\alpha_s \propto \frac{1}{\log(q^2/\Lambda^2)} \ll 1 \tag{1.1}$$

for $q^2 \gg \Lambda^2$. This fact has had a crucial impact on the formation of the hadron phenomenology during the last decades. A natural explanation was thus found for the scale invariance in deeply inelastic processes [45], and the formation of (QCD) jets was predicted [46]. At the same time, it can be seen from (1.1) that the weak-coupling perturbation theory has a limited applicability domain because of the existence of a pole in the expression for α_s at $q^2 = \Lambda^2$. This difficulty can be eliminated by introducing a certain analyticity condition in the theory [47]. "Correcting" the theory by taking power corrections into account was also attempted [48].

But there remains the problem of constraints, in particular gauge ones, that are essential at long distances, where the fields are strong and must therefore influence the spectrum of "soft" particles. Namely, such particles are produced in the domain of very high multiplicities, and, as noted above, it is desirable to learn to describe them first because they are likely to be particularly simple [7]. To take the constraints into account, we use the idea in [49], which is quite popular because of its transparency. It can be observed that the invariant hypersurface W preserving the symmetry group constraints is determined by a partial solution of the Lagrange equations.[5] The quantization problem can then be reduced to quantization of the invariant hypersurface W, which is considerably simpler, because this hypersurface can coincide with the quotient space $\mathcal{G}/\mathcal{H}$, which is homogeneous and isotropic in the quasi-classical approximation by definition. Indeed, $\mathcal{G}$ is the symmetry group in the problem, and $\mathcal{H}$ is the symmetry group of a given solution. The hypersurface $W = \mathcal{G}/\mathcal{H}$ is therefore determined by the quasi-classically conserved generator of the subgroup that is broken by the chosen solution.

We must note that both approaches (the direct one via a straightforward account of constraints in quantization and the indirect one via the mapping of the problem into the space W) must be equivalent. The main aspects of our scheme and several examples of its application are given in [7], [50]–[53]. The scheme incorporates the indirect method of taking the constraints into account through the mapping of the quantum problem into the space W.

Many works are devoted to the quantization of constrained systems. We note that this problem is essential whenever the kinetic and potential parts of the Lagrangian are equally significant from the dynamic standpoint. Precisely this kinematics is realized in the production of "soft" particles. In the earliest works, the Wentzel–Kramers–Brillouin (WKB) quasi-classical expansion [54] was considered, which is a direct generalization of the well-known stationary phase method. This formalism was subsequently developed in [55], where convenient boundary conditions that essentially simplified the calculations were proposed. These works were important because they allowed fully realizing the difficulties entailed by the "naive" approach to the problem of quantizing theories with a high-symmetry group. But we see in what follows that the WKB quantization scheme is the only possible scheme because only it preserves the full probability [50], [51], [56].

In [57], the problem of separating the nondynamic degrees of freedom, the *zero modes*, was placed in the foreground. Using the notion of *collective variables* was proposed for this based on the earlier works (see, e.g., [58]).

For integrable (1+1)-dimensional field theories, the inverse scattering problem is a canonical transformation to variables of the "action–angle" type (see details in [59]). It is then natural to quantize extended soliton-like objects precisely in terms of collective variables if these are in involution [60], [61]. Precisely the collective variables were used as local coordinates on the space W in [52].

We must find and describe the full set of quantum states in the space W.[6] This problem is very complicated if we do not know the inverse scattering problem, as is the case with the (3+1)-dimensional field theory in the Minkowski metric considered in Secs. 4 and 5. We see in what follows that the key role in this process is played by precisely the relation $W = \mathcal{G}/\mathcal{H}$.

In the conventional formulations of quantum theory, the problem of mapping into the space W is practically unsolvable [63]. Attempts to use the lattice expansion of the path integral for this purpose involve an essential uncertainty, which becomes noticeable in the transition to the continuum limit [64].

Taking the above experience into account, we proceed as follows. In the example of particle motion in a potential well, the spectral representation for the corresponding amplitude is given by

$$A(E, x_1, x_2) = \sum_l \frac{\psi(x_1)\psi^*(x_2)}{E - E_l + i\varepsilon}.$$

[5] If the constraints are insufficient to select the hypersurface W, then they play no role in the dynamics (see the selection rule in Proposition 8 in what follows).

[6] For the quantum theory of solitons, it is important that the soliton S-matrices are factorable [62].

If we are not interested in coordinates, it is useful to consider the quantity [55]

$$a(E) = \int dx\, A(E,x,x) = \sum_l \frac{1}{E-E_l+i\varepsilon} = \sum_l \mathcal{P}\frac{1}{E-E_l} + i\pi\sum_l \delta(E-E_l),$$

where we use the orthonormalizability of the wave functions $\psi(x)$. We note that in reality, we are not interested in unobservable values $E \neq E_l$. It then suffices to evaluate only the absorption part

$$\operatorname{Im} a(E) = \pi \sum_l \delta(E-E_l).$$

This means that we drop the continuum of states (not realized in nature) with $E \neq E_l$. But we do not know how to formulate the theory in terms of the absorption parts of the amplitude. To circumvent this difficulty, we consider the integral probability

$$r(E) = \int dx_1\, dx_2\, |A(E,x_1,x_2)|^2 = \sum_l \left|\frac{1}{E-E_l+i\varepsilon}\right|^2$$

and use the optical theorem (the unitarity condition)

$$\varepsilon r(E) = \varepsilon \sum_l \left|\frac{1}{E-E_l+i\varepsilon}\right|^2 = \frac{1}{2i}\sum_l \left\{\frac{1}{E-E_l-i\varepsilon} - \frac{1}{E-E_l+i\varepsilon}\right\} =$$

$$= \pi\sum_l \delta(E-E_l) = \operatorname{Im} a(E),$$

which shows that the observables are determined by the absorption parts of the amplitudes. This is a general assertion and must always be true.

The unitarity condition ensuring the conservation of the full probability is formally realized as the result of canceling the real part. We want to use this cancellation to refine the definition of the functional measure in the integral for the amplitude [56]. In Secs. 3 and 4, we generalize this cancellation mechanism to the field theory problem. Namely, we show that the functional measure for $\rho(\alpha, z)$ contains a functional δ function that determines the full set of contributions. This in turn opens the possibility to map the *quantum* theory into any manifold and, in particular, into the quotient space.

We once again stress that we solve the constrained problem of calculating the probabilities that are given by absolute values of the corresponding amplitudes by definition. Or, using the unitarity condition, we restrict ourselves to calculating the absorption parts of the amplitudes. But we must note that if quantum perturbations are turned on adiabatically [10], then we can also evaluate the full amplitudes by applying the dispersion relations.

We stress that the generating function $\rho(\alpha, z)$, as noted above, is defined by integrals of precisely $|a_n|^2$. We then use the optical theorem to express $\rho(\alpha, z)$ through the absorption part, thus closing our formalism, because the absorption part is defined on a δ-like functional measure. Thus, we first determine the structure of the perturbation theory in terms of field variables; this structure coincides with the standard WKB scheme. We then use the fact that the functional measure is δ-like to map the perturbation theory into the space $W = \mathcal{G}/\mathcal{H}$.

In Secs. 3 and 4, we describe the perturbation theory in the quotient space $W = \mathcal{G}/\mathcal{H}$ of a simpler conformal (3+1)-dimensional scalar theory in real time. We do not know the general structure of W, and the realization in the space $W = O(4,2)/O(4) \times O(2)$ [65] considered in this review is therefore only an example. Otherwise, Eq. (5.38) for the generating functions of the multiple-production cross sections given in what follows is exact.

The Yang–Mills theory is considered in Sec. 5. The most important result is a perturbation theory that does not require gauge fixing (Sec. 5.4). This is obviously achieved by describing quantum perturbations in the space W instead of the space of fields (more precisely, of the Yang–Mills gauge field potentials).

1.4. The main points and results. We now briefly describe the contents of this review. In Sec. 2, we describe the relation to the thermodynamics in real time. The main result in Sec. 2.1 is the factored representation for $\rho(\alpha, z)$ (see Proposition 1), which allows not discriminating between mechanical and thermodynamic perturbations. This result is important because, strictly speaking, quantum perturbations can affect the thermodynamics and vice versa. This is why there has always been the problem of the time ordering of these perturbations [66]. For example, it was proposed in [67] to consider "thermal" perturbations separately from the "mechanical" ones. The result obtained in Sec. 2.1 therefore has an independent importance. In Sec. 2.2, we use the possibility to write $\rho(\alpha, z)$ in the factored form and show that $\rho(\alpha, z)$ is defined on a δ-like functional Dirac measure, which proves the unitarity of the WKB scheme.

In Secs. 3 and 4, we construct the perturbation theory in the quotient space. We consider the simplest example of a mapping into the space W in Sec. 3.1 and give the general theory of mappings in Sec. 3.2. It is important to demonstrate the possibility to reduce to W and the fibering mechanism $W = T^*W \times R$, where the q-numbers belong to T^*W and the c-number zero modes are involved in R.

All this is first demonstrated with an example of a simpler $O(4, 2)$-invariant scalar field theory. Here, it is important to be able to find the measure $\rho(\alpha, z)$ that would take the energy–momentum conservation laws into account. This is a nontrivial problem because we describe the inelastic scattering of particles through an extended soliton-like object (see Sec. 4.3).

Finally, in Sec. 5, we give an explicit expression for the generating functional $\rho(\alpha, z)$ in the Yang–Mills theory. The result is a strong-coupling perturbation theory (the expansion in the inverse powers of the coupling constant) for $\rho(\alpha, z)$. We show that the new perturbation theory is free of divergences, at least in the vector-field sector, and is formulated such that it does not require gauge fixing (see Sec. 5.4). This spares us the introduction of the Faddeev–Popov *ghosts* and the struggle with Gribov ambiguities.

2. Field theory in real time and at finite temperatures

In the S-matrix interpretation of thermodynamics, the role of a particle is played by a point from which a particle with a given momentum is emitted (or into which it is absorbed). It must then be kept in mind that the four-coordinate of this point, generally speaking, has no meaning because the uncertainty relation must be taken into account. This picture is dual because, on one hand, the emitted particles are free, being on the mass shell, but on the other hand, the momentum distribution of these particles does not coincide with the black-body radiation, because the particles are emitted by interacting fields. These peculiarities of the interpretation developed here should be remembered in reading [3], [7].

2.1. The S-matrix theory at finite temperatures. We show here that under certain conditions, an isomorphism can be established between the thermodynamic description of a system with a large number of particles and the S-matrix formalism accepted in the description of the multiple-production process. For simplicity, we begin by considering the simplest massive real scalar field theory. The specific form of the Lagrangian is then irrelevant.

We must first introduce the notion of the generating functional for cross sections $\rho(\alpha, z) \equiv \rho(\alpha_{\rm i}, \alpha_{\rm f}; z_{\rm i}, z_{\rm f})$. In the simplest version of the theory considered here, we can trace only the momenta q_j and p_j of the particles with a given mass m with $p_j^2 = q_j^2 = m^2$. We assume that

$$|a_{mn}(p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_n)|^2 \delta\Big(P - \sum_{j=1}^{m} p_j\Big)\delta\Big(P - \sum_{j=1}^{n} q_j\Big) =$$

$$= \int \frac{d^4\alpha_{\rm i}}{(2\pi)^4}\frac{d^4\alpha_{\rm f}}{(2\pi)^4} e^{iP(\alpha_{\rm i}+\alpha_{\rm f})} \prod_{j=1}^{m} \frac{\epsilon(p_j)}{(2\pi)^3}\frac{\delta}{\delta z_{\rm i}(p_j)} \prod_{j=1}^{n} \frac{\epsilon(q_j)}{(2\pi)^3}\frac{\delta}{\delta z_{\rm f}(q_j)} \rho(\alpha, z)\Big|_{z_{\rm i}=z_{\rm f}=0}, \quad (2.1)$$

where $a_{mn}(p_1, p_2, \ldots, p_m; q_1, q_2, \ldots, q_n) = a_{mn}(p; q)$ is the transition amplitude from an m-particle state to an n-particle one and

$$P = \sum_j p_j = \sum_j q_j$$

is the total four-momentum of the colliding particles. Inverting Eq. (2.1), we find $\rho(\alpha, z)$.

We now show that the following proposition holds (see also [68]).

Proposition 1. *If the reduction formula holds and if the surface term vanishes,*

$$\int dx\, \partial_\mu \{u\, \partial^\mu u\} = \int_{\sigma_\infty} dx_\mu\, \{u\, \partial^\mu u\} = 0, \tag{2.2}$$

where σ_∞ is the infinitely remote hypersurface, then the generating functional $\rho(\alpha, z)$ for the multiple-production cross sections can be represented in the factored form

$$\rho(\alpha, z) = e^{-\mathbf{N}(\varphi;\alpha,z)} \rho_0(\varphi), \tag{2.3}$$

where the functional $\rho_0(\varphi)$ is defined in Eq. (2.14) and $\mathbf{N}(\varphi; \alpha, z)$ is defined in Eq. (2.15).

This representation of the generating functional plays a key role in what follows because all the information about the external conditions (the dependence on the parameters α and z) is contained in the operator $\mathbf{N}(\varphi; \alpha, z)$ and all the information about the interacting fields is included in the functional $\rho_0(\varphi)$. It can be assumed that the operator $\mathbf{N}(\varphi; \beta, z)$ projects the system of interacting fields onto observable states. Moreover, because the external effect is turned on adiabatically, it is assumed that the functional $\rho_0(\varphi)$ and all its derivatives exist.

Proof of Proposition 1. We introduce the standard definition of amplitudes (see [3] and the references therein) through the reduction formula [69]

$$a_{mn}(p; q) = \prod_{k=1}^{m} \widehat{\varphi}(p_k) \prod_{k=1}^{n} \widehat{\varphi}^*(q_k) Z(\varphi), \tag{2.4}$$

where the hat denotes the variational (or the usual) derivative at zero. For example,

$$\widehat{\varphi}(q) \equiv \int dx\, e^{-iqx} \frac{\delta}{\delta\varphi(x)} \equiv \int dx\, e^{-iqx} \widehat{\varphi}(x), \tag{2.5}$$

and the auxiliary field φ must be set equal to zero at the end of the calculation.

The vacuum-to-vacuum transition amplitude in the external (auxiliary) field $\varphi(x)$ is given by

$$Z(\varphi) = \int Du\, e^{iS_0(u)} e^{-iV(u+\varphi)}, \tag{2.6}$$

where S_0 is the free part of the action,

$$S_0(u) = \frac{1}{2} \int_{C_+} dx\, \left((\partial_\mu u)^2 - m^2 u^2\right), \tag{2.7}$$

and V describes interactions,

$$V(u) = \int_{C_+} dx\, v(u). \tag{2.8}$$

The time integrals in (2.7) and (2.8) are defined on the Mills time contour [70] chosen as

$$C_{\pm} : t \to t \pm i\epsilon, \quad \epsilon \to +0, \quad -\infty \le t \le +\infty, \tag{2.9}$$

which is equivalent to the Feynman $i\epsilon$ prescription.

We consider the quantity

$$r(P;z) = \sum_{n,m} \frac{1}{m!\,n!} \int d\omega_m(p;z_{\rm i})\, d\omega_n(q;z_{\rm f})\, \delta\Big(P - \sum_{k=1}^{m} p_k\Big)\delta\Big(P - \sum_{k=1}^{n} q_k\Big)|a_{mn}|^2, \tag{2.10}$$

where the phase volume element

$$d\omega_m(q;z) = \prod_{k=1}^{m} \frac{dq_k\, z(q_k)}{(2\pi)^3\, 2\varepsilon(q_k)}, \qquad \varepsilon(q) = (q^2 + m^2)^{1/2},$$

involves a "good" weight function $z(q)$. Substituting (2.4) in (2.10) and using the Fourier decomposition of the δ functions, we obtain

$$r(P;z) = \int \frac{d\alpha_{\rm i}}{(2\pi)^4} \frac{d\alpha_{\rm f}}{(2\pi)^4} e^{iP(\alpha_{\rm i}+\alpha_{\rm f})} \rho(\alpha, z), \tag{2.11}$$

where

$$\rho(\alpha, z) = e^{-\mathbf{N}_+(\widehat{\varphi};\alpha_{\rm i},z_{\rm i}) - \mathbf{N}_-(\widehat{\varphi};\alpha_{\rm f},z_{\rm f})} \rho_0(\varphi), \tag{2.12}$$

$$\mathbf{N}_{\pm}(\widehat{\varphi};\alpha,z) \equiv \int d\omega_1(q;z)\, e^{-iq\alpha} \widehat{\varphi}_{\pm}(q) \widehat{\varphi}^*_{\mp}(q), \tag{2.13}$$

$$\rho_0(\varphi) = Z(\varphi_+) Z^*(-\varphi_-). \tag{2.14}$$

If we introduce the notation

$$\mathbf{N}(\widehat{\varphi};\alpha,z) = \mathbf{N}_+(\widehat{\varphi};\alpha_{\rm i},z_{\rm i}) + \mathbf{N}_-(\widehat{\varphi};\alpha_{\rm f},z_{\rm f}), \tag{2.15}$$

then Eqs. (2.13) and (2.15) define the operator $\mathbf{N}(\widehat{\varphi};\alpha,z)$ and Eq. (2.14) determines $\rho_0(\varphi)$.

We emphasize that the main quantity under consideration, the squared modulus $|a_{mn}|^2$, actually contains the doubled number of the degrees of freedom, i.e., is a more complicated quantity than just the amplitude a_{mn}. At first glance, it may therefore seem natural to draw an analogy with the thermodynamics using only amplitudes. Such has been attempted, but it led to unphysical *pinch singularities* that were canceled only after doubling the number of the degrees of freedom. Moreover, the experience with the thermofield description shows that this is a necessary complication. A sufficiently detailed discussion of this problem can be found in [5], [71].

Proposition 2. *Representation* (2.3) *admits, instead of* (2.2), *the periodic boundary condition*

$$\int_{C_+} dx\, \partial_\mu \{u_+ \partial^\mu u_+\} - \int_{C_-} dx\, \partial_\mu \{u_- \partial^\mu u_-\} = 0, \tag{2.16}$$

where u_+ *and* u_- *are completely independent fields on the respective contours* C_+ *and* C_-.

