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UNCONSTRAINED *SU*(2) YANG–MILLS THEORY WITH TOPOLOGICAL TERM IN THE LONG-WAVELENGTH APPROXIMATION

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1 Introduction

For a complete understanding of the low-energy quantum phenomena of Yang-Mills theory, it is necessary to have a nonperturbative, gauge invariant description of the underlying classical theory including the θ-dependent Pontryagin term [1]-[4]. Several representations of Yang-Mills theory in terms of local gauge invariant fields have been proposed [5]-[24] during the last decades, implementing Gauss law as a generator of small gauge transformations. However, dealing with such local gauge invariant fields special consideration is needed, when the topological term is included, since it is the 4-divergence of a current changing under large gauge transformations. In particular, consistency of constrained and unconstrained formulations of gauge theories with topological term requires to verify that, after projection to the reduced phase space, the classical equations of motion for the unconstrained variables remain θ-independent[1]. Furthermore the question, which trace the large gauge transformations with nontrivial Pontryagin topological index leave on the local gauge invariant fields, has to be addressed.

Having this in mind, we extend in the present paper our approach [22, 27, 28], *to construct the unconstrained form of $SU(2)$ Yang-Mills theory*, to the case when the topological term is included in the classical action. We generalize the Hamiltonian reduction of classical $SU(2)$ Yang-Mills field theory to arbitrary θ-angle by reformulating the original degenerate Yang-Mills theory as a nonlocal theory of a self-interacting symmetric second-rank tensor field. The consistency of the Hamiltonian reduction in the presence of the Pontryagin term is demonstrated by constructing the canonical transformation, well-defined on the reduced phase space, that eliminates the θ-dependence of the classical equations of motion for the unconstrained variables.

With the aim to obtain a practical form of the nonlocal unconstrained Hamiltonian, we perform an expansion in powers of the inverse coupling constant, equivalent to an expansion in the number of spatial derivatives. We find that a straightforward application of the derivative expansion violates the principle of θ-independence of the classical observables. To cure this problem, we propose to exploit the property of chromoelectro-magnetic duality of pure Yang-Mills theory, symmetry under exchange of chromoelectric and -magnetic fields. The electric and magnetic fields are subject to dual constraints, the Gauss-law and Bianchi identity, and only when both are fulfilled, the classical equations of motion are θ-independent. Thus any approximation in resolving the Gauss law constraints should be consistent with the Bianchi identity. We show how to use the Bianchi identity to rearrange the derivative expansion in such a way, that the θ-independence is restored to all orders on the classical level.

[1]The question of consistency of the elimination of redundant variables in theories containing both constraints and pure divergencies, the so-called "divergence problem", has for the first time been analyzed in the context of the canonical reduction of General Relativity by P. Dirac [25] and by R. Arnowitt, S. Deser, C.W. Misner [26].

In order to have a representation of the gauge invariant degrees of freedom suitable for the study of the low energy phase of Yang-Mills theory, we perform a main-axis transformation of the symmetric tensor field and obtain the unconstrained Hamiltonian in terms of the main-axis variables in the lowest order in $1/g$. Carrying out an inverse Legendre transformation to the corresponding unconstrained Lagrangian, we find the explicit form of the unconstrained analog of the Chern-Simons current, linear in the derivatives.

Finally we expand the action around the minimum of the classical potential and derive an effective classical theory of a unit vector field interacting with a scalar field. Using typical boundary conditions for the unit-vector field at spatial infinity, the Pontryagin topological charge density reduces to the Abelian Chern-Simons invariant density [4]. We discuss its relation to the Hopf number of the mapping from the 3-sphere $\mathbb{S}^3$ to the unit 2-sphere $\mathbb{S}^2$ in the Whitehead representation[29]. The Abelian Chern-Simons invariant is known from different areas in Physics, in fluid mechanics as "fluid helicity", in plasma physics and magnetohydrodynamics as "magnetic helicity" [30]-[33]. In the context of 4-dimensional Yang-Mills theory a connection between non-Abelian vacuum configurations and certain Abelian fields with nonvanishing helicity established already in [34, 35].