Proof. We recall that Eq. (2.4) is a reduction formula based on an important assumption regarding the "sufficiently good" behavior of fields at infinity. Proposition 2 implies that in considering the quantities

$|a_{mn}|^2$, we can use condition (2.16), which is weaker than (2.2) and which also ensures the absence of the surface term but does not assume the fields (and their first derivatives) to decrease sufficiently rapidly at infinity.

In our case, it is necessary and sufficient to assume that the fields u_+ and u_- and their first derivatives coincide on σ_∞,

$$u_+(x \in \sigma_\infty) = u_-(x \in \sigma_\infty). \tag{2.17}$$

We call this condition the *periodic boundary condition*. It is the most general one and is kept to the end of the calculation.

Equation (2.12) can also be written as

$$\rho(\alpha, z) = \exp\left\{ i \int dx\, dx' \left(\hat{\phi}_+(x) D_{+-}(x - x'; z_f, \alpha_f)\hat{\phi}_-(x') - \right.\right.$$
$$\left.\left. - \hat{\phi}_-(x) D_{-+}(x - x'; z_i, \alpha_i)\hat{\phi}_+(x')\right) \right\} \rho_0(\phi), \tag{2.18}$$

where D_{+-} and D_{-+} with $z = 1$ are the standard positive- and negative-frequency Green's functions [10]. The function

$$D_{+-}(x - x'; z, \alpha) = -i \int d\omega_1(q)\, z e^{iq(x - x' - \alpha)}$$

describes the propagation process of a particle produced at the moment x_0 and absorbed at the moment x_0', $x_0 > x_0'$, where α coincides with the four-coordinate of the system. These functions satisfy the homogeneous equations

$$(\partial^2 + m^2)_x D_{+-} = (\partial^2 + m^2)_x D_{-+} = 0.$$

We now assume that the generating functional $Z(\phi)$ can be calculated as a power series in the coupling constant. It is then convenient to use the transformation (we recall that $\widehat{X}$ denotes the derivative with respect to X at zero)

$$e^{-iV(\phi)} = \exp\left\{ -i \int dx\, \hat{j}(x)\hat{\phi}'(x) \right\} \exp\left\{ i \int dx\, j(x)\phi(x) \right\} e^{-iV(\phi')} =$$
$$= \exp\left\{ \int dx\, \phi(x)\hat{\phi}'(x) \right\} e^{-iV(\phi')} = e^{-iV(-i\hat{j})} \exp\left\{ i \int dx\, j(x)\, \phi(x) \right\}. \tag{2.19}$$

Choosing the first equation in (2.19), we then obtain

$$Z(\phi) = \exp\left\{ -i \int dx\, \hat{j}(x)\widehat{\Phi}(x) \right\} e^{-iV(\Phi + \phi)} \exp\left\{ -\frac{i}{2} \int dx\, dx'\, j(x) D_{++}(x - x') j(x') \right\}, \tag{2.20}$$

where D_{++} is the standard causal Green's function,

$$(\partial^2 + m^2)_x D_{++}(x - y) = \delta(x - y).$$

Substituting (2.20) in (2.18) and doing simple manipulations with differential operators, we obtain the expression

$$\rho(\alpha, z) = e^{-iV(-i\hat{j}_+) + iV(-i\hat{j}_-)} \exp\left\{ \frac{i}{2} \int dx\, dx' \left(j_+(x) D_{+-}(x - x'; \alpha_1) j_-(x') - \right.\right.$$
$$- j_-(x) D_{-+}(x - x'; \alpha_2) j_+(x') - j_+(x) D_{++}(x - x') j_+(x') +$$
$$\left.\left. + j_-(x) D_{--}(x - x') j_-(x') \right) \right\}, \tag{2.21}$$

where $D_{--} = (D_{++})^*$ is the anticausal Green's function.

For a system of a large number of particles, the problem can be simplified by choosing the center-of-mass system $P = (P_0 = E, \vec{0})$. It can then be assumed that $\alpha_{0,k} = -i\beta_k$, $\mathrm{Im}\,\beta_k = 0$, $k = \mathrm{i}, \mathrm{f}$ [72], [73]. In this case, we have $\rho = \rho(\beta, z)$.

In what follows, we intend to use the quantity β along with the energy as one more parameter characterizing the system of the produced particles. But it must be remembered that this requires special conditions under which β can be measured simultaneously with the energy. These conditions can be formulated as follows [7].

Proposition 3. *The energy spectrum of secondary particles is described by the Boltzmann exponent if and only if the momenta K_l that are central with respect to energy are sufficiently small,*

$$|K_l(n)|^{2/l} \ll K_2(n), \quad l = 3, 4, \ldots. \tag{2.22}$$

Proof. We note that these conditions resemble the Bogoliubov correlation relaxation principle in the course of approaching the equilibrium state [29].[7] We recall that $K_1(n) = \langle\varepsilon; n\rangle$ is the mean energy of the produced particles, $K_2(n) = \langle\varepsilon^2; n\rangle - \langle\varepsilon; n\rangle^2$ is the dispersion of the energy distribution of secondary particles, and $K_3(n) = \langle\varepsilon^3; n\rangle - 2\langle\varepsilon^2; n\rangle\langle\varepsilon; n\rangle + 3\langle\varepsilon; n\rangle^3$ is the third moment that is central with respect to energy, and so on.

We now show how to obtain conditions (2.22) and to prove the validity of the definition

$$\langle\varepsilon^l; n\rangle = \frac{1}{\sigma_n} \int d\omega_l(q) \prod_{i=1}^{l} \varepsilon(q_i) \frac{d^l \sigma_n}{dq_1 \cdots dq_l}, \qquad \varepsilon(q) = \sqrt{q^2 + m_{\mathrm{h}}^2}, \tag{2.23}$$

where σ_n is the cross section of the production of n particles and $d^l\sigma_n/(dq_1 \cdots dq_l)$ is the differential cross section (see Eq. (2.29)).

To derive conditions (2.22), we consider the integrals

$$a_{mn}(E; z) = \oint \frac{dz_{\mathrm{i}}}{2\pi i z_{\mathrm{i}}^{m+1}} \frac{dz_{\mathrm{f}}}{2\pi i z_{\mathrm{f}}^{n+1}} \int \frac{d\beta_{\mathrm{i}}}{2\pi i} \frac{d\beta_{\mathrm{f}}}{2\pi i} e^{(\beta_{\mathrm{i}}+\beta_{\mathrm{f}})E} e^{-F(z,\beta)}, \tag{2.24}$$

where z varies over a closed contour that encompasses the point $z = 0$ and

$$F(z, \beta) = -\log \rho(\beta, z). \tag{2.25}$$

We now use the stationary phase method to evaluate the integrals in (2.24). For this, as in the microcanonical formalism, we must find a solution of the equation of state

$$E = \frac{\partial}{\partial \beta_k} F(\beta, z), \quad k = \mathrm{i}, \mathrm{f}, \tag{2.26}$$

which determines the most probable values of β_k for a given E (and z). Equations (2.26) always have positive real solutions [73]. Because of the energy conservation law, the solutions coincide,

$$\beta_k = \tilde{\beta}(E, z), \quad \tilde{\beta} > 0.$$

[7] The term "vanishing correlation principle" is used in the English literature [74].

To find the most probable values of z, we must solve the equations

$$m = -z_{\mathrm{i}}\frac{\partial}{\partial z_{\mathrm{i}}}F(\beta, z), \qquad n = -z_{\mathrm{f}}\frac{\partial}{\partial z_{\mathrm{f}}}F(\beta, z). \tag{2.27}$$

Equations (2.26) and (2.27) must be solved simultaneously. In particle physics, the number of the incident particles is $m = 2$, and it therefore suffices to know only the solution $z_{\mathrm{c}} = \tilde{z}(\beta_{\mathrm{c}}, n) = z_{\mathrm{c}}(E, n)$ of the equation for z_{f}.

The expansion of the integrand in (2.24) in a vicinity of $\tilde{\beta}(E, z) = \beta_{\mathrm{c}}(E, n)$ gives an asymptotic series because function (2.25) is essentially nonlinear. This implies that fluctuations in the vicinity of $\beta_{\mathrm{c}}(E, n)$, in general, are arbitrarily large. It must then be assumed that the expansion in the vicinity of $\beta_{\mathrm{c}}(E, n)$ exists, for example, in the Borel sense.[8] This allows finding an asymptotic estimate for the series. The conditions for the validity of this estimate are given by the inequalities

$$\left|\frac{\partial^l}{\partial \beta^l}F(\beta_{\mathrm{c}}, z_{\mathrm{c}})\right|^{2/l} \ll \frac{\partial^2}{\partial \beta^2}F(\beta_{\mathrm{c}}, z_{\mathrm{c}}), \quad l > 2. \tag{2.28}$$

This makes it easy to derive conditions (2.22) and the corresponding definition in Eq. (2.23).

We now find the explicit form of the derivative $\partial^l F(z, \beta)/\partial \beta^l$. We start with the case where $l = 1$,

$$\frac{\partial}{\partial \beta}F(\beta, z) = \frac{1}{\rho(\beta, z)}\frac{\partial}{\partial \beta}\rho(\beta, z) =$$
$$= \frac{1}{\rho(\beta, z)}\sum_n n \int d\omega_n(q; z) \exp\Big\{-\beta\sum_j \varepsilon(q_j)\Big\}\varepsilon(q_1)|a_n|^2.$$

The coefficient n appears here because the particles are identical. We next introduce the differential cross section that is not normalized to the flow of colliding particles,

$$\frac{d^l\tilde{\sigma}(\beta, z)}{dq_1\, dq_2 \cdots dq_l} = \sum_{n\geq l} n(n-1)\cdots(n-l+1)\int d\omega_{n-l}(q; z) \exp\Big\{-\beta\sum_j \varepsilon(q_j)\Big\}|a_n|^2,$$

where we again take into account that the particles are identical. If the total energy E and the number of particles n are given, we must set $z = \mathrm{const}$ and consider the quantity

$$\frac{d^l\sigma_n(E)}{dq_1\, dq_2 \cdots dq_l} = \frac{1}{2\pi i}\oint \frac{dz}{z^{n+1}}\int \frac{d\beta}{2\pi i}e^{\beta E}\frac{d^l\tilde{\sigma}(\beta, z)}{dq_1\, dq_2 \cdots dq_l}. \tag{2.29}$$

In general, the integrals with respect to z and β fix the exact conservation laws for the number of particles and the energy. But we now assume that these conservation laws can be taken into account approximately. In this case, the sum of energies of the produced particles is equal to E only with exponential accuracy. We then assume the same for the number of produced particles. To find the neighborhood of energy values $1/\beta_{\mathrm{c}}$ and of the number $1/\log z_{\mathrm{c}}$ of produced particles where the energy values and the number of produced particles are concentrated, we must solve Eqs. (2.26) and (2.27). Then

$$\frac{d^l\sigma_n(E)}{dq_1 dq_2 \cdots dq_l} = z_{\mathrm{c}}^{-(n+1)}e^{\beta_{\mathrm{c}}E}\frac{d^l\tilde{\sigma}(\beta_{\mathrm{c}}, z_{\mathrm{c}})}{dq_1 dq_2 \cdots dq_l}\Upsilon_l(E, n),$$

[8] This problem is practically not studied. but we can use the analogy between the high-temperature expansion in equilibrium thermodynamics, i.e., the expansion in powers of β, and the expansion with respect to the coupling constant. In this sense, a positive answer to the question of the existence of power series with respect to β seems natural.

where $\Upsilon_l(E,n)$ includes Gaussian corrections coming from the integration in the neighborhoods of $\beta_c(E,n)$ and $z_c(E,n)$. Thus,

$$\frac{\partial}{\partial\beta_c}F(\beta_c,z_c)=\frac{\Upsilon_1(E,n)e^{\beta_c E}}{z_c^{n+1}\rho(\beta_c,z_c)}\int\varepsilon(q_1)\,d\omega_1(q_1;z_c)\frac{d\tilde{\sigma}(\beta_c,z_c)}{dq_1}=$$
$$=\frac{1}{\rho(\beta_c,z_c)}\int\varepsilon(q_1)\,d\omega_1(q_1;1)\frac{d\sigma_n(E)}{dq_1}=\langle\varepsilon^1;n\rangle=K_1(n).$$

Proceeding similarly, we can obtain

$$\frac{\partial^2}{\partial\beta_c^2}F(\beta_c,z_c)=\frac{1}{\rho(\beta_c,z_c)}\int\varepsilon(q_1)\,d\omega_1(q_1;1)\,\varepsilon(q_2)\,d\omega_1(q_2;1)\,\frac{d^2\sigma_n(E)}{dq_1\,dq_2}-$$
$$-\left\{\frac{1}{\rho(\beta_c,z_c)}\int\varepsilon(q_1)\,d\omega_1(q_1;1)\frac{d\sigma_n(E)}{dq_1}\right\}^2=K_2(n). \tag{2.30}$$

In the general case, we then have

$$\frac{\partial^l}{\partial\beta_c^l}F(\beta_c,z_c)=K_l(n). \tag{2.31}$$

Substituting this in (2.28), we obtain (2.22).

The same argument can be applied to z. If the conditions for the validity of asymptotic estimates are then satisfied, we could interpret β_c as the temperature, $\mu_c=(\log z_c)/\beta_c$ as the chemical potential, and finally $F(z_c,\beta_c)/\beta_c$ as the free energy. If this interpretation is valid, then the point from which a particle with the momentum q is emitted can be interpreted as a "particle" with the momentum q. In general, however, this interpretation involves an inaccuracy related to the impossibility of simultaneously introducing the coordinate of this point [3].

The proposed interpretation of the β_c and z_c parameters seems natural if we note that function (2.21) has the same *structure* as the Niemi–Semenoff generating function [75] derived in the framework of the Schwinger–Keldysh theory [76], [77]. The only difference is in the definition of the Green's functions D_{ij}, $i,j=+,-$, which can be essential.

We now clarify which conditions allow the above interpretation of $\rho(\beta_c,z_c)$ as the partition function with the respective parameters $1/\beta_c$ and z_c interpreted as the temperature and activity.

Proposition 4. *If periodic boundary condition* (2.17) *is valid and, moreover, if fluctuations in the vicinity of the solution of Eq.* (2.26) *are Gaussian and correlations vanish on the hypersurface* σ_∞, *then the thermodynamics admits an* S*-matrix interpretation with* $\rho(\beta,z)$ *playing the role of the grand partition function.*

Proof. We assume [68] that our system of colliding particles is a subsystem of a larger system that also includes noninteracting (free) particles (which model the heat bath). As a result, the Boltzmann exponential $e^{-\beta\varepsilon}$ is replaced with the occupation number $\bar{n}(\beta\varepsilon)$ corresponding to the statistics. This changes only the form of the Green's functions D_{ij}. The strategy that we use in the proof is as follows. The Green's functions occurring in the formalism must satisfy the equations

$$\begin{aligned}(\partial^2+m^2)_x G_{+-}(x-y)&=(\partial^2+m^2)_x G_{-+}(x-y)=0,\\(\partial^2+m^2)_x G_{++}(x-y)&=(\partial^2+m^2)_x^* G_{--}(x-y)=\delta(x-y).\end{aligned} \tag{2.32}$$

Next, it is important to take into account that boundary condition (2.17) admits a more general solution of these equations,

$$G_{ii} = D_{ii} + g_{ii}, \qquad G_{ij} = g_{ij}, \quad i \neq j, \tag{2.33}$$

where g_{ij} is the solution of the homogeneous equations

$$(\partial^2 + m^2)_x g_{ij}(x-y) = 0, \quad i, j = +, -, \tag{2.34}$$

that must be distinguished in accordance with the property of belonging to different time contours $C_\pm$. Therefore,

$$g_{ij}(x-x') = \int d\omega(q)\, e^{iq(x-x')} n_{ij}(q), \tag{2.35}$$

where we recall that $q^2 = m^2$. The unknown functions $n_{ij}(q)$ are determined as the means of the fields,

$$u(\sigma_\infty):\ n_{ij} \sim \big\langle u(\sigma_\infty) \cdots u(\sigma_\infty) \big\rangle.$$

The simplest of these is given by

$$n_{ij} \sim \langle u_i u_j \rangle \sim \big\langle u^2(\sigma_\infty) \big\rangle, \tag{2.36}$$

where we take periodic boundary condition (2.17) into account. But the functions $n_{ij}(q)$ must then be identical to the occupation number of the black-body radiation.

The formal derivation of the final formulas repeats the argument in [68] (see also [3]). We find that

$$n_{++}(q_0) = n_{--}(q_0) = \left(\exp\left(|q_0|\frac{\beta_1+\beta_2}{2}\right) - 1\right)^{-1} \equiv \tilde{n}\left(|q_0|\frac{\beta_1+\beta_2}{2}\right), \tag{2.37}$$

$$n_{+-}(q_0) = \Theta(q_0)\big(1 + \tilde{n}(q_0\beta_1)\big) + \Theta(-q_0)\tilde{n}(-q_0\beta_1), \tag{2.38}$$

$$n_{-+}(q_0) = \Theta(q_0)\tilde{n}(q_0\beta_2) + \Theta(-q_0)\big(1 + \tilde{n}(-q_0\beta_2)\big). \tag{2.39}$$

This leads to the Green's functions

$$i\tilde{G}_{ij}(q,\beta) = \begin{pmatrix} \dfrac{i}{q^2 - m^2 + i\epsilon} & 0 \\ 0 & -\dfrac{i}{q^2 - m^2 - i\epsilon} \end{pmatrix} + \\ + 2\pi\delta(q^2 - m^2) \begin{pmatrix} \tilde{n}(\frac{\beta_1+\beta_2}{2}|q_0|) & \tilde{n}(\beta_2|q_0|)a_+(\beta_2) \\ \tilde{n}(\beta_1|q_0|)a_-(\beta_1) & \tilde{n}(\frac{\beta_1+\beta_2}{2}|q_0|) \end{pmatrix}, \tag{2.40}$$

where

$$a_\pm(\beta) = -e^{\beta(|q_0| \pm q_0)/2}.$$

According to Proposition 2, the generating functional can be written in the standard form

$$\rho_{\mathrm{cp}}(\beta) = e^{-iV(-i\hat{j}_+) + iV(-i\hat{j}_-)} \exp\left\{\frac{i}{2}\int dx\, dx'\, j_i(x) G_{ij}\big(x - x', (\beta)\big) j_j(x')\right\}, \tag{2.41}$$

where summation over repeated indices is assumed. If conditions (2.22) are satisfied, it follows that

$$D_{+-}(t-t') = D_{-+}(t-t'-i\beta),$$
$$D_{-+}(t-t') = D_{+-}(t-t'+i\beta). \tag{2.42}$$

These relations are the Kubo–Martin–Schwinger periodic boundary conditions. They are typically used to determine the temperature dependence in the canonical formulation of thermodynamics.

As a result, we have obtained the grand partition function defined on the Mills time contour [70]

$$C_\infty : C_+ + C_-; \qquad C_\pm : t \to t \pm i\varepsilon, \quad \varepsilon \to +i0, \quad |t| \le \infty. \tag{2.43}$$

For the theory defined on the Niemi–Semenoff time contour to be obtained from the above, we must have the right to add contributions defined on the imaginary-time contour

$$C_{\text{Im}} : t \in \lim_{t_f \to +\infty} (t_f + i\varepsilon, t_f - i\varepsilon).$$

But this is possible only in the case where the Green's functions vanish on C_{Im}. It can be shown that this condition is satisfied at least within the canonical perturbation theory [71].

Whenever $C = C_+ + C_- + C_{\text{Im}}$ and the Kubo–Martin–Schwinger periodic boundary condition holds [3], the functional $\rho(\beta, z)$ found in (2.21) has the standard integral representation

$$\rho(\beta, z) = \int D_C\phi \, e^{iS_C(\phi)},$$

where all the variables are defined on the contour C. This representation can be recast by analytic continuation into the Matsubara representation for the grand partition function in the imaginary time [75].

We note here that in our S-matrix formulation of field theory at finite temperatures, the contributions from C_{Im} are absent from the very beginning. In other words, our approach and the approach based on the canonical Gibbs–Boltzmann formalism differ by the contributions from C_{Im}. As noted above, the existence of contributions from C_{Im} is determined by correlation properties on the infinitely remote hypersurface.

By definition, the generating functional $\rho(\beta, z)$ can be used as the event generator for the description of accelerator experiments [3], [78]. For example, if

$$\rho_{nm}(q_1, \ldots, q_n; p_1, \ldots, p_m) = \prod_{j=1}^{m} \frac{\delta}{\delta z_i(p_j)} \prod_{j=1}^{n} \frac{\delta}{\delta z_f(q_j)} \rho(\beta, z) \Big|_{z=0}, \tag{2.44}$$

then

$$\frac{1}{J(s)} \sum_n \int_s \rho_{2n}(q_1, \ldots, q_n; p_1, p_2) = \sigma_{\text{tot}}(s),$$

where σ_{tot} is the total cross section and J is the standard normalization factor. In this expression, the integration over particle momenta is taken under the constraint $s = (p_1 + p_2)^2$.

In addition, the grand partition function can be expressed through

$$\sum_{n,m} \int_{(\beta_i, z_i; \beta_f, z_f)} \rho_{nm}(q_1, \ldots, q_n; p_1, \ldots, p_m), \tag{2.45}$$

where ρ_{nm} is defined in (2.44). The summation over the number of particles and the integration over the particle momenta are taken under restrictions: the respective mean particle energies in the initial and final states are $1/\beta_{\rm i}$ and $1/\beta_{\rm f}$ and the activities of the initial and final states are $z_{\rm i}$ and $z_{\rm f}$. We only add that this description coincides with the microcanonical approach, where the temperature is introduced as a Lagrange multiplier.

In [3], an attempt was made to obtain the above interpretation in the case where the system has not yet reached the *hydrodynamic phase*, where knowing the mean particle energy suffices for completely describing the particle energy spectrum. It can be conjectured that only local temperatures can be introduced at the *kinetic stage* preceding the hydrodynamic one. Using the Wigner formalism, we must then replace $\beta_k \to \beta_k(x)$, $k = {\rm i}, {\rm f}$, in expression (2.40), where x is the Wigner coordinate [4]. We emphasize that this interpretation is only possible because we define the temperature through the mean energy of noninteracting particles [3].

2.2. The unitary definition of the measure. We show that the following proposition holds.

Proposition 5. *If factored structure* (2.3) *occurs, then by the unitarity of the S-matrix, the generating function* (*or the generating functional*) $\rho(\beta, z)$ *is given by*

$$\rho(\beta, z) = e^{-i\mathbf{K}(j\varphi)} \int DM(u) e^{-iU(u,\varphi)} e^{-N(\beta,z;u)}, \tag{2.46}$$

where

$$N(\beta, z; u) = n(\beta_{\rm i}, z_{\rm i}; u) + n^*(\beta_{\rm f}, z_{\rm f}; u),$$

$$n(\beta, z; u) = \int d\omega_1(q; z) e^{-\beta\epsilon(q)} \Gamma(q, u)\Gamma^*(q, u), \tag{2.47}$$

$$\Gamma(q, u) = \int dx\, e^{-iqx} (\partial^2 + m^2) u(x), \qquad q^2 = m^2, \tag{2.48}$$

$$U(u, \varphi) = V(u + \varphi) - V(u - \varphi) - 2\,{\rm Re} \int_{C_+} dx\, \varphi(x) v'(u) = O(\varphi^3), \tag{2.49}$$

$$DM(u) = \prod_x du(x)\, \delta\big(\partial_\mu^2 u + m^2 u + v'(u) - j\big), \tag{2.50}$$

$$2\mathbf{K}(j\varphi) = {\rm Re} \int_{C_+} dx\, \hat{j}(x) \hat{\varphi}(x). \tag{2.51}$$

After all the calculations, we must set $j = \varphi = 0$.

Proof. The derivation of representation (2.46) is given, e.g., in [7]. Factoring, we can separately consider

$$\rho_0(\phi) = \int Du_+\, Du_-\, e^{iS_0(u_+) - iV(u_+ + \varphi_+)} e^{-iS_0(u_-) + iV(u_- - \varphi_-)}, \tag{2.52}$$

where u_- and φ_- are defined on the complex-conjugate contour C_-. The fields $\varphi_\pm$ carry all the information about external conditions, and the integrals must include only closed trajectories.

Instead of the two independent fields u_+ and u_-, we introduce [56]

$$u(x)_\pm = u(x) \pm \varphi(x) \tag{2.53}$$

with the relation

$$\int_{\sigma_\infty} dx_\mu\, \varphi(x)\partial^\mu u(x) = 0 \tag{2.54}$$

(where σ_∞ is the infinitely remote timelike hypersurface) ensuring the periodic boundary condition. We choose the solution of Eq. (2.54) as

$$\varphi(x \in \sigma_\infty) = 0, \tag{2.55}$$

which guarantees the validity of condition (2.17). Then the full action $S_0(u_+) - V(u_+) - S_0(u_-) + V(u_-)$ describes the motion along a closed path starting with the *turning points* $u(x \in \sigma_\infty)$. The integration over $u(x \in \sigma_\infty)$ is assumed because precisely the periodic boundary condition is chosen [3]. For simplicity from here until Sec. 4.3, we assume that

$$\lim_{u_\pm \to u} \big(S_0(u_+) - V(u_+) - S_0(u_-) + V(u_-)\big) = 0. \tag{2.56}$$

We consider φ as a virtual field. Introducing the auxiliary field $\phi(x,t)$, $\phi(x, t \in C_\pm) = \phi_\pm(x, t \in C_\pm)$, and assuming that the variational derivatives are defined as

$$\frac{\delta\phi(x, t \in C_i)}{\delta\phi(x', t' \in C_j)} = \delta_{ij}\delta(x - x')\delta(t - t'), \quad i, j = +, -,$$

we can write (see (2.13))

$$\mathrm{N}_\pm(\varphi; \beta, z) = \int d\omega_1(q; z)\, e^{-\beta\varepsilon(q)} \int_{C_+} dx \int_{C_-} dy\, \widehat{\varphi}_\pm(x)\widehat{\varphi}_\mp(y) e^{\pm iq(x-y)}. \tag{2.57}$$

Using this notation, we separate the term that is linear with respect to $\phi + \varphi$ in exponent (2.52),

$$V\big(u + (\phi + \varphi)\big) - V\big(u - (\phi + \varphi)\big) = U(u, \phi + \varphi) + 2\,\mathrm{Re} \int_{C_+} dx\, \big(\phi(x) + \varphi(x)\big)v'(u) \tag{2.58}$$

and

$$S_0(u + \varphi) - S_0(u - \varphi) = S_0(u) - 2i\,\mathrm{Re} \int_{C_+} dx\, \varphi(x)(\partial_\mu^2 + m^2)u(x). \tag{2.59}$$

The expansion with respect to $\phi + \varphi$ can be written as

$$e^{-iU(u, \phi+\varphi)} = \exp\left\{\frac{1}{2i}\,\mathrm{Re}\int_{C_+} dx\, \hat{\jmath}(x)\widehat{\varphi}'(x)\right\} \exp\left\{2i\,\mathrm{Re}\int_{C_+} dx\, dt\, j(x)\big(\phi(x) + \varphi(x)\big)\right\} e^{-iU(u, \varphi')}, \tag{2.60}$$

where, as usual, $\hat{\jmath}(x)$ and $\widehat{\varphi}'(x)$ are the corresponding variational derivatives. The auxiliary variables (j, φ') must be set equal to zero at the end of the calculation. As a result, we have

$$\rho_0(\varphi) = \exp\left\{\frac{1}{2i}\,\mathrm{Re}\int_{C_+} dx\, \hat{\jmath}(x)\widehat{\varphi}(x)\right\} \int \prod_x du(x)\, \delta\big(\partial_\mu^2 u + m^2 u + v'(u) - j\big) \times$$

$$\times\, e^{is_0(u)} e^{-iU(u, \varphi)} \exp\left\{2i\,\mathrm{Re}\int_{C_+} dx\, \big(j(x) - v'(u)\big)\varphi(x)\right\}, \tag{2.61}$$

where the δ function is defined by

$$\prod_x \delta(\partial_\mu^2 u + m^2 u + v'(u) - j) = \int D\varphi \exp\left\{-2i \operatorname{Re} \int_{C_+} dx \, (\partial_\mu^2 u + m^2 u + v'(u) - j(x))\varphi(x)\right\}. \tag{2.62}$$

It must be remembered here that in view of the chosen periodic boundary condition and solution (2.55), the product of δ functions (2.62) does not involve the values $x \in \sigma_\infty$. This implies that the turning points $u(x \in \sigma_\infty)$ are completely arbitrary. This leads to the appearance of integrals over the volume of the quotient space $\mathcal{G}/\mathcal{H}$, i.e., over zero modes.

Equation (2.61) can also be written in the equivalent form

$$\rho_0(\phi) = e^{-i\hat{K}(j\varphi)} \int DM(u) \, e^{is_0(u) - iU(u,\varphi)} \exp\left\{2i \operatorname{Re} \int_{C_+} dx \, \phi(x)(\partial_\mu^2 + m^2)u(x)\right\} \tag{2.63}$$

if we use the fact that the relation

$$\partial_\mu^2 u + m^2 u = -v'(u) + j \tag{2.64}$$

is exact.

To conclude, we note that the contour C_+ in Eq. (2.51) cannot be displaced to the real axis. Next, because the exponent in Eq. (2.63) is linear in $\varphi(t \in C_+)$ and in $\varphi(t \in C_-)$ separately, acting with the operator $e^{-\mathbf{N}_+(\hat{\varphi};\alpha_i,z_i) - \mathbf{N}_-(\hat{\varphi};\alpha_f,z_f)}$ gives (2.46).

To conclude this section, we give the most important consequence of the fact that the functional measure is δ-like.

Proposition 6. *Only the exact solutions u_c of the equation*

$$\partial_\mu^2 u + m^2 u + v'(u) = 0 \tag{2.65}$$

must be taken into account.

Proof. This is obvious, because the path integral defined on δ-like measure (2.50) and perturbations of the trajectory u_c by the source j are taken into acccount within the perturbation theory. We assume here that as follows from the definition of the operator $e^{-i\mathbf{K}(j\varphi)}$ generating the perturbation series, the exact meaning of Eq. (2.64) is preserved for all values of $j(x)$ and, in particular, for $j(x) = 0$.

This important consequence means that contributions from approximate solutions of the equations of motion are eliminated from consideration. In this sense, the resulting formalism is *simple*, being free of uncertainties.

Proposition 7. *The generating functional ρ is given by the sum of all solutions of Eq.* (2.65), *including the trivial solution.*

Proof. This statement is true independently of "distances" between the critical points of the action because the δ function has zero width,

$$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{\pi\sigma^2}} e^{-x^2/\sigma^2}.$$

We note that the expression for ρ does not involve interferential contributions of different solutions of Eq. (2.65), which necessarily occur if the amplitudes are written as the sum of contributions of the critical points of the action. This implies that in our approach, the orthogonality of Hilbert spaces spanned by solutions of Eq. (2.65) is taken into account.

But it should be kept in mind that we must be able to select the "physical" solution if there are several solutions and no external conditions determine which of them must be taken into account, i.e., in the case of the general position. For this, we introduce the following selection rule.

Proposition 8. *Let u_c and u'_c be solutions of Eq. (2.65), and let W and W' be the corresponding quotient spaces. Also let V and V' be the respective volumes of W and W'. Assuming that $V > V'$, the contributions of u'_c can be omitted with the accuracy $\sim V'/V$. If this ratio is equal to zero, we say that the contributions of u'_c are realized on measure zero.*

Proof. This selection rule is obvious if Proposition 7 is taken into account. Because a δ-like measure determines the full set of contributions and all the contributions must be summed over, we must drop all the contributions realized on measure zero from the sum in the case of the general position. The selection by the dimension of W implies that the largest contribution is given by those field configurations that maximally violate the symmetry of the classical action.

It must be remembered that in field theory, tunneling vacuum-to-vacuum transitions can occur that do not correspond to any dynamics. These contributions do not enter the full system defined by Eq. (2.65) in real time (we do not discuss the contributions that can be obtained by *analytic continuation* into the imaginary time domain here). We believe that they must be added to the contributions discussed here because they correspond to a different topology of fields [79], [80].

We must begin the analysis with the trivial solution, whose quotient space is a point, i.e., has the dimension $\dim W_0 = 0$. The next exact solution that is regular in real time is the eight-parameter $O(4)\times O(2)$-invariant solution, with $\dim W = 8$ [65]. The existence of a nontrivial solution implies that trivial field configurations must be neglected in quantizing the Yang–Mills fields. As follows from the selection rule formulated in Proposition 8, the imaginary-time "vacuum contributions" can be dropped if and only if they are realized on the measure zero. In the Yang–Mills theory, there is an exact instanton solution that is regular in imaginary time and is such that $\dim W_{\text{inst}} = 5$ [81]. Therefore, this solution can be neglected. There also exist approximate multi-instanton solutions, but the dimension of their quotient space does not exceed $\dim W_{\text{inst}}$ [82].

3. Quantization on quotient manifolds

3.1. Introduction to the theory of transformations. Having obtained the theory defined on a δ-like measure, we must first find solutions of Eq. (2.64) as power series in $j(x)$. At this stage, if we forget about the selection rule in Proposition 8, our approach has practically no differences from the standard WKB stationary phase method [50], [51]. It is even somewhat more complicated than the latter because, as noted above, it involves the doubled number of the degrees of freedom. In reality, however, the subsequent calculations are problematic in the framework of this WKB scheme.

In our representation, the problem is as follows. We assume that we know the solution $u_c(x)$ of Eq. (2.65). In the first order in $j(x)$, we must then solve the equation

$$(\partial_\mu^2 + v''(u_c))G(x,x';u_c) = \delta(x-x'), \tag{3.1}$$

which constitutes a certain difficulty.

Despite the apparent simplicity, the problem of describing the motion of a particle in the external field ($u_c(\mathrm{x},t)$ in the present case) whose configuration depends on time is in fact as complicated as the original problem because a particle can freely acquire or loose energy in this field (see the discussion of this problem, e.g., in [83]). Formally, the problem is that because the field depends on the four-coordinate, the space in which the particle propagates looses its homogeneity and isotropy (see (3.1)).

In this section, we show how this problem can be bypassed by passing to new dynamic variables in (2.63). Namely, we choose the variables such that the space becomes homogeneous and isotropic.

In what follows, we are interested in the motion in phase space. For this, instead of (2.46), we consider the generating function

$$\rho(\beta, z) = e^{-i\mathbf{K}(j\varphi)} \int DM(u,p)\, e^{-iU(u,\varphi)} e^{N(\beta,z;u)}, \tag{3.2}$$

where

$$DM(u,p) = \prod_x du(x)\, dp(x)\, \delta\Big(\dot{u}(x) - \frac{\delta H_j}{\delta p(x)}\Big)\delta\Big(\dot{p}(x) - \frac{\delta H_j}{\delta u(x)}\Big) \tag{3.3}$$

and the full Hamiltonian

$$H_j(u,p) = \int d^3x \left\{\frac{1}{2}p^2 + \frac{1}{2}(\nabla u)^2 + m^2u^2 + v(u) - ju\right\} \tag{3.4}$$

involves the energy of quantum perturbations $j(\mathbf{x},t)u(\mathbf{x},t)$.

We note that the transition to the phase space affects only the measure $DM(u,p)$, and it can be easily verified that representation (3.2) coincides with (2.46) identically. This allows assuming that we have simply passed to a more convenient *first-order formalism.*

To evaluate integral (3.2), we must first find all solutions of the equation

$$\dot{u}(x) = \frac{\delta H_j}{\delta p(x)}, \qquad \dot{p}(x) = -\frac{\delta H_j}{\delta u(x)}. \tag{3.5}$$

We can next show that the following proposition holds.

Proposition 9. *If the conditions that*

a. *the functional measure is δ-like (the Dirac measure),*
b. *the operator* $\mathbf{K}$ *generating the perturbation theory series is known, and*
c. *the functional describing the interactions $U(u,\varphi)$ is given*

are satisfied, then the formalism admits arbitrary nonlinear canonical transformations.