The paper is organized as follows. In Section II the θ-independence of classical Yang-Mills theory in the framework of the constrained Hamiltonian formulation is revised. Section III is devoted to the derivation of unconstrained $SU(2)$ Yang-Mills theory for arbitrary θ-angle. The consistency of our reduction procedure is demonstrated by explicitly quoting the canonical transformation, which removes the θ-dependence from the unconstrained classical theory. In Section IV the unconstrained Hamiltonian up to order $o(1/g)$ is obtained. Section V presents the long-wavelength classical Hamiltonian in terms of main-axis variables. Performing an inverse Legendre transformation to the corresponding Lagrangian up to second order in derivatives, the unconstrained analog of the Chern-Simons current, linear in the derivatives, is obtained. In Section VI, the unconstrained action is expanded around the minimum of the classical potential, and a non-linear σ-model type effective theory of a unit vector field coupled to a scalar field with an Abelian Chern-Simons term obtained. Section VII finally gives our conclusions.

2 Constrained Hamiltonian formulation

Yang-Mills gauge fields are classified topologically by the value of Pontryagin index[2]

$$p_1 = -\frac{1}{8\pi^2}\int \operatorname{tr} F \wedge F\,. \tag{1}$$

[2]Necessary notations and definitions for $SU(2)$ Yang-Mills theory used in the text have been collected in Appendix A.

Its density, the so-called topological charge density $Q = -(1/8\pi^2)\,\mathrm{tr}\, F \wedge F$, being locally exact $Q = dC$, can be added to the conventional Yang-Mills Lagrangian with arbitrary parameter θ

$$\mathcal{L} = -\frac{1}{g^2}\,\mathrm{tr}\, F \wedge {}^*F - \frac{\theta}{8\pi^2 g^2}\,\mathrm{tr}\, F \wedge F\,, \tag{2}$$

without changing the classical equations of motion. In the Hamiltonian formulation, this shifts the canonical momenta, conjugated to the field variables A_{ai},

$$\Pi_{ai} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{ai}} = \dot{A}_{ai} - (D_i(A))_{ac}\,A_{c0} + \frac{\theta}{8\,\pi^2}\,B_{ai}\,, \tag{3}$$

by the magnetic field $(\theta/8\,\pi^2)\,B_{ai}$. As a result, the total Hamiltonian [36, 37] of Yang-Mills theory with θ-angle as a functional of canonical variables (A_{a0}, Π_a) and (A_{ai}, Π_{ai}) obeying the Poisson bracket relations

$$\{A_{ai}(t,\vec{x})\,, \Pi_{bj}(t,\vec{y})\} = \delta_{ab}\,\delta_{ij}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \tag{4}$$
$$\{A_{a0}(t,\vec{x})\,, \Pi_{b}(t,\vec{y})\} = \delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \tag{5}$$

takes the form

$$H_T = \int d^3x \left[\frac{1}{2}\left(\Pi_{ai} - \frac{\theta}{8\pi^2}B_{ai}\right)^2 + \frac{1}{2}\,B_{ai}^2 - A_{a0}\,(D_i(A))_{ac}\,\Pi_{ci} + \lambda_a\,\Pi_a\right]\,. \tag{6}$$

Here, the linear combination of three primary constraints

$$\Pi_a(x) = 0\,, \tag{7}$$

with arbitrary functions $\lambda_a(x)$ and the secondary constraints, the non-Abelian Gauss law

$$(D_i(A))_{ac}\,\Pi_{ci} = 0 \tag{8}$$

reflect the gauge invariance of the theory.

Based on the representation (6) for the total Hamiltonian, one can immediately verify that classical theories with different value of the θ-angle are equivalent. Performing the canonical transformation

$$\begin{aligned} A_{ai}(x) &\longmapsto A_{ai}(x)\,, \\ \Pi_{bj}(x) &\longmapsto E_{bj} := \Pi_{bj}(x) - \frac{\theta}{8\pi^2}\,B_{bj}(x)\,, \end{aligned} \tag{9}$$

to the new variables A_{ai} and E_{bj}, and using the Bianchi identity

$$(D_i(A))_{ab}\,B_{bi}(A) = 0\,, \tag{10}$$

one can then see that the θ-dependence completely disappears from the Hamiltonian (6). Note that the canonical transformation (9) can be represented in the form

$$E_{ai} = \Pi_{ai} - \theta \frac{\delta}{\delta A_{ai}} W[A] , \tag{11}$$

where $W[A]$ denotes the winding number functional,

$$W[A] = \int d^3x \, K^0[A] , \tag{12}$$

constructed from the zero component of the Chern-Simons current

$$K^\mu[A] = -\frac{1}{16\,\pi^2} \varepsilon^{\mu\alpha\beta\gamma} \operatorname{tr} \left(F_{\alpha\beta}\, A_\gamma - \frac{2}{3} A_\alpha\, A_\beta\, A_\gamma \right) . \tag{13}$$

The question now arises, whether, after reduction of Yang-Mills theory including topological term to the unconstrained system, a transformation analogous to (9) can be found, that correspondingly eliminates any θ-dependence on the reduced level, proving the consistency of the Hamiltonian reduction.