This statement is based on the fact that a δ-like functional measure determines the complete set of contributions to the path integral. (But see the discussion of selection rules in Proposition 8. In relation to this, we emphasize that Proposition 9 does not hold if the theory cannot be defined on the Dirac measure, as is the case, e.g., in Euclidean field theories.)

Proof of Proposition 9. We consider a problem in quantum mechanics that is a (0+1)-dimensional analogue of the field theory. The corresponding measure is given by

$$DM(u,p) = \prod_t du\, dp\, \delta\Big(\dot{u} - \frac{\partial H_j}{\partial p}\Big)\delta\Big(\dot{p} + \frac{\partial H_j}{\partial u}\Big), \tag{3.6}$$

where the full Hamiltonian

$$H_j = \frac{1}{2}p^2 + v(u) - ju \tag{3.7}$$

turns out to explicitly depend on time through $j(t)$.

We can introduce a new pair of conjugate coordinates (ξ,η) instead of (u,p). For this, we substitute

$$1 = \int D\xi\, D\eta \prod_t \delta\Big(\eta - \frac{1}{2}p^2 - m^2u^2 - v(u)\Big)\delta\Big(\xi - \int^u dx\, \big(2(\eta - v(x))\big)^{-1/2}\Big) \tag{3.8}$$

in (3.2). Here, it is important to take into account that both measures, the one over (u,p) in (3.6) and the one over (ξ,η) in (3.8), are δ-like, i.e., have the same power. This allows confidently changing the integration order and integrating over (u,p) first. But this is true under the following condition. As noted above, because the measure is δ-like, it is necessary to sum over all solutions of the Lagrange equation. In other words, the entire phase space (u,p) must be split into subspaces separated by bifurcation lines [80]. Trajectories belonging to each of these subspaces then have different topologies. In this sense, each trajectory (phase flow) *completely* belongs to its subspace. It is assumed that we know the structure of the phase space and we consider a specific subspace in changing the above integration order.

To evaluate the integrals, we can use the δ functions in (3.6). In this case, the δ functions in (3.8) impose restrictions on the dynamics, i.e., determine constraints, namely, those imposed by the initial conditions that we specify using the coordinates ξ and η. Using the δ functions in (3.8), we perform the mapping $(u,p)\to(\xi,\eta)$. We note that the algebraic equations

$$\eta=\frac{1}{2}p^2+m^2u^2+v(u),\qquad \xi=\int^u dx\,\big(2(\eta-v(x))\big)^{-1/2} \tag{3.9}$$

completely determine the trajectory $u_c(\xi,\eta)$ and $p_c(\xi,\eta)$ in the phase space. The dynamics in the quotient space W are determined by the product of two δ functions $\delta(\dot u_c-\partial H_j/\partial p_c)\delta(\dot p_c+\partial H_j/\partial u_c)$ remaining after the integrations over u and p. It can therefore be asserted that the mapping into W automatically takes the constraints into account because the calculation methods described above are completely equivalent to each other.

Indeed, using the δ functions in (3.8), we obtain

$$DM(\xi,\eta)=\prod_t d\xi\,d\eta\,\delta\Big(\dot\xi-\frac{\partial h_j}{\partial\eta}\Big)\delta\Big(\dot\eta+\frac{\partial h_j}{\partial\xi}\Big). \tag{3.10}$$

The Jacobian of the transformation is equal to one because our mapping is canonical, $\{\xi(u,p),\eta(u,p)\}=1$,

$$h_j(\xi,\eta)=\eta-ju_c(\xi,\eta)=H_j(u_c,p_c) \tag{3.11}$$

is the transformed Hamiltonian, and $(u,p)_c(\xi,\eta)$ is a solution of algebraic equations (3.9). We note that the transformation of the measure does not affect the structure of the operator $\mathbf{K}$ and of the functional U.

The above solution mimics canonical mappings in classical mechanics [80]. It is based on the assumption that the algebraic equations (see (3.9)) completely solve the mechanical problem. In this case, the problem is usually said to be completely integrable.

In the above example, we have split the problem into two parts. In the first part, we found the phase flow $(u,p)_c(\xi,\eta)$; in the second part, we solved the dynamic problem of finding $(\xi,\eta)=(\xi,\eta)(t)\in W$ from the equations

$$\dot\xi=\frac{\partial h_j}{\partial\eta}=1-j\frac{\partial u_c}{\partial\eta},\qquad \dot\eta=-\frac{\partial h_j}{\partial\xi}=j\frac{\partial u_c}{\partial\eta}, \tag{3.12}$$

which corresponds to the quantization of the quotient space W. We note that the explicit form of p_c was not needed.

We now expand the solution of Eqs. (3.12) in j,

$$\begin{aligned}\xi_j(t)&=\xi_0(t)+\int dt'\,\xi_1(t,t')j(t')+\dots,\\ \eta_j(t)&=\eta_0(t)+\int dt'\,\eta_1(t,t')j(t')+\dots.\end{aligned} \tag{3.13}$$

Substituting these expansions, we find that

$$\xi_0 = t_0 + t, \qquad \eta_0 = \text{const}. \tag{3.14}$$

Then

$$\dot{\xi}_1(t,t') = -\delta(t-t')\frac{\partial u_c(\xi_0,\eta_0)}{\partial \eta_0(t)}, \qquad \dot{\eta}_1(t,t') = \delta(t-t')\frac{\partial u_c(\xi_0,\eta_0)}{\partial \xi_0(t)}. \tag{3.15}$$

The Green's function $g(t,t')$ of these equations is translationally invariant,

$$\partial_t g(t,t') = \delta(t-t'). \tag{3.16}$$

It can be easily seen that the following proposition holds.

Proposition 10. *If the Feynman $i\varepsilon$ prescription applies, then*

$$g(t-t') = \theta(t-t'), \qquad g(0) = 1, \tag{3.17}$$

where $\theta(t-t')$ is the step function.

Proof. Indeed, with the $i\varepsilon$ prescription, the Fourier transform of Eq. (3.16) is given by

$$(\omega + i\varepsilon)\tilde{g}(\omega) = 1, \tag{3.18}$$

which then leads to (3.17). We note that in contrast to the causal Green's function $G(t,t')$, which is the sum of the advanced and the retarded parts, the function $g(t-t')$ depends on only the order of t and t'. But, as we see in what follows, the final theory is time-reversible [52]. We also note the uniqueness of solution (3.17).

We use the relations

$$g(t-t')g(t'-t) = 0, \qquad 1 = g(t-t') + g(t'-t), \quad t \neq t', \tag{3.19}$$

where $g(t-t')$ is considered a distribution. We also note that the boundary condition $g(0) = 1$ (see (3.17)) has not been justified. We have chosen it based on the experience in solving quantum mechanical problems [50].

Having thus mapped the problem into the quotient space W, we have found a way to solve equations for the Green's function. But the problem of mapping into the quotient space W remains incomplete because the dependence on j, the *Lagrangian* source of quantum fluctuation, is preserved.

Proposition 11. *If the perturbation theory series generated by the operator* **K** *exists, then there also exists representation* (3.2) *with*

$$DM(u,p) \to DM(\xi,\eta) = \prod_t d\xi\, d\eta\, \delta(\dot{\xi} - 1 - j_\xi)\delta(\dot{\eta} - j_\eta), \tag{3.20}$$

$$2\mathbf{K} = \operatorname{Re}\int_{C_+} dt\, \left(\hat{j}_\eta(t)\hat{e}_\eta(t) + \hat{j}_\xi(t)\hat{e}_\xi(t)\right), \tag{3.21}$$

$$\varphi \to \varphi_c = e_\eta \frac{\partial u_c}{\partial \xi} - e_\xi \frac{\partial u_c}{\partial \eta} \equiv (\varphi_\eta \hat{\xi} - \varphi_\xi \hat{\eta})u_c. \tag{3.22}$$

Proof. For simplicity, we consider the (0+1)-dimensional theory. Acting with the operator generating the perturbation series gives

$$\exp\left\{-\frac{i}{2}\operatorname{Re}\int_{C_+} dt\,\hat{j}(t)\hat{\varphi}(t)\right\}e^{-iU_T(x_c,\varphi)}\prod_t \delta\left(\dot{\xi}-1+j\frac{\partial x_c}{\partial\eta}\right)\delta\left(\dot{\eta}-j\frac{\partial x_c}{\partial\xi}\right)=$$

$$=\int D\varphi_\xi\,D\varphi_\eta\,\exp\left\{2i\operatorname{Re}\int_{C_+} dt\,((\dot{\xi}-1)\varphi_\xi+\dot{\eta}\varphi_\eta)\right\}e^{-iU_T(u_c,\varphi_c)}, \quad (3.23)$$

where φ_c is defined in (3.22). The integrals over $(\varphi_h, \varphi_\theta)$ are then calculated as usual using the series expansion

$$e^{-iU_T(u_c,\varphi_c)}=\sum_{n_\xi,n_\eta=0}^{\infty}\frac{1}{n_\xi!\,n_\eta!}\int\prod_{k=1}^{n_\xi}(dt_k\,\varphi_\xi(t_k))\prod_{k=1}^{n_\eta}(dt'_k\,\varphi_\eta(t'_k))\times$$

$$\times P_{n_\xi,n_\eta}(u_c;t_1,\ldots,t_{n_\xi},t'_1,\ldots,t_{n_\eta}), \quad (3.24)$$

where

$$P_{n_\xi,n_\eta}(u_c;t_1,\ldots,t_{n_\xi},t'_1,\ldots,t_{n_\eta})=\prod_{k=1}^{n_\xi}\hat{\varphi}'_\xi(t_k)\prod_{k=1}^{n_\eta}\hat{e}'_\eta(t'_k)e^{-iU_T(u_c,\varphi'_c)}, \quad (3.25)$$

$\varphi'_c\equiv\varphi_c(\varphi'_\xi,\varphi'_\eta)$, and the derivatives are evaluated at $\varphi'_h=0$, $\varphi'_\theta=0$. On the other hand,

$$\prod_{k=1}^{n_\xi}\varphi_\xi(t_k)\prod_{k=1}^{n_\eta}\varphi_\eta(t'_k)=\prod_{k=1}^{n_\xi}(i\hat{j}_\xi(t_k))\prod_{k=1}^{n_\eta}(i\hat{j}_\eta(t'_k))\exp\left\{-2i\operatorname{Re}\int_{C_+} dt\,(j_\xi(t)\varphi_\xi(t)+j_\eta(t)\varphi_\eta(t))\right\}, \quad (3.26)$$

where the limit $(j_\xi, j_\eta)=0$ is assumed. Substituting (3.25) and (3.26) in (3.24), we find a new representation for $\rho(E)$ with DM, $\mathbf{K}$, and φ_c given by Eqs. (3.20)–(3.22). In this expression, perturbations of all the variables are related to separate sources, and their renormalizations can therefore be analyzed separately. Obviously, the dimensionality of the theory cannot affect the derivation of the final formula.

3.2. The general theory of transformations. The above example demonstrates a special role of canonical transformations of the integration variables. First, this allows obtaining a functional measure that is free of ghosts. Second, the quotient space W turns out to be homogeneous and isotropic (see (3.14)), which results in the equation for the Green's functions becoming solvable because we then pass to variables of the action–angle type. But this solution of the mapping problem seems inapplicable in the general case. First, we cannot be sure that the transformation to W is canonical. For example, in the Coulomb problem, the effect of the hidden $O(4)$ symmetry is that the corresponding quotient space is not symplectic [50], [51]. At the same time, the general quantum theory principles (the uncertainty relation) force the condition that quantum degrees of freedom must belong to a symplectic subspace T^*W. In other words, in general, we must have

$$W=T^*W\times R, \quad (3.27)$$

where R is the space of c-number zero modes. All this also implies that the dimension of T^*W can differ from the dimension of the original phase space. Equation (3.27) implies that we must be able to separate the quantum degrees of freedom belonging to T^*W if we wish to map the dynamics into the quotient space $\mathcal{H}/\mathcal{G}$.

Moreover, in integrable field theory models, there is an *infinite* number of (polynomial) conservation laws. Because of this, the above scheme of transformations to the cotangent bundle T^*W (the moment mapping [80]) based on solving algebraic equations of type (3.9) is inconvenient in general. In this section, we therefore attempt to generalize the transformation scheme. We formulate a more general way to replace variables in the path integral that can also be used in the field theory, i.e., in a system with an infinite number of degrees of freedom, and even in case where the mapping is not canonical.

We use the following idea. As could be noticed, the classical phase flow $(u,p)_c$ completely belongs to $W \ni (\xi,\eta)$. We can then attempt to invert the problem, assuming that

a. the manifold $W \neq \varnothing$ can be reconstructed if the corresponding flows $(u,p)_c$ are known and
b. quantum perturbations do not take $(u,p)_c$ outside W.

In what follows, we show that these assumptions are justified.

Proposition 12. *The formalism of the* (0+1)*-dimensional field theory based on a δ-like measure allows separating the description of phase flows on the cotangent bundle T^*W from the dynamics in the quotient space W of an arbitrary dimension if*

$$W = T^*W = \mathcal{G}/\mathcal{H}. \tag{3.28}$$

Proof. We let

$$\Delta(u,p) = \int \prod_t d\xi\,\delta\eta\,\delta\big(u(t) - u_c(\xi,\eta)\big)\delta\big(p(t) - p_c(\xi,\eta)\big) \tag{3.29}$$

be a functional of u and p. We assume that $(\xi,\eta) \in W$. In this expression, u_c and p_c are arbitrary fixed functions of $(\xi,\eta)(t)$. Our aim is to reduce the problem to the level where u_c and p_c coincide with a solution of the Hamilton equation.

We note that the relations

$$u(t) = u_c(\xi,\eta), \qquad p(t) = p_c(\xi,\eta)$$

can always be satisfied for arbitrary ξ and η. This is indeed so because the integration over u and p also involves the case where these relations are valid. Therefore, $\Delta(u,p) \neq 0$ in general. More precisely, it is necessary and sufficient to assume the validity of the inequalities

$$\Delta_c(\xi,\eta) = \int \prod_t d\bar{\xi}\,d\bar{\eta}\,\delta\Big(\frac{\partial u_c}{\partial \xi}d\bar{\xi} + \frac{\partial u_c}{\partial \eta}d\bar{\eta}\Big)\delta\Big(\frac{\partial p_c}{\partial \xi}d\bar{\xi} + \frac{\partial p_c}{\partial \eta}d\bar{\eta}\Big) \neq 0. \tag{3.30}$$

We note that this is a condition only on u_c and p_c. It means that the derivatives of u_c and p_c in the direction of the vector $(\bar{\xi},\bar{\eta})$ are equal to zero if and only if all the components of this vector are equal to zero.

To perform the mapping, we must substitute the unity

$$1 = \frac{\Delta(u,p)}{\Delta_c(\xi,\eta)}$$

in the integral and then integrate over u and p using the δ function in (3.29). As a result, we find the measure of the form

$$DM(\xi,\eta) = \frac{1}{\Delta_c(\xi,\eta)} \prod_t d\xi\,d\eta\,\delta\Big(\dot{u}_c - \frac{\partial H_j}{\partial p_c}\Big)\delta\Big(\dot{p}_c + \frac{\partial H_j}{\partial u_c}\Big). \tag{3.31}$$

We next obtain

$$\delta\left(\dot{u}_c - \frac{\partial H_j}{\partial p_c}\right)\delta\left(\dot{p}_c + \frac{\partial H_j}{\partial u_c}\right) = \delta\left(\frac{\partial u_c}{\partial \xi}\dot{\xi} + \frac{\partial u_c}{\partial \eta}\dot{\eta} - \frac{\partial H_j}{\partial p_c}\right)\delta\left(\frac{\partial p_c}{\partial \xi}\dot{\xi} + \frac{\partial p_c}{\partial \eta}\dot{\eta} + \frac{\partial H_j}{\partial u_c}\right) =$$

$$= \int \prod_t d\bar{\xi}\, d\bar{\eta}\, \delta\left(\bar{\xi} - \left[\dot{\xi} - \frac{\partial h_j}{\partial \eta}\right]\right)\delta\left(\bar{\eta} - \left[\dot{\eta} + \frac{\partial h_j}{\partial \xi}\right]\right) \times$$

$$\times\, \delta\left(\frac{\partial u_c}{\partial \xi}\bar{\xi} + \frac{\partial u_c}{\partial \eta}\bar{\eta} + \{u_c, h_j\} - \frac{\partial H_j}{\partial p_c}\right) \times$$

$$\times\, \delta\left(\frac{\partial p_c}{\partial \xi}\bar{\xi} + \frac{\partial p_c}{\partial \eta}\bar{\eta} + \{p_c, h_j\} + \frac{\partial H_j}{\partial u_c}\right),$$

where the Poisson brackets are

$$\{X, h_j\} = \frac{\partial X}{\partial \xi}\frac{\partial h_j}{\partial \eta} - \frac{\partial X}{\partial \eta}\frac{\partial h_j}{\partial \xi} \tag{3.32}$$

and we introduce an auxiliary function $h_j = h_j(\xi, \eta)$ defined by

$$\{u_c, h_j\} - \frac{\partial H_j}{\partial p_c} = 0, \qquad \{p_c, h_j\} + \frac{\partial H_j}{\partial u_c} = 0. \tag{3.33}$$

These relations can always be satisfied if the functions u_c and p_c are arbitrary.