3 Unconstrained Hamiltonian formulation

3.1 Hamiltonian reduction for arbitrary θ-angle

In order to derive the unconstrained form of $SU(2)$ Yang Mills theory with θ-angle we follow the method developed in [22] . We perform the point transformation

$$A_{ai}\,(q,S) = O_{ak}(q)\, S_{ki} + \frac{1}{2g}\, \varepsilon_{abc} \, \left(\partial_i O(q)\, O^T(q) \right)_{bc} \tag{14}$$

from the gauge fields $A_{ai}(x)$ to the new set of three fields $q_j(x)$, $j = 1,2,3$, parameterizing an orthogonal 3×3 matrix $O(q)$ and the six fields $S_{ik}(x) = S_{ki}(x)$, $i,k = 1,2,3$, collected in the positive definite symmetric 3×3 matrix $S(x)$. Eq. (14) can be seen as a gauge transformation to new field configuration $S(x)$ which satisfy the "symmetric gauge" condition $\varepsilon_{abc}\, S_{bc} = 0$[3]. The transformation (14) induces a point canonical transformation linear in the new momenta $P_{ik}(x)$ and $p_i(x)$, conjugated to $S_{ik}(x)$ and $q_i(x)$, respectively. Their expressions in terms of the old variables $(A_{ai}(x)\,, \Pi_{ai}(x))$ can be obtained from the requirement of the canonical invariance of the symplectic 1-form

$$\sum_{i,a=1}^{3} \Pi_{ai}\, \dot{A}_{ai}\, dt = \sum_{i,j=1}^{3} P_{ij}\, \dot{S}_{ij}\, dt + \sum_{i=1}^{3} p_i\, \dot{q}_i\, dt , \tag{15}$$

[3]The decomposition (14) is a generalization of the well-known polar decomposition valid for arbitrary quadratic matrices used in a similar form in [10].

with the fundamental brackets

$$\{S_{ij}(t,\vec{x})\,,P_{kl}(t,\vec{y})\} = \frac{1}{2}\,(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk})\;\delta^{(3)}(\vec{x}-\vec{y})\,, \tag{16}$$

$$\{q_i(t,\vec{x})\,,p_j(t,\vec{y})\} = \delta_{ij}\,\delta^{(3)}(\vec{x}-\vec{y})\;, \tag{17}$$

for the new canonical pairs $(S_{ij}(x)\,,P_{ij}(x))$ and $(q_i(x)\,,p_i(x))$. The brackets (16) account for the second class symmetry-constraints $S_{ij} = S_{ji}$ and $P_{ij} = P_{ji}$ and therefore are Dirac brackets. As result we obtain the expression

$$\Pi_{ai} = O_{ak}(q)\left[\,P_{\,ki} + g\,\varepsilon_{kin}{}^{*}D^{-1}_{nm}(S)\,\left(\mathcal{S}_m - \Omega^{-1}_{jm}p_j\right)\right]\,, \tag{18}$$

of the old momenta Π_{ai} in terms of the new canonical variables, (for a detailed derivation see [22]). Here ${}^{*}D^{-1}_{mn}(S)$ denotes the inverse of the dual covariant derivative,

$${}^{*}D_{mn}(S) = \varepsilon_{njc}\,(D_j(S))_{mc}\;, \tag{19}$$

the vector $\mathcal{S}$ is defined as

$$\mathcal{S}_m = \frac{1}{g}\,(D_j(S))_{mn}P_{nj}\;, \tag{20}$$

and the matrix Ω^{-1} the inverse of

$$\Omega_{ni}(q) \;:=\; -\frac{1}{2}\,\varepsilon_{nbc}\left(O^T(q)\,\frac{\partial O(q)}{\partial q_j}\right)_{bc}. \tag{21}$$

The main advantage of introducing the variables S_{ij} and q_i is, that they Abelianise the non-Abelian Gauss law constraints (8). In terms of the new variables the Gauss's law constraints

$$g\,O_{as}(q)\,\Omega^{-1}_{is}(q)\,p_i = 0\,, \tag{22}$$

depend only on (q_i, p_i), showing that the variables $(S_{ij}\,,P_{ij})$ are gauge-invariant, physical fields. Hence the reduced Hamiltonian, defined as the projection of the total Hamiltonian onto the constraint shell, can be obtained from (6) by imposing the equivalent set of Abelian constraints

$$p_i = 0\,. \tag{23}$$

Due to gauge invariance, the reduced Hamiltonian is independent of the coordinates q_i canonically conjugated to p_i and is hence a function of the unconstrained gauge-invariant variables S_{ij} and P_{ij} only

$$H = \int d^3x\left[\frac{1}{2}\left(P_{ai} - \frac{\theta}{8\pi^2}\,B^{(+)}_{ai}(S)\right)^2 + \left(P_a - \frac{\theta}{8\pi^2}\,B^{(-)}_{a}(S)\right)^2 + \frac{1}{2}\,V(S)\right]. \tag{24}$$

Here the P_a denotes the nonlocal functional, according to (18) defined as solution of the system of differential equations

$$ {}^*D_{ks}(S)P_s = (D_j(S))_{kn}P_{nj}\,. \tag{25} $$

Furthermore,

$$ B^{(+)}_{ai}(S) := \frac{1}{2}\left(B_{ai}(S) + B_{ia}(S)\right), \qquad B^{(-)}_{a}(S) := \frac{1}{2}\,\varepsilon_{abc}\,B_{bc}(S)\,, \tag{26} $$

denote the symmetric and antisymmetric parts of the reduced chromomagnetic field

$$ B_{ai}(S) = \varepsilon_{ijk}\,\left(\partial_j S_{ak} + \frac{g}{2}\,\varepsilon_{abc}\,S_{bj}\,S_{ck}\right)\,. \tag{27} $$

It is the same functional of the symmetric field S as the original $B_{ai}(A)$, since the chromomagnetic field transforms homogeneously under the change of coordinates (14). Finally the potential $V(S)$ is the square of the reduced magnetic field (27),

$$ V(S) = \frac{1}{2}\,\mathrm{tr}\,{}^*F^{(3)}\,\wedge\,F^{(3)}\,, \tag{28} $$

with the curvature 2-form in 3-dimensional Euclidean space

$$ F^{(3)} = dS + S\,\wedge\,S\,, \tag{29} $$

in terms of the symmetric 1-form

$$ S = g\tau_k\,S_{kl}\,dx_l, \qquad k,l = 1,2,3\,, \tag{30} $$

whose 6 components depend on the time variable as an external parameter. The reduced chromomagnetic field (27) is given in terms of the dual field strength ${}^*F^{(3)}$ as $B_{ai}(S) = \frac{1}{2}\,\varepsilon_{ijk}\,F^{(3)}_{ajk}$.

3.2 Canonical equivalence of unconstrained theories with different θ-angles

For the original degenerate action in terms of the A_μ fields the equivalence of classical theories with arbitrary value of θ-angle has been reviewed in Section 2. Let us now examine the same problem for the derived unconstrained theory considering the analog of the canonical transformation (9) after projection onto the constraint surface,

$$ \begin{aligned} S_{ai}(x) &\longmapsto S_{ai}(x)\,, \\ P_{bj}(x) &\longmapsto \mathcal{E}_{bj}(x) := P_{bj}(x) - \frac{\theta}{8\,\pi^2}\,B^{(+)}_{bj}(x)\,. \end{aligned} \tag{31} $$