As a result, taking (3.33) into account and using the fact that the neighborhoods of $\bar{\xi} = 0$ and $\bar{\eta} = 0$ are essential in (3.30), we obtain

$$\delta\left(\dot{u}_c - \frac{\partial H_j}{\partial p_c}\right)\delta\left(\dot{p}_c - \frac{\partial H_j}{\partial u_c}\right) = \prod_t \delta\left(\dot{\xi} - \frac{\partial h_j}{\partial \eta}\right)\delta\left(\dot{\eta} - \frac{\partial h_j}{\partial \xi}\right) \times$$

$$\times \int \prod_t d\bar{\xi}\, d\bar{\eta}\, \delta\left(\frac{\partial u_c}{\partial \xi}\bar{\xi} + \frac{\partial u_c}{\partial \eta}\bar{\eta}\right)\delta\left(\frac{\partial p_c}{\partial \xi}\bar{\xi} + \frac{\partial p_c}{\partial \eta}\bar{\eta}\right) =$$

$$= \prod_t \delta\left(\dot{\xi} - \frac{\partial h_j}{\partial \eta}\right)\delta\left(\dot{\eta} - \frac{\partial h_j}{\partial \xi}\right)\Delta_c(\xi, \eta).$$

Using this expression, we find the sought transformed measure

$$DM(\xi, \eta) = \prod_t \delta\left(\dot{\xi} - \frac{\partial h_j}{\partial \eta}\right)\delta\left(\dot{\eta} + \frac{\partial h_j}{\partial \xi}\right), \tag{3.34}$$

where the functional determinant has been canceled.

We now recall that the variables $(\xi, \eta) \in T^*W$, i.e., Eqs. (3.33) describe the motion on the cotangent bundle. In reality, the above transformation is a simple replacement of $(u,p)(t)$ with composite functions $(u,p)_c(\xi(t), \eta(t))$. This replacement still does not have the dynamic meaning of the original problem in the sense that either $(u,p)_c(\xi(t), \eta(t))$ or $h_j(\xi, \eta)$ remain arbitrary up to condition (3.30). We now must refine these quantities. The following proposition is obvious.

Proposition 13. *If the relation*

$$h_j(\xi, \eta) = H_j(u_c, p_c) \tag{3.35}$$

holds, then $(u,p)_c(\xi, \eta)$ *on measure* (3.34) *describes a phase flow in the original phase space.*

Proof. A formal proof of this relation is sufficiently simple. For example,

$$\dot{u}_c(\xi, \eta) = \frac{\partial u_c}{\partial \xi}\dot{\xi} + \frac{\partial u_c}{\partial \eta}\dot{\eta} \equiv \{u_c, h_j\} = \frac{\partial H_j}{\partial p_c}. \tag{3.36}$$

Here, we first use the fact that measure (3.34) is δ-like and then use the first equation in (3.33). The same can then be easily obtained for $p_c(\xi, \eta)$. In this sense, the above derivation repeats the statement in Proposition 12.

Therefore, with Eq. (3.35) taken into account, if the functions $(u,p)_c(\xi(t), \eta(t))$ satisfy Eqs. (3.33), then the measure $DM(\xi, \eta)$ has form (3.34), and the canonical system of equations

$$\dot{\xi} = \frac{\partial h_j}{\partial \eta}, \qquad \dot{\eta} = -\frac{\partial h_j}{\partial \xi} \tag{3.37}$$

describes a flow in the quotient space W that has a symplectic structure. This completes the proof of Proposition 12.

We note that we have considered a version of the theory where the quotient space coincides with its cotangent bundle, i.e., the case where W is a symplectic manifold (see (3.28)). But the following proposition holds.

Proposition 14. *The formalism of the* (0+1)*-dimensional field theory based on a δ-like measure also allows extending the space to a symplectic manifold $(\xi, \eta) \in T^*V$ of an arbitrary dimension,* $\dim T^*V \geq \dim W$.

Proof. In other words, we want to show that we can consider a mapping into a space of an arbitrary dimension but one endowed with a symplectic metric. There must then exist a mechanism for separating "redundant" degrees of freedom of the extended quotient space T^*V from the dynamic degrees of freedom such that the dependence on them can be canceled in what follows. This implies the possibility of reducing T^*V to the physical quotient space W. At this stage, we assume that W is a symplectic space, $W = T^*W$.

We note that we could assume the dimension of T^*W to be arbitrary until condition (3.35). To formulate the reduction scheme, we assume that W is a subspace in T^*V, $W \subset T^*V$, i.e., that $\dim T^*V \geq \dim W$. We choose the physical group of variables (ξ, η) such that

$$(\xi, \eta) \in W, \qquad \dim\{\xi\} = \dim\{\eta\}, \tag{3.38}$$

with the other variables (ξ', η') assumed to be "nonphysical." We can then assume that

$$\frac{\partial u_c}{\partial \xi'} \sim \frac{\partial u_c}{\partial \eta'} \sim \varepsilon \to 0, \qquad \frac{\partial p_c}{\partial \xi'} \sim \frac{\partial p_c}{\partial \eta'} \sim \varepsilon \to 0, \tag{3.39}$$

i.e., the dependence on the unphysical degrees of freedom disappears in the $\varepsilon = 0$ limit.

In view of the property selected in (3.39), the operator $\mathbf{K}$ can be written as

$$2\mathbf{K} = \mathrm{Re} \int_{C_+} dt\, \{\hat{j}_\xi \cdot \hat{e}_\xi + \hat{j}_\eta \cdot \hat{e}_\eta + \hat{j}_{\xi'} \cdot \hat{e}_{\xi'} + \hat{j}_{\eta'} \cdot \hat{e}_{\eta'}\}.$$

But because of (3.39), the last two terms can be omitted in the $\varepsilon = 0$ limit. As a result, we obtain

$$2\mathbf{K} = \mathrm{Re} \int_{C_+} dt\, \{\hat{j}_\xi \cdot \hat{e}_\xi + \hat{j}_\eta \cdot \hat{e}_\eta\}. \tag{3.40}$$

We now consider the measure DM. Taking the property in (3.39) and the explicit form of operator (3.40) into account, we obtain

$$DM(\xi, \eta) = \prod_t \delta\Big(\dot{\xi} - \frac{\partial h_j}{\partial \eta}\Big) \delta\Big(\dot{\eta} + \frac{\partial h_j}{\partial \xi}\Big) \delta(\dot{\xi}) \delta(\dot{\eta})$$

because the dependence on the auxiliary variables disapears in the $\varepsilon = 0$ limit. We now note that

$$\int \prod_t dX(t)\, \delta(\dot{X}) = \int dX(0).$$

Therefore,

$$DM(\xi, \eta) = d\xi'(0)\, d\eta'(0) \prod_t \delta\Big(\dot{\xi} - \frac{\partial h_j}{\partial \eta}\Big) \delta\Big(\dot{\eta} + \frac{\partial h_j}{\partial \xi}\Big). \tag{3.41}$$

In what follows, we assume that the integrals over $\xi'(0)$ and $\eta'(0)$ cancel because of normalization, which was to be shown.

The following generalization of Proposition 14 is obvious.

Proposition 15. *Quantum degrees of freedom can span only even-dimensional symplectic manifolds.*

This conclusion naturally fits into the classical quantization scheme based on the uncertainty relation. But it is remarkable that the discussed quantization scheme is capable of selecting a subset of q-numbers. We let T^*W denote this subset.

Proof of Proposition 15. We intend to show that it is possible to define the splitting

$$W = T^*W \times R, \tag{3.42}$$

where R is a subspace of c-numbers. To obtain this, we assume that instead of Eq. (3.38), we have, e.g.,

$$(\xi, \eta) \in W, \qquad \dim\{\xi\} > \dim\{\eta\}. \tag{3.43}$$

We recall that by definition, T^*V is an even-dimensional symplectic manifold. All this means that the operator $\mathbf{K}$ is given by

$$2\mathbf{K} = \mathrm{Re} \int_{C_+} dt\, \{(\hat{j}_\xi \cdot \hat{e}_\xi)_{N_\eta} + (\hat{j}_\eta \cdot \hat{e}_\eta)_{N_\eta} + (\hat{j}_\xi \cdot \hat{e}_\xi)_{(N_\xi - N_\eta)} + (\hat{j}_{\eta'} \cdot \hat{e}_{\eta'})_{(N_\xi - N_\eta)}\},$$

where $N_X = \dim\{X\}$ and the scalar product $(X \cdot Y)_N$ contains N terms. We have thus added $N_\xi - N_\eta$ missing variables η'. Derivatives with respect to the remaining "unphysical" variables are omitted in view of Proposition 14.

We now note that in the expression for $\mathbf{K}$, the last term that is proportional to $\hat{e}_{\eta'}$ can be omitted in the $\varepsilon = 0$ limit. We must therefore assume that $j_{\eta'} = 0$ in what follows. As a result, in the $\varepsilon = 0$ limit, we find the measure

$$DM(\xi, \eta) = \prod_t d^{(N_\eta)}\xi\, d^{(N_\xi - N_\eta)}\xi'\, d^{(N_\eta)}\eta\, d^{(N_\xi - N_\eta)}\eta' \times$$
$$\times\, \delta^{(N_\xi)}\Big(\dot{\xi} - \frac{\partial h_j}{\partial \eta}\Big) \delta^{(N_\eta)}\Big(\dot{\eta} + \frac{\partial h_j}{\partial \xi}\Big) \delta^{(N_\xi - N_\eta)}(\dot{\xi}') \delta^{(N_\xi - N_\eta)}\Big(\dot{\eta}' + \frac{\partial h_j}{\partial \xi}\Big),$$

where we take $\partial h_j / \partial \eta' = 0$ into account. Next, we can always replace $\dot{\eta}' + \partial h_j / \partial \xi \to \dot{\eta}'$ because there is no dependence on η'. As a result, we obtain

$$DM(\xi, \eta) = d^{(N_\xi - N_\eta)}\xi'(0) \prod_t d^{(N_\eta)}\xi\, d^{(N_\eta)}\eta\, \delta^{(N_\eta)}\Big(\dot{\xi} - \frac{\partial h_j}{\partial \eta}\Big) \delta^{(N_\eta)}\Big(\dot{\eta} + \frac{\partial h_j}{\partial \xi}\Big),$$

where for simplicity of notation, we omit the differential measure $d^{(N_\xi - N_\eta)}\eta'$ because none of the variables depend on η'.

We note that the expression for the measure no longer depends on j_ξ, where ξ is conjugate to η', $\dim\{\xi\} = N_\xi - N_\eta$. Precisely this effect leads to the reduction of those quantum degrees of freedom that do not constitute canonically conjugate pairs.

Obviously, if it turns out that $\dim\{\xi\} < \dim\{\eta\}$, then ξ and η must be transposed in the original expression for the operator $\mathbf{K}$. Therefore, in the general case,

$$DM(\xi,\eta) = d\Omega_{(N_\xi - N_\eta)} \prod_t \delta^{(\min\{N\})}\left(\dot{\xi} - \frac{\partial h_j}{\partial \eta}\right)\delta^{(\min\{N\})}\left(\dot{\eta} + \frac{\partial h_j}{\partial \xi}\right), \tag{3.44}$$

where under the condition that $\Theta(0) = 1/2$,

$$d\Omega_{(N_\xi - N_\eta)} = \Theta(N_\xi - N_\eta) d^{(N_\xi - N_\eta)}\xi(0) + \Theta(N_\eta - N_\xi) d^{(N_\eta - N_\xi)}\eta(0) \tag{3.45}$$

is the measure for the integrals over c-number variables, and accordingly

$$2\mathbf{K} = \mathrm{Re} \int_{C_+} dt \left\{(\hat{j}_\xi \cdot \hat{e}_\xi)_{(\min\{N\})} + (\hat{j}_\eta \cdot \hat{e}_\eta)_{(\min\{N\})}\right\}, \tag{3.46}$$

where $\min\{N\} = \min(N_\xi, N_\eta)$.

We have thus shown that if the variable X does not have a canonically conjugate pair, it must be considered a c-number and $\dim T^*W = \min\{N\}$. The possibility to isolate $d\Omega_{(N_\xi - N_\eta)}$ implies that W is factored into direct product (3.42).

We can now pass to considering field variables. As an intermediate model, we can take the example where $u(t)$ is an N-component quantity. The function $u_{\mathbf{i}}(t) = u(\mathbf{i}, t)$ can then be considered the image of the field on a *spatial* lattice, where $\mathbf{i}$ is the coordinate of a cell. As a result, we obtain the following proposition.

Proposition 16. *If $u_c(\mathbf{x}, t; \xi, \eta)$, $p_c(\mathbf{x}, t; \xi, \eta)$ is an exact nonsingular solution of Eqs. (3.33) satisfying condition (3.30) and if Eq. (3.35) is satisfied with the full Hamiltonian given by*

$$H_j(u_c, p_c) = \int d^3x\, \widetilde{H}_j(u_c, p_c) = \\ = \int d^3x \left\{\frac{1}{2}p^2 + \frac{1}{2}(\nabla u_c)^2 + m^2u^2 + v(u) - ju\right\}, \tag{3.47}$$

then the differential measure of the scalar theory on the quotient space $(\xi, \eta) \in W$ is given by

$$DM(\xi,\eta) = d\Omega_{(N_\xi - N_\eta)} \prod_t \delta\left(\dot{\xi}(t) - \frac{\delta h_j}{\delta \eta(t)}\right)\delta\left(\dot{\eta}(t) + \frac{\delta h_j}{\delta \xi(t)}\right), \tag{3.48}$$

where $\dim\{\xi\} = \dim\{\eta\} = \min\{N_\xi, N_\eta\}$, the differential measure is given by (3.45), and the operator generating quantum perturbations is

$$2\mathbf{K} = \mathrm{Re} \int_{C_+} dt \left\{\hat{j}_\xi(t) \cdot \hat{e}_\xi(t) + \hat{j}_\eta(t) \cdot \hat{e}_\eta(t)\right\}. \tag{3.49}$$

Proof. This statement directly follows from Propositions 12–15. It is remarkable that a field theory problem on the quotient space coincides with a quantum mechanical problem.

Substituting (3.35) in (3.33), we find the relations

$$\begin{aligned}
&\{u_c(\mathbf{x};\xi(t),\eta(t)),u_c(\mathbf{y};\xi(t),\eta(t))\}=0,\\
&\{p_c(\mathbf{x};\xi(t),\eta(t)),p_c(\mathbf{y};\xi(t),\eta(t))\}=0,\\
&\{u_c(\mathbf{x};\xi(t),\eta(t)),p_c(\mathbf{y};\xi(t),\eta(t))\}=\delta(\mathbf{x}-\mathbf{y})
\end{aligned}\tag{3.50}$$

that must be satisfied for arbitrary values of $j(\mathbf{x},t)$. The quantization scheme found above thus results in transferring the canonical scheme into the quotient space.

4. The $O(4,2)$-invariant scalar theory

We start with a scalar theory, which is simpler than the theory of the vector Yang–Mills fields but possesses the highest conformal symmetry group $O(4,2)$ (which is higher than the group of general coordinate transformations).

4.1. Generating functional for a massless scalar field. We evaluate the $2N$-dimensional path integral

$$\rho(\beta,z)=e^{-i\mathbf{K}(je)}\int DM(\xi,\eta)\,e^{-iU(u_c,\varphi_c)}e^{-N(\beta,z;u_c)}\tag{4.1}$$

for the $O(4,2)$-invariant scalar theory. Here,

$$N(\beta,z;u_c)=n(\beta_i,z_i;u_c)+n^*(\beta_f,z_f;u_c),\tag{4.2}$$

$$n(\beta,z;u_c)=\int d\omega_1(q;z)\,e^{-\beta\varepsilon(q)}\Gamma(q,u_c)\Gamma^*(q,u_c),\tag{4.3}$$

$$\Gamma(q,u_c)=\int dx\,e^{-iqx}\partial^2u_c(x),\qquad q^2=0,\tag{4.4}$$

$$U(u_c,\varphi_c)=2g\,\mathrm{Re}\int_{C_+}d^3x\,dt\,\varphi_c^3(\mathbf{x},t)u_c(\mathbf{x};\xi(t),\eta(t)),\tag{4.5}$$

$$\varphi_c(\mathbf{x},t)=e_\xi(t)\frac{\partial u_c(\mathbf{x};\xi(t),\eta(t))}{\partial\eta(t)}-e_\eta(t)\frac{\partial u_c(\mathbf{x};\xi(t),\eta(t))}{\partial\xi(t)},\tag{4.6}$$

$$DM(u)=d\Omega\prod_t d\xi(t)\,d\eta(t)\,\delta\Big(\dot\xi-\frac{\partial h}{\partial\eta}-j_\xi\Big)\delta\Big(\dot\eta+\frac{\partial h}{\partial\xi}-j_\eta\Big),\tag{4.7}$$

where

$$h(\xi,\eta)=h_j(\xi,\eta)\big|_{j=0}$$

and finally

$$2\mathbf{K}(je)=\mathrm{Re}\int_{C_+}dt\,\{\hat{j}_\xi(t)\cdot\hat{e}_\xi(t)+\hat{j}_\eta(t)\cdot\hat{e}_\eta(t)\}.\tag{4.8}$$

It is assumed that

$$W=T^*W\times R,\qquad(\xi,\eta)\in T^*W,\qquad\dim W=8,\qquad\dim T^*W=2N\le 8.\tag{4.9}$$

We note that, generally speaking, $\Gamma(q, u_c) \neq 0$ because even if the fields sufficiently rapidly decrease at σ_∞, $u_c = u_c(\mathbf{x}; \xi(t), \eta(t))$ depends on singular (generalized) functions $\xi(t)$ and $\eta(t)$ that are defined through Green's function (3.17) (see Proposition 10).

We find it convenient to replace variables in $2N$-dimensional path integral (4.1) such that the dependence on the $2N$ sources j_x, j_η is eliminated from measure (4.7). If we make the shift

$$\begin{aligned} \xi(t) &\to \xi(t) + \int dt_1\, g(t-t_1) j_\xi(t) \equiv \xi(t) + \xi_j(t), \\ \eta(t) &\to \eta(t) + \int dt_1\, g(t-t_1) j_\eta(t) \equiv \eta(t) + \eta_j(t), \end{aligned} \tag{4.10}$$

where $g(t-t_1)$ is Green's function (3.17), then everywhere

$$u_c = u_c\big(\mathbf{x}; \xi(t) + \xi_j(t), \eta(t) + \eta_j(t)\big). \tag{4.11}$$

Therefore,

$$DM(\xi, \eta) = d\Omega \prod_t d\xi(t)\, d\eta(t)\, \delta\big(\dot{\xi} - \omega(\eta + \eta_j)\big)\delta(\dot{\eta}), \tag{4.12}$$

where we take into account that ξ and η are canonically conjugate variables and accordingly $\partial h/\partial\xi = 0$. We also introduce the notation $\omega_i(\eta) = \partial h(\eta)/\partial\eta_i$.