One can easily check that this transformation to new variables S_{ai} and $\mathcal{E}_{bj}$ is canonical with respect to the Dirac brackets (16). In terms of the new variables S_{ai} and $\mathcal{E}_{bj}$ the Hamiltonian (24) can be written as

$$H = \int d^3x \left[\frac{1}{2}\,\mathcal{E}_{ai}^2 + \mathcal{E}_a^2 + \frac{1}{2}\,V(S) \right], \tag{32}$$

with $\mathcal{E}_a$ defined as

$$\mathcal{E}_a := P_a - \frac{\theta}{8\,\pi^2}\,B_a^{(-)}\,. \tag{33}$$

Now, if P_a is a solution of equation (25), then $\mathcal{E}_a$ is a solution of the same equation

$$^*\!D_{ks}(S)\mathcal{E}_s = (D_j(S))_{kn}\mathcal{E}_{nj}\,, \tag{34}$$

with the replacement $P_{ai} \longmapsto \mathcal{E}_{ai}$, since the reduced field B_{ai} satisfies the Bianchi identity

$$(D_i(S))_{ab}\,B_{bi}(S) = 0\,. \tag{35}$$

Hence we arrive at the same unconstrained Hamiltonian system (32) and (34) with vanishing θ-angle. Note that after the elimination of the three unphysical fields $q_j(x)$ the projected canonical transformation (31) that removes the θ-dependence from the Hamiltonian can be written as

$$\mathcal{E}_{bj}(x) = P_{bj}(x) - \theta\,\frac{\delta}{\delta S_{bj}}\,W[S]\,, \tag{36}$$

which is of the same form as (11) with the nine gauge fields $A_{ik}(x)$ replaced by the six unconstrained fields $S_{ik}(x)$.

In summary, the exact projection to reduced phase space leads to an unconstrained system, whose equations of motion are consistent with the original degenerate theory in the sense that they are θ-independent. Thus if our consideration is restricted only to the classical level of the exact nonlocal unconstrained theory, the generalization to arbitrary θ-angle can be avoided[4]. However, in order to work with such a complicated nonlocal Hamiltonian it is necessary to make approximations, such as for example expansion in the number of spatial derivatives, which we shall carry out in the next section. For these one has to check that this approximation is free of the "divergence problem", that is all terms in the corresponding truncated action containing the θ-angle can be collected into a 4-divergence and all dependence on θ disappears from the classical equations of motion.

4 Expansion of the unconstrained Hamiltonian in $1/g$

Let us now consider the regime when the unconstrained fields are slowly varying in space-time and expand the nonlocal part of the kinetic term in the unconstrained

[4]The extension of the proof of θ-independence to quantum theory requires to show the unitarity of the operator corresponding to transformation (31).

Hamiltonian (24) as a series of terms with increasing powers of inverse coupling constant $1/g$, equivalent to an expansion in the number of spatial derivatives of field and momentum. Our expansion is purely formal and we shall in this work not study the question of its convergence. We shall see, that for nonvanishing θ-angle, a straightforward expansion in $1/g$ leads to the above mentioned "divergence problem", and suggest an improved form of the expansion in $1/g$ of the unconstrained Hamiltonian exploiting the Bianchi identity.

4.1 Divergence problem in lowest-order approximation

According to [22], the nonlocal funtional P_a in the unconstrained Hamiltonian (32), defined as solution of the system of linear differential equations (25), can formally be expanded in powers of $1/g$. The vector P_a is then given as a sum of terms containing an increasing number of spatial derivatives of field and momentum

$$P_s(S,P) = \sum_{n=0}^{\infty} (1/g)^n \, a_s^{(n)}(S,P) \,. \tag{37}$$

The zeroth-order term is

$$a_s^{(0)} = \gamma_{sk}^{-1} \varepsilon_{klm} \, (PS)_{lm} \,, \tag{38}$$

with $\gamma_{ik} := S_{ik} - \delta_{ik} \operatorname{tr} S$, and the first-order term is determined as

$$a_s^{(1)} = -\,\gamma_{sl}^{-1} \, \left[(\mathrm{rot}\ \vec{a}^{(0)})_l + \partial_k P_{kl} \right] \tag{39}$$

from the zeroth-order term. The higher terms are then obtained by the simple recurrence relations

$$a_s^{(n+1)} = -\,\gamma_{sl}^{-1} (\mathrm{rot}\, \vec{a}^{\,(n)})_l \,. \tag{40}$$

Inserting these expressions into (24) we obtain the corresponding expansion of unconstrained Hamiltonian as a series in higher and higher numbers of derivatives.