Making shift (4.10), we must redefine the perturbation-generating operator as

$$\begin{aligned} 2\mathbf{K} &= \mathrm{Re} \int_{C_+} dt\, \{\hat{\xi}_j(t) \cdot \hat{e}_\xi(t) + \hat{\eta}_j(t) \cdot \hat{e}_\eta(t)\} = \\ &= \mathrm{Re} \int_{C_+} dt\, dt_1\, \theta(t - t_1) \{\hat{j}_\xi(t_1) \cdot \hat{e}_\xi(t) + \hat{j}_\eta(t_1) \cdot \hat{e}_\eta(t)\}. \end{aligned} \tag{4.13}$$

4.2. The structure of the $O(4,2)/O(4)\times O(2)$ quotient space. We are interested in the quantization of the $O(4)\times O(2)$-invariant solution [84]

$$u_c(x) = \left\{ \frac{-(\varsigma - \sigma)^2}{g(x-\varsigma)^2 (x-\sigma)^2} \right\}^{1/2}. \tag{4.14}$$

It is regular if the four-vectors ς and σ are complex and is real if

$$\varsigma^* = \sigma = x_0 + i\lambda'. \tag{4.15}$$

Expression (4.14) can then be written as

$$\sqrt{g}\, u_c(x) = \left\{ \frac{4\eta^2}{\left(\eta^2 (x + x_0)^2 - 1\right)^2 + 4\left(\eta (x + x_0)_\mu \lambda^\mu\right)^2} \right\}^{1/2}, \tag{4.16}$$

where $\lambda^2 = \lambda_0^2 - \lambda_i^2 = 1$. This solution depends on eight parameters $(x_{0\mu}, \lambda_i, \eta)$, where $\mu = 0, 1, 2, 3$ and $i = 1, 2, 3$. Substituting these expressions in the formula

$$h = \int d^3x \left\{ \frac{1}{2} p_c^2 + \frac{1}{2} (\nabla u_c)^2 + \frac{g}{4} u_c^4 \right\}, \tag{4.17}$$

we obtain

$$h = \eta\Phi(\lambda_i^2). \tag{4.18}$$

Equation (4.18) can be taken as the *definition* of the set of values of the variable η.

We assume that the physical quotient space W is constrained by the inequalities

$$\eta^2 \geq 0, \qquad -\infty \leq \lambda_i \leq +\infty, \qquad -\infty \leq x_{0\mu} \leq +\infty. \tag{4.19}$$

The first of these ensures the positivity of energy of the classical field.

We must now find the reduction of quantum degrees of freedom that leads to Eq. (3.27). For this, we must once again consider Eq. (3.33),

$$\{u_c, h\} - \frac{\delta H}{\delta p_c} = 0, \qquad \{p_c, h\} + \frac{\delta H}{\delta u_c} = 0. \tag{4.20}$$

As noted above, Eqs. (3.33) must be satisfied for any j_ξ and j_η, in particular, for $j_\xi = 0$ and $j_\eta = 0$. Therefore, with Eq. (4.18) taken into account, the first equation in (4.20) gives

$$\frac{\partial u_c}{\partial \xi}\frac{\partial h}{\partial \eta} = \frac{\delta H}{\delta p_c} = p_c. \tag{4.21}$$

As a result, we find that the sought parameterization of the field is given by (with $\lambda \equiv |\lambda_i|$)

$$\begin{aligned} u_c(\mathbf{x}; \xi, \eta) = \frac{2\eta\Phi^2(\lambda)}{\sqrt{g}} \{ & \left(\eta^2\xi^2 - \Phi^2(\lambda)\eta^2(x - x_0)^2 - \Phi^2(\lambda)\right)^2 + \\ & + 4\eta^2\Phi^2(\lambda)\left(\xi(1 + \lambda^2)^{1/2} - \Phi(\lambda)\lambda_i(x - x_0)_i\right)^2\}^{-1/2}. \end{aligned} \tag{4.22}$$

It can be easily verified that solution (4.22) identically satisfies the first equation in (4.20).

Parameterization (4.22) is useful because in this case,

$$DM = d^3\lambda\, d^3x_0 \prod_t d\xi\, d\eta\, \delta\left(\dot{\xi} - \Phi(\lambda)\right)\delta(\dot{\eta}). \tag{4.23}$$

Then the equations

$$\dot{\xi}(t) = \Phi(\lambda), \qquad \dot{\eta}(t) = 0 \tag{4.24}$$

determine the dynamics in the space W and have the solutions

$$\xi(t) = \Phi(\lambda)(t - t_0), \qquad \eta(t) = \eta_0 = \text{const}, \tag{4.25}$$

which completes the definition of the functional measure in the quotient space $W = O(4,2)/O(4) \times O(2)$. Substituting (4.25) in (4.22), we find that the solution thus derived identically satisfies the original Lagrange equation. As a result, we obtain

$$\{\xi, \eta\} \in T^*W, \quad \dim T^*W = 2, \qquad \{x_{0i}, \lambda_i\} \in R, \quad \dim R = 6. \tag{4.26}$$

4.3. Conservation laws in the quotient space. We return to boundary condition (2.17). It implies that

$$u_c(x \in \partial\sigma_\infty^+) = u_c(x \in \partial\sigma_\infty^-), \tag{4.27}$$

where $\partial\sigma_\infty^\pm$ are the boundaries at the infinitely remote hypersurface σ_∞ of the respective branches $C_\pm$. Depending on the topology of the field $u_c(\mathbf{x}, t)$, conditions (4.27) can constrain solutions of Eq. (4.24).

To clarify this remark, we consider the simplest problem of the motion of a particle in a potential well $v(u)$. The action is then given by

$$S_T(u) = \int_0^T dt \left\{ \frac{1}{2}\dot{u}^2 - v(u) \right\},$$

where $(0, T)$ is the interval of motion (the initial coordinate is therefore not specified). If the particle energy E is fixed, it is necessary to integrate over the time T. The Dirac measure DM of this problem contains two δ functions [56]. The standard (one-dimensional) δ function leads to the equation

$$E = -\eta_q + H_j\big(u(T)\big),$$

where η_q is the energy of quantum corrections and $H_j\big(u(T)\big)$ is the full Hamiltonian at the instant T. The second (functional) δ function gives the equation of motion

$$\ddot{u} + m^2 u + v'(u) = j.$$

The solution of this equation at $j = 0$ is specified, e.g., by the energy $\eta(0)$ and the initial instant $\xi(0)$. We recall that the integrals over $\eta(0)$ and $\xi(0)$ must be taken in general.

In deriving the δ-like Dirac measure, we used the periodic boundary condition

$$u\big(t \in \partial C_+(T)\big) = u\big(t \in \partial C_-(T)\big).$$

Next, because we wish to describe a periodic motion in the potential well, this boundary condition can be satisfied for a set of values of $\xi(0)$. It can be easily found that if $\xi(t \in C_\pm) \equiv \xi_\pm$, then

$$\xi_+ - \xi_- = \Delta\xi = kP(E) + t_0, \quad k = 0, \pm 1, \pm 2, \dots,$$

where $P(E)$ is the period and $0 \le t_0 \le P(E)$. Precisely because it is necessary to sum over all k, the energy level quantization arises [56]. We also note that the boundary condition gives the relation $\eta_+ = \eta_-$, where $\eta(t \in C_\pm) \equiv \eta_\pm$.

Because η and ξ constitute a canonically conjugate pair (see Sec. 3.1), they can be used as the respective generalized momentum and coordinate. Normalizing ξ to the period, we then find that the above uncertainty in choosing the initial condition corresponds to a rotation with the number of laps $k = 0, \pm 1, \pm 2, \dots$, i.e., the number k determines how many times the circle is covered by the mapping $(u, p) \to (\xi, \eta)$. It is then necessary to sum over all integers k. The same effect results in the momenta and positions of topological solitons in the sine-Gordon model defined on the branches C_+ and C_- coinciding, but only up to a certain number. Summation over its values leads to the quantization of the topological charge of solitons and hence to their "sterility" with respect to the emission and absorption of particles [52].

If a problem with a closed classical trajectory is investigated, then our boundary conditions lead to periodic relations between the integration constants $\xi(0)_+$ and $\xi(0)_-$ pertaining to the respective contours

C_+ and C_-. This property of trajectories in the configuration space is usually formulated as the condition for the existence of the homotopy group [85].

There is also another possibility where the trajectory is nonclosed, i.e., does not have topological properties. But it is still necessary to take boundary condition (2.17) into account, ensuring the closedness of the trajectory on the full time contour $C = C_+ + C_-$. In this case, we conventionally speak about a homotopy group singlet. Therefore, we must first determine the relation of the contribution under consideration to the homotopy group. It can be easily seen that the topological charge of the $O(4)\times O(2)$-invariant solution discussed here is equal to zero [65], and we therefore consider a homotopy group singlet. But we show that the following proposition holds.

Proposition 17. *The renormalized energy–momentum tensor* $Q_\mu = T_{\mu\mu}(u_c)$ *coincides with the total four-momentum of incident particles.*

Proof. With solution (4.25) substituted in (4.22), it follows from (4.27) that

$$\eta_-^2\left\{\left(\eta_+^2(t+t_+)^2 - \eta_+^2(\boldsymbol{x}-\boldsymbol{x}_+)^2 - 1\right)^2 + 4\eta_+^2\left((t+t_+)(1+\lambda_+^2)^{1/2} - \lambda_{i+}(x-x_+)_i\right)^2\right\} =$$

$$= \eta_+^2\left\{\left(\eta_-^2(t+t_-)^2 - \eta_-^2(\boldsymbol{x}-\boldsymbol{x}_-)^2 - 1\right)^2 + 4\eta_-^2\left((t+t_-)(1+\lambda_-^2)^{1/2} - \lambda_{i-}(x-x_-)_i\right)^2\right\},$$

which must be satisfied for $x_\mu \in \sigma_\infty$. This directly implies that

$$\eta_+ = \eta_-, \qquad \lambda_{i-} = \lambda_{i-}, \quad i = 1,2,3. \tag{4.28}$$

But no conditions on $x_{\mu,\pm}$ arise. As the result, differential measure (4.23) is in fact given by

$$dM = d^3\lambda\, d\eta\, d^4x_+\, d^4x_- = \frac{1}{16} d^3\lambda\, d\eta\, d^4x_0\, d^4\Delta, \tag{4.29}$$

where

$$\Delta = x_+ - x_-, \qquad x_0 = x_+ + x_-.$$

Making the shift $x \to x - x_\pm$, we can isolate the dependence on $x_\pm$ in $\Gamma(q, u_c)$,

$$\left|\Gamma(q,u_c)\right|^2 = e^{iq\Delta}\left|\Gamma'(q,u_c)\right|^2, \tag{4.30}$$

where $\Gamma'(q, u_c)$ is already independent of x_0. We then see that the variation of the field with respect to $x_\pm$ gives the factor $e^{-iQ_\mu\Delta^\mu}$, where $Q_\mu(u_c) = T_{\mu\mu}(u_c)$ is the renormalized energy–momentum tensor.

Expanding the integrand in (4.1) in powers of $\left|\Gamma(q,u_c)\right|^2$ and taking (4.30) into account, we find that the integration over Δ results in the δ function

$$\delta^{(4)}\Big(\sum_i p_i - Q(u_c)\Big), \tag{4.31}$$

which fixes the energy–momentum conservation laws, thus relating the total four-momentum of the incident (or produced) particles to the four-momentum of the classical field $Q_\mu(u_c)$.

We now show that the tensor Q_μ involved in (4.31) is an energy–momentum tensor. By definition,

$$U(u_c, e) = \left(S_{C_+}(u_c+e) - S_{C_-}(u_c - e)\right) + 2\,\mathrm{Re}\int_{C_+} dx\,\left(\partial^2 u_c + v'(u_c)\right)e \tag{4.32}$$

or, equivalently,

$$U(u_c, e) = \left\{ S(u_c) - g \int dx\, u_c(x) e(x)^3 \right\}_{C_+} - \left\{ S(u_c) + g \int dx\, u_c(x) e(x)^3 \right\}_{C_-}. \tag{4.33}$$

We consider the difference $\delta S(u_c) = S_{C_+}(u_c) - S_{C_-}(u_c)$, which was neglected in the previous expressions (see (2.56)). This difference is the variation of the action $\delta S(u_c)$ with respect to the translation group

$$x_\mu \to x_\mu + \delta x_\mu, \qquad \delta x_\mu = \Delta_\mu.$$

In the lowest order in Δ, we then have

$$\delta S(u_c) = \Delta^\mu T_{\mu\mu}(u_c) + \widetilde{\delta S}(\Delta), \tag{4.34}$$

where $T_{\mu\mu}(u_c)$ is the energy–momentum tensor of the field and the last term includes higher powers of Δ. It can be easily verified that this term plays no role. Indeed, we can write

$$e^{-i\widetilde{\delta S}(\Delta)} = e^{i\hat{\tau}\hat{\sigma}} e^{-i\varsigma\Delta} e^{-i\widetilde{\delta S}(\tau)}, \tag{4.35}$$

where $\hat{\sigma}$ denotes the corresponding derivative at zero. Using this expression, we find that the δ function fixing the conservation laws is given by

$$e^{i\hat{\tau}\hat{\varsigma}} \delta\Big(\sum_i p_i - Q - \varsigma \Big) e^{-i\widetilde{\delta S}(\tau)}. \tag{4.36}$$

But it is impossible to shift the argument of a δ function because the identity

$$(e^{-i\hat{\varsigma}\hat{\tau}} - 1) \delta\Big(\sum_i p_i - Q - \varsigma \Big) e^{-i\widetilde{\delta S}(\tau)} \equiv 0 \tag{4.37}$$

holds. Here, we use the property of the δ function

$$\frac{\partial}{\partial \varsigma} \delta\Big(\sum_i p_i - Q - \varsigma \Big)\Big|_{\varsigma=0} = -\delta\Big(\sum_i p_i - Q \Big) \frac{\partial}{\partial Q}.$$

This completes the proof of Proposition 17.

As a result, we have

$$\rho(\alpha, z) = e^{-i\mathbf{K}} \int dM\, e^{iQ_\mu(u_c)\Delta^\mu} e^{-iU(u_c,\varphi)} e^{-N(\alpha,z;u_c)}, \tag{4.38}$$

where dM is defined in (4.29).

5. Non-Abelian gauge theories

In what follows, we consider only vector gauge fields, assuming that interactions with matter fields (quarks) can be taken into account within the perturbation theory. Therefore, a part of the results in this section pertaining to the Yang–Mills field theory have limited applicability. For example, we cannot claim that the renormalization of quark masses is absent. This problem requires additional discussion.

5.1. The Yang–Mills theory on the Dirac measure. The action

$$S(A) = \frac{1}{2g} \int d^4x \, F_{\mu\nu a}(A) F_a^{\mu\nu}(A) \tag{5.1}$$

of the theory under consideration is $O(4,2)$-invariant. The Yang–Mills fields

$$F_{\mu\nu a}(A) = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} - C_a^{bc} A_{\mu b} A_{\nu c} \tag{5.2}$$

are covariant under non-Abelian gauge transformations. We do not specify the gauge group.

For simplicity, we begin with the integral

$$\mathcal{N} = e^{-i\mathbf{K}(je)} \int DM \, e^{-2iU(A,e)}, \tag{5.3}$$

where the measure

$$DM(A) = \prod_{\mu,a} \prod_x dA_\mu^a(x,t) \, \delta(D_a^{\nu b} F_{\nu\mu b} - j_{\mu a}) \tag{5.4}$$

is explicitly conformally and gauge invariant if $j_{\mu a} = 0$. The covariant derivative is given by

$$D_a^{\mu b} = \partial^\mu \delta_a^b + C_a^{bc} A_c^\mu,$$

and the perturbation-generating operator is

$$2\mathbf{K}(je) = \operatorname{Re} \int_{C_+} d^4x \, \frac{\delta}{\delta j_a^\mu(x,t)} \frac{\delta}{\delta e_{\mu a}(x,t)}. \tag{5.5}$$

As usual, $j_{\mu a}$ and e_a^μ must be set equal to zero at the very end of the calculations. The functional

$$-2U(A,e) = S_{C_+}(A+e) - S_{C_-}(A-e) - 2\operatorname{Re} \int_{C_+} d^4x \, e_a^\mu(x) \frac{\delta S(A)}{\delta A_a^\mu} + O(\varepsilon) \tag{5.6}$$

describes interactions. All the quantities are defined on a complex Mills contour. The terms $\sim \epsilon \to +0$ can be omitted in (5.6). Therefore, $U(A,e) = O(e^3)$ contains only odd powers of $e_{a\mu}$. This implies that the functional $U(A,e)$ can be written as

$$U(A,e) = -\int d^4x \left\{ e_a^\mu(x) \frac{\delta}{\delta A_a^\mu(x)} \right\}^3 S(A). \tag{5.7}$$

5.2. The first-order formalism. The noncovariant formalism involving the electric field $E_a^i = F_a^{i0}$ provides an introduction to the Hamiltonian description that we need here. In this case, the action is given by

$$S_{C\pm}(A,F) = \frac{1}{g} \int_{C\pm} d^4x \left\{ \dot{\mathbf{A}}_a \cdot \mathbf{E}_a + \frac{1}{2}(\mathbf{E}_a^2 + \mathbf{B}_a^2(\mathbf{A})) - A_{0a}(\mathbf{D} \cdot \mathbf{E})_a \right\}, \tag{5.8}$$

where the magnetic field

$$\mathbf{B}_{ia}(\mathbf{A}) = (\operatorname{rot} \mathbf{A})_{ia} + \frac{1}{2} \varepsilon_{ijk} [\mathbf{A}_j, \mathbf{A}_k]_a \tag{5.9}$$

is not an independent quantity and is introduced only for simplicity of notation. We note that A_{0a} does not have a conjugate pair and the action S is linear in this quantity.

Measure (5.4) can be written in the first-order formalism,

$$DM(\mathbf{A},\mathbf{P}) = \prod_a \prod_x d\mathbf{A}_a(x)\, d\mathbf{P}_a(x)\, \delta(\mathbf{D}_a^b \cdot \mathbf{P}_b) \times$$

$$\times\, \delta\Big(\dot{\mathbf{P}}_a(x) + \frac{\delta H_j(\mathbf{A},\mathbf{P})}{\delta \mathbf{A}_a(x)}\Big)\delta\Big(\dot{\mathbf{A}}_a(x) - \frac{\delta H_j(\mathbf{A},\mathbf{P})}{\delta \mathbf{P}_a(x)}\Big), \tag{5.10}$$

where

$$d\mathbf{A}_a(x)\, d\mathbf{P}_a(x) = \prod_i dA_{ia}\, d\mathbf{P}_{ai}(x), \tag{5.11}$$

$H_j(\mathbf{A},\mathbf{P})$ is the full Hamiltonian,

$$H_j = \frac{1}{2g}\int d^3x\, \big(\mathbf{P}_a^2 + \mathbf{B}_a^2(\mathbf{A})\big) + \int d^3x\, \mathbf{j}_a \mathbf{A}_a, \tag{5.12}$$

$\mathbf{P}_a(x) \equiv \mathbf{E}_a(x)$ is the momentum conjugate to $\mathbf{A}_a(x)$, and $\mathbf{B}_a(\mathbf{A})$ is defined in (5.9). In the expression for the measure DM, we can introduce an additional δ function

$$\prod_a \prod_x \delta\Big(\mathbf{B}_a^i - (\mathrm{rot}\,\mathbf{A})_a^i - \frac{1}{2}\varepsilon^i_{jk}[A^j, A^k]_a\Big). \tag{5.13}$$

Hamiltonian (5.12) then becomes symmetric with respect to the fields $\mathbf{E}_a$ and $\mathbf{B}_a$.

We note that the first δ function involved in (5.10) follows from the linearity of the action in A_{0a}. The time component A_{0a} indeed has the meaning of a Lagrange multiplier for the Gauss law

$$\mathbf{D}_a^b \cdot \mathbf{P}_b = 0. \tag{5.14}$$

It must be stressed that there is no equation for A_{0a}. Moreover, the dependence on A_{0a} has entirely disappeared because the functional $U(A,e)$ describing interactions is determined by the third derivative with respect to $A_{\mu a}$ (see (5.7)).

5.3. Mapping into the quotient space. Measure (5.10) does not have a physical meaning, because for a given a, it depends on the three-vector potentials $\mathbf{A}_a(x)$. The "unphysical" degrees of freedom are usually eliminated using the gauge-fixing Faddeev–Popov ansatz. But we prefer another approach.

As in Sec. 2, we introduce the quantity

$$\Delta(\mathbf{A},\mathbf{P}) = \int D\xi\, D\eta \prod_a \delta\big(\mathbf{A}_a(x) - \mathbf{u}_a(x;\xi(x),\eta(x))\big)\delta\big(\mathbf{P}_a(x) - \mathbf{p}_a(x;\xi(x),\eta(x))\big) \tag{5.15}$$

to realize the transformation

$$u\colon (\mathbf{A},\mathbf{P})_a(x) \longrightarrow (\xi,\eta)(x) \tag{5.16}$$

to *space–time* local functions $(\xi,\eta)(x)$ using the composite vector functions $(\mathbf{u},\mathbf{p})_a(x;\xi(x),\eta(x))$. It is assumed that $\Delta \neq 0$.

After transformation (5.16), we obtain

$$DM(\xi,\eta) = \frac{1}{\Delta_c(u)}\prod_a \prod_x d\xi\, d\eta\, d\lambda_a\, dq_a\, \delta(\mathbf{D}_a^b \cdot \mathbf{p}_b)\delta\Big(\dot{\mathbf{u}}_a(x) - \frac{\delta H_j}{\delta \mathbf{p}_a(x)}\Big)\delta\Big(\dot{\mathbf{p}}_a(x) + \frac{\delta H_j}{\delta \mathbf{u}_a(x)}\Big). \tag{5.17}$$

In general, the set ξ, η is arbitrary (see Proposition 14 in Sec. 3.2). Along with ξ and η, we can therefore consider the phase of gauge transformations λ_a and its conjugate charge q_a. The dependence on $\lambda_a(\mathbf{x},t)$ and $q_a(\mathbf{x},t)$, however, was separated from the sets $\xi(t)$ and $\eta(t)$ for convenience.

Repeating the calculations in Sec. 2, we obtain

$$DM(\xi,\eta,\lambda,Q) = \prod_{x,t,a} d\xi\, d\eta\, d\lambda\, dq\, \delta\big(\mathbf{D}_a^b(\mathbf{u})\cdot\mathbf{p}_b\big) \times \\ \times\, \delta\Big(\dot\lambda_a - \frac{\delta h_j}{\delta q_a}\Big)\delta\Big(\dot q_a + \frac{\delta h_j}{\delta\lambda_a}\Big)\delta\Big(\dot\xi - \frac{\partial h_j}{\partial\eta}\Big)\delta\Big(\dot\eta + \frac{\partial h_j}{\partial\xi}\Big). \tag{5.18}$$

Equation (5.18) is satisfied if and only if the functions h_j are determined by the Poisson equations (with given three-vectors $\mathbf{u}_a$ and $\mathbf{p}_a$)

$$\{\mathbf{u}_a(x), h_j\} = \frac{\delta H_j}{\delta\mathbf{p}_a(x)}, \qquad \{\mathbf{p}_a(x), h_j\} = -\frac{\delta H_j}{\delta\mathbf{u}_a(x)}, \tag{5.19}$$

where (ξ,η) and (λ,q) are chosen as canonically conjugate pairs in the Poisson brackets.

If Eq. (5.19) is supplemented by the additional relation

$$h_j(\xi,\eta,\lambda,q) = H_j(\mathbf{u}_a,\mathbf{p}_a), \tag{5.20}$$

then, as shown above, $\mathbf{u}_a$ and $\mathbf{p}_a$ must coincide with solutions of the original equations under the condition that Eqs. (5.19) are satisfied on measure (5.18). It then follows that

$$\mathbf{D}_a^b(u)\cdot\mathbf{p}_b \equiv 0 \tag{5.21}$$

because $\mathbf{p}_b$ is a solution of Eqs. (5.19) for arbitrary $j_{\mu a}$. This remarkable result is a consequence of the mapping into the invariant space $\mathcal{G}/\mathcal{H}$ to which the classical phase flow completely belongs. Therefore, the δ function in (5.18) corresponding to (5.21) becomes $\prod_x \delta(0)$ identically. This infinite factor is canceled by the normalization and is not explicitly written in what follows.

The mapping described above thus gives

$$DM(\xi,\eta,\lambda,Q) = \prod_{x,t,a} d\lambda_a\, dq_a\, d\xi\, d\eta\, \delta(\dot\lambda_a)\,\delta\Big(\dot q_a + \frac{\delta h_j}{\delta\lambda_a}\Big)\delta\Big(\dot\xi - \frac{\partial h_j}{\partial\eta}\Big)\delta\Big(\dot\eta + \frac{\partial h_j}{\partial\xi}\Big). \tag{5.22}$$

Here, we take into account that the functions $(u,p)_a$ are independent of q_a. The Hamiltonian h_j is defined by Eq. (5.20),

$$2gh_j = \int d^3x\,(p_a^2 + \mathbf{B}_a^2(u)) + \int d^3x\,\mathbf{j}_a\mathbf{u}_a \equiv h + J, \tag{5.23}$$

where h is the Hamiltonian unperturbed by the force $\mathbf{j}_a$.

In accordance with Proposition 15, we can eliminate the dependence on q_a,

$$DM(\xi,\eta,\lambda) = dR\prod_{x,a} d\lambda_a\, d\xi\, d\eta\, \delta(\dot\lambda_a)\,\delta(\dot\xi - \omega - j_\xi)\,\delta(\dot\eta - j_\eta) \tag{5.24}$$

with the "velocity" $\omega = \partial h/\partial\eta$. Otherwise, nothing changes as compared with the scalar theory considered in the previous section.

As follows from (5.24), we must consider time-independent gauge transformations $\dot{\lambda}_a(x) = 0$. To drop this restriction, we must generalize Eqs. (5.19). If we therefore consider the relation

$$\{\mathbf{u}_a(x;\xi,\eta,\lambda), h_j\} = \frac{\delta H_j}{\delta \mathbf{p}_a(x)} - \Omega_a(x)\frac{\partial \mathbf{u}(x;\xi,\eta,\lambda)}{\partial \lambda_a} \tag{5.25}$$

instead of the first equation in (5.19), then we must replace

$$\prod_{x,a} d\lambda_a(x)\, \delta\big(\dot{\lambda}_a(x)\big) \to \prod_{x,a} d\lambda_a(x)\, \delta\big(\dot{\lambda}_a(x) - \Omega_a(x)\big) \tag{5.26}$$

in (5.24), where $\Omega_a(x)$ is an arbitrary function of y and t. This is the most general representation for the gauge measure in our formalism.

As a result, the basic elements of the Yang–Mills theory in the quotient space $\mathcal{G}/\mathcal{H}$ are as follows:

1. The measure is

$$DM(\xi,\eta,\lambda) = dR \prod_{x,a} d\lambda_a\, d\xi\, d\eta\, \delta\big(\dot{\lambda}_a(x) - \Omega_a(x)\big)\delta(\dot{\xi} - \omega - j_\xi)\delta(\dot{\eta} - j_\eta). \tag{5.27}$$

It can be noted that

$$\int \prod_{x,a} d\lambda_a\, \delta\big(\dot{\lambda}_a(x) - \Omega_a(x)\big)$$

implies the integration over all functions $\lambda_a(\mathbf{x},t)$ arbitrarily depending on time. On the other hand,

$$\frac{\int \prod_{x,a} d\lambda_a\, \delta\big(\dot{\lambda}_a(x) - \Omega_a(x)\big)}{\int \prod_{x,a} d\lambda_a} \equiv 0. \tag{5.28}$$

Therefore, our normalization to the gauge group volume differs from the standard one. But this must not affect the final result, because only gauge-invariant quantities are evaluated.

2. The quantum perturbation-generating operator is

$$2\mathbf{K}(je) = \int dt\, \{\hat{j}_\xi \hat{e}_\xi + \hat{j}_\eta \hat{e}_\eta\}. \tag{5.29}$$

3. The functional $U(\mathbf{u}, \mathbf{e}_a)$ describing the interaction depends on the auxiliary field

$$\mathbf{e}_a = e_{\xi_i}\frac{\partial \mathbf{u}_a}{\partial \eta_i} - e_{\eta_i}\frac{\partial \mathbf{u}_a}{\partial \xi_i}, \tag{5.30}$$

where summation over repeated indices is understood. We note that $\lambda_a(x)$ is a c-number function and the dependence on nondynamic variables has been dropped as a result of the reduction.

5.4. Gauge invariance and divergences. If the perturbation theory is formulated in gauge-invariant terms of the color electric and magnetic fields $\mathbf{E}_a$ and $\mathbf{B}_a$, then "unphysical" degrees of freedom are automatically eliminated. We can therefore formulate the following proposition within the above formalism.

Proposition 18. *Each order of the new perturbation theory with respect to $1/g$ is explicitly gauge invariant.*

Proof. For the proof, we use the fact that the operator $\mathbf{K}(je)$ acts in the quotient subspace TW^*. It then suffices to show that the functional $U(\mathbf{u}, \mathbf{e}_a)$ is gauge invariant. For this, we use the representation

$$U(\mathbf{u}, \mathbf{e}_a) = \frac{1}{g} \int dx \prod_{k=1}^{3} \left\{ \left(e_{\xi_i} \frac{\partial \mathbf{u}_{a_k}}{\partial \eta_i} - e_{\eta_i} \frac{\partial \mathbf{u}_{a_k}}{\partial \xi_i} \right) \cdot \frac{\partial}{\partial \mathbf{u}_{a_k}} \right\} F^{\mu\nu a} F_{\mu\nu a}, \tag{5.31}$$

which can be obtained from the explicit form of $\mathbf{e}_a$ (see Eq. (5.30)). This expression is manifestly gauge invariant because the operator

$$\left\{ \left(\mathbf{e}_\xi \cdot \frac{\partial \mathbf{u}_{a_k}}{\partial \eta} - \mathbf{e}_\eta \cdot \frac{\partial \mathbf{u}_{a_k}}{\partial \xi} \right) \cdot \frac{\partial}{\partial \mathbf{u}_{a_k}} \right\}$$

is a singlet of the gauge transformation group.

Indeed, representation (5.31) can be written as the recursive relation

$$U(\mathbf{u}, \mathbf{e}_a) = \frac{1}{g} \int dx \left\{ \mathbf{e}_\xi \cdot \frac{\partial}{\partial \eta} - \mathbf{e}_\eta \cdot \frac{\partial}{\partial \xi} \right\} F_2(\mathbf{u}), \tag{5.32}$$

where the scalar product symbol indicates summation over all canonically conjugate pairs (ξ, η) and it must be assumed that $F_2(\mathbf{u}) = F_2(\mathbf{u}(\xi, \eta))$ is a composite function of ξ and η. In precisely the same way, we have

$$\begin{aligned} F_2(\mathbf{u}(\xi, \eta)) &= \left\{ \mathbf{e}_\xi \cdot \frac{\partial}{\partial \eta} - \mathbf{e}_\eta \cdot \frac{\partial}{\partial \xi} \right\} F_1(\mathbf{u}(\xi, \eta)), \\ F_1(\mathbf{u}(\xi, \eta)) &= \left\{ \mathbf{e}_\xi \cdot \frac{\partial}{\partial \eta} - \mathbf{e}_\eta \cdot \frac{\partial}{\partial \xi} \right\} F^{\mu\nu a}(\mathbf{u}(\xi, \eta)) F_{\mu\nu a}(\mathbf{u}(\xi, \eta)). \end{aligned} \tag{5.33}$$

We now note that the differential operator in $F_1(\mathbf{u})$ is independent of the field $\mathbf{u}_a$. Therefore, $F_1(\mathbf{u})$ is a gauge-invariant quantity. For the same reason, all $F_l(\mathbf{u})$, $l = 2, 3$, are gauge invariant. The proposition is proved.

This result implies that perturbation theory contributions cannot violate the non-Abelian gauge symmetry. Next, we can show the following important property of the perturbation theory considered here.

Proposition 19. *The perturbation theory in the quotient space does not contain divergences, at least in the vector-field sector, if*

$$|S(u)| < \infty. \tag{5.34}$$

Proof. The action of the operator of quantum perturbations gives

$$\mathcal{N} = \int DM \, {:}e^{-2i\mathbf{U}(\mathbf{u},\mathbf{j})}{:}, \tag{5.35}$$

where

$$\mathbf{U}(\mathbf{u}, \mathbf{j}) = \int \frac{dt}{3!\,(2i)^3} \left\{ \hat{j}_\xi \cdot \frac{\partial}{\partial \eta} - \hat{j}_\eta \cdot \frac{\partial}{\partial \xi} \right\} \widetilde{F}_2(\mathbf{u}) \tag{5.36}$$

and

$$\widetilde{F}_2(\mathbf{u}) = \int d^3x \, F_2(\mathbf{u}). \tag{5.37}$$

Equations (5.36) and (5.33) directly imply condition (5.34).

5.5. Generating functional in the Yang–Mills theory. We propose constructing the event generator as

$$\rho(\alpha, z) = \int dM(\xi_0, \eta_0; \lambda_a) :e^{-i\mathbf{U}(\mathbf{u}_c, \mathbf{e})}: e^{iQ_\mu(\mathbf{u}_c)\Delta^\mu} e^{-N(\alpha, z; \mathbf{u}_c)}, \tag{5.38}$$

where the operator $\mathbf{U}(\mathbf{u}_c, \mathbf{e})$ is defined in (5.36). The functional $N(\alpha, z; \mathbf{u}_c)$ was introduced in (4.2) and is expressed (see (4.3)) through the "vertex function"

$$\Gamma(q, \mathbf{u}_c) = \int dx\, e^{iqx} \frac{\delta S_0(\mathbf{u}_c)}{\delta \mathbf{u}_c(x)}.$$

In reality, if the color charge is confined, the generating function thus defined is trivial, $\partial\rho(\alpha, z)/\partial z \equiv 0$.

Therefore, in the perturbation theory formalism described here, which is closed in the sense of being free of divergences and therefore being applicable at any distance, the main problem is to find asymptotic states, i.e., the fundamental Lagrangian of the theory and the corresponding functional $N(\alpha, z; \mathbf{u}_c)$.

6. Conclusions

It may seem that we enter a new stage in constructing the Yang–Mills theory, where the main formula for the generating functional $\rho(a, z)$ that is capable of describing the interaction at any distance becomes a one-line relation. The calculations are then so complicated that they are accessible only to sufficiently powerful computers. As a result, all the intermediate stages of the calculations are done by computer and do not require our interference.

In reality, this is not the case. The exact evaluation of integrals (5.38) is most probably beyond the power of modern computers and is unlikely to ever become accessible. This highlights the remark that the previous formulas do not make it totally obvious that the color charge is permanently confined within hadrons. We are currently investigating this problem. In doing so, we must of course start with simpler problems that admit approximations justified by specific conditions in the problem. One of these is the asymptotic regime with respect to multiplicity as $n \to n_{\max}$. The analysis of this asymptotic regime shows that the process must then be "hard" in the sense that we can expect the mean transverse momentum to be $\pi/4$ greater than the mean longitudinal one. But this is then the asymptotic freedom domain, where $\alpha_s \ll 1$, and QCD predictions can be used.

Analysis of QCD predictions in this regime shows that the ideology of the leading logarithmic approximation is unacceptable in this case because the logarithmic accuracy of the contribution estimates is insufficient to describe the kinematic conditions under which the particle momenta are only insignificantly different (the inelasticity coefficient is close to unity) [7].[9] For this reason, QCD has low predicting power in the domain of very high multiplicities. The proposed perturbation theory is "superconvergent," and we therefore hope to obtain a higher (power-law) accuracy of the predictions.