Let us check, whether the truncation of the expansion (37) to lowest order is consistent with θ-independence, that is, whether all θ-dependent terms can be collected into 4-divergence after Legendre transformation to the corresponding Lagrangian. In $o(1/g)$ approximation (38), the Hamiltonian reads[5]

$$H^{(2)} = \int d^3x \left[\frac{1}{2} \mathrm{tr} \left(P - \frac{\theta}{8\pi^2} B^{(+)} \right)^2 + \left(a_s^{(0)}(S,P) - \frac{\theta}{8\pi^2} B_s^{(-)} \right)^2 + \frac{1}{2} V(S) \right], \tag{41}$$

where $M = SP - PS$ is the spin part of the angular momentum tensor of the gluon field and $B^{(+)}$ and $B^{(-)}$ denote the symmetric and antisymmetric parts of the chromomagnetic field, defined in (26).

[5] When all spatial derivatives of the fields and momenta are neglected, Yang-Mills theory reduces to the so-called Yang-Mills mechanics and its θ-independence has been shown in [27].

After inverse Legendre transformation of the Hamiltonian (41), the θ-dependent terms in the corresponding Lagrangian cannot be collected to a total 4-divergence, as is shown in Appendix B, and therefore contribute to the unconstrained equations of motion. Hence applying a straightforward derivative expansion to the Yang-Mills theory with topological term after projection to reduced phase space we face the "divergence problem" dicussed above.

4.2 Improved $1/g$ expansion using the Bianchi identity

In order to avoid the "divergence problem" one can proceed as follows. Let us consider additionally to the differential equation (25), which determines the non-local term P_a, the Bianchi identity (35) as an equation for determination of the antisymmetric part $B_s^{(-)}$ of the chromomagnetic field

$$ {}^*\!D_{ks}(S)\, B_s^{(-)} = (D_i(S))_{kl}\, B_{li}^{(+)} \,, \tag{42} $$

in terms of its symmetric part $B_{bc}^{(+)}$. The complete analogy of this equation with (25) expresses the duality of chromoelectric and chromagnetic fields on the unconstrained level. Hence one can write

$$ {}^*\!D_{ks}(S) \left[P_s - \frac{\theta}{8\pi^2}\, B_s^{(-)}\right] = (D_i(S))_{kl} \left[P_{li} - \frac{\theta}{8\,\pi^2}\, B_{li}^{(+)}\right] . \tag{43} $$

Using the same type of the spatial derivative expansion as before in (38)-(40), we obtain

$$ P_s - \frac{\theta}{8\,\pi^2}\, B_s^{(-)} = \sum_{n=0}^{\infty} (1/g)^n\, a_s^{(n)}(S, P - \frac{\theta}{8\pi^2}\, B^{(+)})\,. \tag{44} $$

In this way we achieve a form of the derivative expansion such that the unconstrained Hamiltonian is a functional of field combination $P_{ai} - (\theta/8\,\pi^2)\, B_{ai}^{(+)}$

$$ H = \int d^3x \left[\frac{1}{2} \left(P_{ai} - \frac{\theta}{8\pi^2}\, B_{ai}^{(+)}\right)^2 + \left(\sum_{n=0}^{\infty}(1/g)^n\, a_i^{(n)}(S, P - \frac{\theta}{8\pi^2}\, B^{(+)})\right)^2 + \frac{1}{2}\, V(S)\right], \tag{45} $$

explicitly showing the chromoelectro-magnetic duality on the reduced level and hence free of the "divergence problem". To obtain the unconstrained Hamiltonian up to leading order $o(1/g)$, only the lowest term $a_s^{(0)}(S, P - (\theta/8\,\pi^2)\, B^{(+)})$ in the sum in (45) has to be taken into account, so that

$$ H^{(2)} = \frac{1}{2} \int d^3x \left[\operatorname{tr} \left(P - \frac{\theta}{8\pi^2}\, B^{(+)}\right)^2 - \frac{1}{\det^2\gamma} \operatorname{tr} \left(\gamma\, [S, P - \frac{\theta}{8\,\pi^2}\, B^{(+)}]\, \gamma\right)^2 + V(S)\right]. \tag{46} $$

The advantage of this Hamiltonian compared with (41), derived before, is that the classical equations of motion following from (46) are θ-independent. In order to obtain a transparent form of the corresponding surface term in the unconstrained action, it is useful to perform a main-axis transformation of the symmetric second-rank tensor field S.

5 Long-wavelength approximation to reduced theory

In this section we shall at first rewrite the unconstrained Hamiltonian (46) in terms of main-axis variables of the symmetric tensor field S_{ij}. The corresponding second-order Lagrangian $L^{(2)}$ is then obtained via Legendre transformation and the form of the corresponding unconstrained total divergence derived in an explicit way.

5.1 Hamiltonian in terms of main-axis variables

In [22] it was shown, that the field $S_{ij}(x)$ transforms as a second-rank tensor under the spatial rotations. This can be used to explicitly separate the rotational degrees of freedom from the scalars in the Hamiltonian (46). Following [22] we introduce the main-axis representation of the symmetric 3×3 matrix field $S(x)$,

$$S(x) = R^T(\chi(x)) \begin{pmatrix} \phi_1(x) & 0 & 0 \\ 0 & \phi_2(x) & 0 \\ 0 & 0 & \phi_3(x) \end{pmatrix} R(\chi(x)) \,. \tag{47}$$

The Jacobian of this transformation is

$$J\left(\frac{S_{ij}[\phi,\chi]}{\phi_k,\chi_l}\right) \propto \prod_{i\neq j} \mid \phi_i(x) - \phi_j(x) \mid \ , \tag{48}$$

and thus (47) can be used as definition of the new configuration variables, the three diagonal fields ϕ_1, ϕ_2, ϕ_3 and the three angular fields χ_1, χ_2, χ_3, only if all eigenvalues of the matrix S are different. To have the uniqueness of the inverse transformation we assume here that

$$\phi_1(x) < \phi_2(x) < \phi_3(x) \,. \tag{49}$$

The variables ϕ_i in the main axes transformation(47) parameterize the orbits of the action under the $SO(3,\mathbb{R})$ group. The configuration (49) belongs to the so-called principle orbit class, whereas all orbits with coinciding eigenvalues of the matrix S are singular orbits [38].

The momenta π_i and p_{χ_i}, canonical conjugate to the diagonal elements ϕ_i and χ_i, can be found using the condition of the canonical invariance of the symplectic 1-form

$$\sum_{i,j=1}^{3} P_{ij}\, \dot{S}_{ij}\, dt = \sum_{i=1}^{3} \pi_i\, \dot{\phi}_i dt + \sum_{i=1}^{3} p_{\chi_i}\, \dot{\chi}_i dt \,. \tag{50}$$

The original physical momenta P_{ik}, expressed in terms of the new canonical variables, read

$$P(x) = R^T(x) \sum_{s=1}^{3} \left(\pi_s(x)\, \overline{\alpha}_s + \frac{1}{2}\mathcal{P}_s(x)\, \alpha_s \right) R(x) \,. \tag{51}$$

Here $\overline{\alpha}_i$ and α_i denote the diagonal and off-diagonal basis elements for symmetric matrices with the orthogonality relations $\operatorname{tr}(\overline{\alpha}_i\,\overline{\alpha}_j) = \delta_{ij}$, $\operatorname{tr}(\alpha_i\,\alpha_j) = 2\,\delta_{ij}$, $\operatorname{tr}(\overline{\alpha}_i\,\alpha_j) = 0$, and

$$\mathcal{P}_i(x) = -\frac{\xi_i(x)}{\phi_j(x) - \phi_k(x)}\,, \qquad \text{(cyclic permutations}\, i \neq j \neq k)\,. \tag{52}$$

The ξ_i are the three $SO(3,\mathbb{R})$ right-invariant Killing vector fields given in terms of the angles χ_i and their conjugated momenta p_{χ_i} via

$$\xi_i = M^{-1}_{ji}\, p_{\chi_j}\,, \tag{53}$$

where the matrix M is

$$M_{ji} := -\frac{1}{2}\,\varepsilon_{jab}\left(\frac{\partial R}{\partial \chi_i}\,R^T\right)_{ab}\,. \tag{54}$$