Another class of problems that we believe to be interesting is related to deeply inelastic scatterings. They also assume interactions at small distances and must therefore be sufficiently simple. Our interest in this problem is related to the fact that the proposed theory is an expansion in powers of the inverse coupling constant. In this formulation, the notion of the running expansion parameter is inapplicable [53]. This makes it particularly interesting to investigate how the asymptotic freedom is formulated in this approach. In addition, our formulation does not involve the notion of a *gluon* and hence does not involve infrared divergences. It is therefore interesting to investigate the so-called "small-x problem" [86] within our approach.

[9] We are especially grateful to L. N. Lipatov for a discussion of this point.

Using the lattice expansion is natural in formulating the theory in terms of path integrals. We note that the integrand in (5.38) does not contain time derivatives; its representation on the temporal lattice is therefore free of ambiguities that are inherent in the representation of path integrals (see, e.g., [87] and the references therein).

The lattice constant can depend on the conditions in the problem under investigation. For example, it is easy to understand that particle momenta are small in the asymptotic regime with respect to multiplicity. In the first approximation, therefore, the classical field configuration $\mathbf{u}_a$ does not play a considerable role. The asymptotic regime with respect to multiplicity is the simplest case in precisely this sense.

Finally, it is well known that the S-matrix interpretation of Wigner functions allows formulating the theory in terms of kinetic equations and also verifying the validity of this description in the light of quantum perturbations [2], [3]. This can serve to relate the field theory description to the description of dissipative structures. For example, the above formalism may be useful in investigating the stability of ordered structures arising in dissipative systems [88] and in clarifying the role played by the topology and the structure of the quotient space in their formation and stability. This may have great importance for applications.

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REFERENCES

1. G. Nicolis and I. Prigogine, *Self-Organization in Nonequilibrium Systems: From Dissipative Structures to Order Through Fluctuations*, Wiley, New York (1977).
2. P. Carruthers and F. Zachariasen, *Phys. Rev. D*, **13**, 950 (1986); P. Carruthers and F. Zachariasen, *Rev. Mod. Phys.*, **55**, 245 (1983).
3. J. Manjavidze, *Part. Nucl.*, **30**, 124 (1999).
4. E. Wigner, *Phys. Rev.*, **40**, 749 (1932); K. Hisimi, *Proc. Phys. Math. Soc. Japan*, **23**, 264 (1940); R. J. Glauber, *Phys. Rev. Lett.*, **10**, 84 (1963); E. C. G. Sudarshan, *Phys. Rev. Lett.*, **10**, 177 (1963); R. E. Cahill and R. G. Glauber, *Phys. Rev.*, **177**, 1882 (1969); S. Mancini, V. I. Man'ko, and P. Tombesi, *Quantum Semiclass. Opt.*, **7**, 615 (1995); V. I. Man'ko, L. Rose, and P. Vitale, "Probability representation in quantum field theory," hep-th/9806164 (1998).
5. H. Umezawa, H. Matsumoto, and M. Tachiki, *Thermo-Field Dynamics and Condensed States*, North-Holland, Amsterdam (1982).
6. E. Fermi, *Progr. Theor. Phys.*, **4**, 570 (1950); *Phys. Rev.*, **81**, 115 (1950); **92**, 452 (1953); L. D. Landau, *Math. USSR, Izv.*, **17**, 85 (1953).
7. J. Manjavidze and A. Sissakian, *Phys. Rep.*, **346**, 1 (2001).
8. V. E. Zakharov, *JETP*, **38**, 108 (1974).
9. D. V. Skobeltzin, *Z. Phys.*, **54**, 68 (1929); *C. R. Acad. Sci.*, **194**, 118 (1932); G. V. Watagin, *Z. Phys.*, **688**, 92 (1934).

10. N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantum Fields* [in Russian] (4th ed.), Nauka, Moscow (1984); English transl. prev. ed., Wiley, New York (1980).
11. V. G. Kadyshevskii, *JETP*, **19**, 443, 597 (1964).
12. A. A. Logunov, M. A. Mestvirishvili, and V. A. Petrov, "General principles of quantum field theory and hadron interactions at high energies," in: *Quantum Field Theory and Its Consequences* [in Russian], Nauka, Moscow (1977), p. 183; A. A. Logunov, M. A. Mestvirishvili, and Nguen Van Hieu, *Phys. Lett. B*, **25**, 611 (1967); A. A. Logunov and M. A. Mestvirishvili, "General principles of the field theory and inclusive reactions at high energies," CERN TH/1707, CERN, Geneva (1973).
13. V. De Alfaro, S. Fubini, G. Furlan, and C. Rossetti, *Currents in Hadron Physics*, North-Holland, Amsterdam (1973).
14. E. Kuraev, L. Lipatov, and V. Fadin, *JETP*, **44**, 443 (1976); L. Lipatov, *Sov. J. Nucl. Phys.*, **20**, 94 (1975); V. Gribov and L. Lipatov, *Sov. J. Nucl. Phys.*, **15**, 438, 675 (1972); G. Altarelli and G. Parisi, *Nucl. Phys. B*, **126**, 298 (1977); I. V. Andreev, *Chromodynamics and Hard Processes at High Energies* [in Russian], Nauka, Moscow (1981).
15. V. N. Gribov, "Space–time description of hadron interactions at high energies [in Russian]," in: *Proc. 8th LNPI School on Nuclear and Elementary Particle Physics* (Leningrad, 1973), Vol. 2 (1973), p. 5; English transl.: hep-ph/0006158 (2000); O. V. Kancheli, *JETP Letters*, **18**, 274 (1973); H. D. I. Abarbanel, J. D. Bronzan, R. L. Sugar, and A. K. White, *Phys. Rep. C*, **21**, 119 (1975); J. Koplik and A. H. Mueller, *Phys. Rev. D*, **12**, 3638 (1975); M. Baker and K. Ter-Martirosyan, *Phys. Rep. C*, **28**, 1 (1976).
16. V. A. Matveev, R. M. Muradyan, and A. N. Tavkhelidze, *Nuovo Cimento Lett.*, **5**, 907 (1973); **7**, 719 (1973); S. Brodski and C. Farrar, *Phys. Rev. Lett.*, **31**, 1153 (1973).
17. A. A. Logunov, A. N. Tavkhelidze, and L. D. Soloviev, *Phys. Lett. B*, **24**, 181 (1967).
18. J. Polchinski, "TASI lectures on *D*-branes," hep-th/9611050 (1996).
19. J. Ellis, "Aspects of M theory and phenomenology," hep-ph/9804440 (1998).
20. V. A. Matveev, A. N. Sisakyan, and L. A. Slepchenko, *Sov. J. Nucl. Phys.*, **23**, 227 (1976); A. N. Sisakyan and N. B. Skachkov, "Multiple processes and the description of the constituent structure of hadrons in the three-dimensional formulation of quantum field theory [in Russian]," in: *Scientific Collaboration in Nuclear Physics* (N. N. Bogoliubov, ed.), Énergoatomizdat, Moscow (1986), p. 56; S. Shch. Mavrodiev, V. K. Mitryushkin, A. N. Sisakyan, and G. T. Torosyan, *Sov. J. Nucl. Phys.*, **30**, 124 (1979); S. V. Chekanov, W. Kittel, and V. I. Kuvshiniv, *J. Phys. G*, **23**, 951 (1997).
21. J. Manjavidze, *Part. Nucl.*, **16**, 44 (1985).
22. J. E. Mayer and M. Goeppert Mayer, *Statistical Mechanics*, Wiley, New York (1940).
23. D. E. Groom et al., *European Phys. J. C*, **15**, 1 (2000).
24. P. V. Landshoff, *Nucl. Phys. B*, **25**, 129 (1992); E. Levin, "Everything about reggeons: Part I. Reggeons in "soft" interaction," Preprint 97-213, DESY, Hamburg (1997); hep-ph/9710546 (1997); "25 years with the Pomeron," Preprint 98-118, DESY, Hamburg (1998); hep-ph/9808483 (1998).
25. E. A. De Wolf, I. M. Dremin, and W. Kittel, *Phys. Rep.*, **270**, 1 (1996); P. Bozek, M. Ploszajczak, and R. Botet, *Phys. Rep.*, **252**, 101 (1995).
26. R. Hagedorn, *Nuovo Cimento*, **35**, 216 (1965); E. L. Feinberg, *Phys. Rep.*, **56**, 237 (1972); I. V. Andreev and I. M. Dremin, *Sov. Phys. Usp.*, **20**, 381 (1977).
27. G. E. Uhlenbeck, "App. I: Boltzmann Equation [in Russian]," in: M. Kac, *Probability and Related Questions in Physics*, Mir, Moscow (1963), p. 227.
28. J. Manjavidze and A. Sissakian, in: *Second VHM Physics Workshop* (Dubna, 7-9 April 2001) (to appear).
29. N. N. Bogoliubov, *Problems of a Dynamical Theory in Statistical Physics* [in Russian], Gostekhizdat, Moscow (1946); English transl., North-Holland, Amsterdam (1962).
30. J. Schwinger, *J. Math. Phys. A*, **9**, 2363 (1994); L. Keldysh, *JETP*, **20**, 1018 (1965); N. P. Landsman and Ch. G. van Weert, *Phys. Rep.*, **145**, 141 (1987).
31. M. Martin and J. Schwinger, *Phys. Rev.*, **115**, 342 (1959); R. Kubo, *J. Phys. Soc. Japan*, **12**, 570 (1957).
32. R. Haag, N. Hugengoltz, and M. Winnink, *Commun. Math. Phys.*, **5**, 5 (1967).

33. D. N. Zubarev, *Nonequilibruim Statistical Thermodynamics* [in Russian], Nauka, Moscow (1971); English transl., Plenum, New York (1974).
34. A. M. Baldin, *Nucl. Phys. A*, **434**, 695 (1985); A. M. Baldin and A. I. Malakhov, *JINR Rapid Commun.*, **1(87)**, 5 (1998).
35. C. N. Yang and R. I. Mills, *Phys. Rev.*, **96**, 191 (1954).
36. N. N. Bogoliubov, V. A. Matveev, and A. N. Tavkhelidze, "Colored quarks [in Russian]," in: *Scientific Collaboration of Socialist Countries in Nuclear Physics*, Énergoatomoizdat, Moscow (1986), p. 33; M. Han and Y. Nambu, *Phys. Lett. B*, **138**, 1005 (1965).
37. P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science), Yeshiva University, New York (1964).
38. J. C. Taylor, *Nucl. Phys. B*, **33**, 436 (1971); A. A. Slavnov, *Theor. Math. Phys.*, **10**, 99 (1972); V. A. Rubakov, *Classical and Gauge Fields* [in Russian], Editorial URSS, Moscow (1999).
39. G. 't Hooft and M. Veltman, "Diagrammar," Rep. No. 73-9-CERN, CERN, Geneva (1973).
40. L. D. Faddeev and V. N. Popov, *Phys. Lett. B*, **25**, 29 (1967).
41. V. N. Gribov, *Nucl. Phys. B*, **139**, 246 (1978).
42. I. M. Singer, *Commun. Math. Phys.*, **60**, 7 (1978); M. F. Atiyah and J. D. S. Jones, *Commun. Math. Phys.*, **61**, 97 (1978).
43. S. V. Shabanov, *Phys. Rep.*, **326**, 1 (2000); B. De Witt and C. Molina-Paris, "Quantum gravity without ghosts," hep-th/9808163 (1998).
44. D. J. Gross and F. Wilczek, *Phys. Rev. Lett.*, **30**, 1343 (1973); H. D. Politzer, *Phys. Rev. Lett.*, **30**, 1346 (1973).
45. Yu. L. Dokshitser, D. I. Dyakonov, and S. I. Troyan, *Phys. Rep.*, **58**, 211 (1980); E. Kuraev, J. Manjavidze, and A. Sissakian, "Multiplicity distribution tails at high energies," hep-ph/0003093 (2000).
46. K. Konishi, A. Ukawa, and G. Veneziano, *Phys. Lett. B*, **80**, 259 (1979); A. Basseto, M. Ciafaloni, and G. Marchesini, *Nucl. Phys. B*, **163**, 477 (1980).
47. D. V. Shirkov, "Analytic perturbation theory for QCD observables," hep-ph/0012283 (2000).
48. V. I. Zakharov, *Progr. Theor. Phys. Suppl.*, **131**, 107 (1998); F. V. Gubarev, M. I. Polikarpov, and V. I. Zakharov, "Physics of the power corrections in QCD," hep-ph/9908292 (1999).
49. V. E. Korepin and L. D. Faddeev, *Theor. Math. Phys.*, **25**, 1039 (1975).
50. I. D. Mandzhavidze and A. N. Sissakian, *Theor. Math. Phys.*, **123**, 776 (2000).
51. J. Manjavidze, *J. Math. Phys.*, **41**, 5710 (2000); hep-th/0104252 (2001).
52. J. Manjavidze and A. Sissakian, *J. Math. Phys.*, **42**, 641 (2001); hep-ph/0104297 (2001).
53. J. Manjavidze and A. Sissakian, "Yang–Mills fields quantization in the factor space," *J. Math. Phys.* (to appear); hep-ph/0104298 (2001).
54. L. D. Faddeev and V. E. Korepin, *Phys. Rep. C*, **42**, 1 (1978).
55. R. Dashen, B. Hasslacher, and A. Neveu, *Phys. Rev. D*, **10**, 4114, 4130 (1974).
56. J. Manjavidze, *Sov. J. Nucl. Phys.*, **45**, 442 (1987).
57. R. Jackiw, *Rev. Mod. Phys.*, **49**, 681 (1977).
58. N. Bogoliubov and S. Tyablikov, *Zh. Eksp. Teor. Fiz.*, **19**, 256 (1949).
59. L. A. Takhtadzhyan and L. D. Faddeev, *The Hamiltonian Methods in the Theory of Solitons* [in Russian], Nauka, Moscow (1986); English transl.: L. D. Faddeev and L. A. Takhtajan, Berlin, Springer (1987).
60. L. D. Faddeev and L. A. Takhtadzhyan, *Usp. Mat. Nauk*, **29**, 249 (1974); R. Jackiw, C. Nohl, and C. Rebbi, "Classical an semi-classical solutions of the Yang–Mills theory," in: *Particles and Fields* (Proc., Banff, Canada, 25 Aug.–3 Sept., 1977, D. H. Boal and A. N. Kamal, eds.), Plenum, New York (1978).
61. G. W. Mackey, *Induced Representation of Groups and Quantum Mechanics*, Benjamin, New York (1969); N. Woodhouse, *Geometric Quantization*, Oxford Univ. Press, Oxford (1980); N. P. Landsman and N. Linden, *Nucl. Phys. B*, **365**, 121 (1991); N. P. Landsman, *Rev. Math. Phys.*, **2**, 45, 73 (1991); C. J. Isham, *Relativity, Groups, and Topology: II*, North-Holland, Amsterdam (1984).
62. R. Rajaraman, *Solitons and Instantons: An Introduction to Solitons and Instantons in Quantum Field Theory*, North-Holland, Amsterdam (1982); A. B. Zamolodchikov and A. B. Zamolodchikov, *Phys. Lett. B*, **72**, 503 (1978).

63. S. F. Edwards and Y. V. Gulyaev, *Proc. Roy. Soc. London A*, **279**, 229 (1964); M. S. Marinov, *Phys. Rep.*, **60**, 1 (1980).
64. B. S. De Witt, *Rev. Mod. Phys.*, **29**, 377 (1957).
65. V. De Alfaro, S. Fubini, and G. Furlan, *Phys. Lett. B*, **65**, 163 (1976); V. De Alfaro, S. Fubini, and G. Furlan, *Acta Phys. Aus. Suppl.*, **22**, 51 (1980).
66. T. Bibilashvili and I. Pasiashvili, *Ann. Phys.*, **220**, 134 (1992).
67. P. Kadanoff and P. C. Martin, *Ann. Phys.*, **24**, 419 (1963).
68. J. Schwinger, *Particles, Sources, and Fields*, Vol. 1, Addison–Wesley, Reading, Mass. (1970).
69. L. Landau and R. Peierls, *Z. Phys.*, **69**, 56 (1931).
70. R. Mills, *Propagators for Many-Particle Systems*, Gordon and Breach, New York (1970).
71. N. P. Landsman and C. G. van Weert, *Phys. Rep.*, **145**, 141 (1987); E. Calsetta and B. L. Hu, *Phys. Rev. D*, **37**, 2878 (1988).
72. E. Byukling and K. Kajantie, *Particle Kinematics*, Wiley, London (1973).
73. M. Martin and J. Schwinger, *Phys. Rev.*, **115**, 342 (1959).
74. N. N. Bogolyubov, *Physica*, **26**, S1 (1960).
75. A. J. Niemi and G. Semenoff, *Ann. Phys.*, **152**, 105 (1984).
76. J. Schwinger, *J. Math. Phys. A*, **9**, 2363 (1994).
77. P. M. Bakshi and K. T. Mahanthappa, *J. Math. Phys.*, **4**, 1, 12 (1961).
78. J. Manjavidze and A. Sissakian, *JINR Rapid Commun.*, **2/281**, 13 (1988).
79. S. Smale, *Invent. Math.*, **11**, No. 1, 45 (1970); R. Abraham and J. E. Marsden, *Foundations of Mechanics*, Benjamin, Reading, Mass. (1978).
80. V. I. Arnold, *Mathematical Methods of Classical Mechanics* [in Russian] (3rd ed.), Nauka, Moscow (1989); English transl. prev. ed., Springer, Berlin (1978).
81. A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, *Phys. Lett. B*, **59**, 85 (1975).
82. A. M. Polyakov, *Nucl. Phys. B*, **120**, 429 (1977).
83. B. M. Barbashov, *JETP*, **21**, 402 (1965).
84. A. Actor, *Rev. Mod. Phys.*, **51**, 461 (1979).
85. L. S. Boya, J. F. Carinena, and J. Mateos, *Fortschr. Phys.*, **26**, 175 (1978).
86. L. V. Gribov, E. M. Levin, and M. G. Ryskin, *Phys. Rep. C*, **100**, 1 (1983).
87. C. Grosche, *Path Integrals, Hyperbolic Spaces, and Selberg Trace Formulae*, World Scientific, Singapore (1995).
88. P. Glansdorff and I. Prigogine, *Thermodynamic Theory of Structure, Stability, and Fluctuations*, Wiley, London (1971).

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Contents