The physical chromomagnetic field $B(S)$ can be regarded as the components of the curvature 2-form $F^{(3)}$, defined in terms of the symmetric 1-form S in (29). Starting from the coordinate basis expression of S in (30), we observe that the main-axis transformation (47) corresponds to the representation

$$S = \sum_{a=1}^{3} e_a\,\phi_a\,\omega_a\,, \tag{55}$$

with the 1-form basis elements

$$\omega_i := R_{ij}\,(\chi(x))dx_j\,, \qquad i = 1,2,3 \tag{56}$$

and the $su(2)$ Lie algebra basis elements

$$e_a := R_{ab}(\chi(x))\,\tau_b\,, \qquad a = 1,2,3\,. \tag{57}$$

In this basis the components of the non-Abelian field strength $F^{(3)}$ read

$$F^{(3)}_{aij} = \delta_{aj}\,X_i\,\phi_j - \delta_{ai}\,X_j\,\phi_i + \phi_i\,\Gamma_{aji} - \phi_j\,\Gamma_{aij} + \Gamma_{a[ij]}\,\phi_a + g\varepsilon_{aij}\,\phi_i\,\phi_j\,, \quad \text{(no summation)}\,, \tag{58}$$

with the components of connection 1-form Γ defined as

$$\Gamma_{aib} := \left(X_i R\,R^T\right)_{ab}\,, \tag{59}$$

and the vector fields

$$X_i := R_{ij}\,\partial_j\,, \tag{60}$$

dual to the 1-forms ω_j, $\omega_i(X_j) = \delta_{ij}$, and acting on the basis elements e_a as

$$X_i\,e_a = -\Gamma_{bia}e_b\,. \tag{61}$$

From the expressions (58) we obtain for the potential (28) (see [22] and Erratum [23]),

$$V(\phi,\chi) = \sum_{i\neq j}^{3} \left(\Gamma_{iij}(\phi_i - \phi_j) - X_j\phi_i\right)^2 + \sum_{cyclic} \left(\Gamma_{ijk}(\phi_i - \phi_k) - \Gamma_{ikj}(\phi_i - \phi_k) - g\phi_j\phi_k\right)^2 . \tag{62}$$

The explicit expressions for the diagonal components β_i and the off-diagonal components b_i of the the symmetric part of the chromomagnetic field

$$B^{(+)} = R^T(\chi) \sum_{i=1}^{3} \left(\beta_i \overline{\alpha}_i + \frac{1}{2} b_i \alpha_i\right) R(\chi) , \tag{63}$$

are given in terms of the diagonal fields ϕ_i and the angular fields χ_i in the cyclic form

$$\beta_i = g\phi_j\phi_k - (\phi_i - \phi_j)\Gamma_{ikj} + (\phi_i - \phi_k)\Gamma_{ijk} , \tag{64}$$

$$b_i = X_i(\phi_j - \phi_k) - (\phi_i - \phi_j)\Gamma_{ijj} + (\phi_i - \phi_k)\Gamma_{ikk} . \tag{65}$$

and the antisymmetric part $B_i^{(-)}$ of the unconstrained magnetic field is

$$B_a^{(-)} = \frac{1}{2} R_{ai}^T \left(X_i(\phi_j + \phi_k) + (\phi_j - \phi_i)\Gamma_{ijj} + (\phi_k - \phi_i)\Gamma_{ikk}\right) . \tag{66}$$

The zeroth-order term of the expansion (44) of the combination $P_a - (\theta/8\pi^2)B_a^{(-)}$, finally, reads

$$a_a^{(0)} = -\frac{R_{ai}^T}{2(\phi_j + \phi_k)} \left(\xi_i + \frac{\theta}{8\pi^2}(\phi_j - \phi_k)\, b_i\right) \quad \text{(cyclic permutations}\, i \neq j \neq k) . \tag{67}$$

Altogether, the $o(1/g)$ Hamiltonian (46), as a functional of main-axis variables, becomes

$$H^{(2)} = \frac{1}{2}\int d^3x \left[\sum_{i=1}^{3} \left(\pi_i - \frac{\theta}{8\pi^2}\beta_i\right)^2 + \sum_{cyclic} k_i \left(\xi_i + \frac{\theta}{8\pi^2}(\phi_j - \phi_k)\, b_i\right)^2 + V(\phi,\chi)\right] , \tag{68}$$

with

$$k_i := \frac{\phi_j^2 + \phi_k^2}{(\phi_j^2 - \phi_k^2)^2} , \quad \text{(cyclic permutations } i \neq j \neq k) . \tag{69}$$

The transformation (31), finally, that excludes the θ-dependence from the Hamiltonian (68) reads

$$\pi_i \longmapsto \pi_i + \frac{\theta}{8\pi^2}\beta_i , \qquad \phi_i \longmapsto \phi_i ,$$

$$\xi_i \longmapsto \xi_i - \frac{\theta}{8\pi^2}(\phi_j - \phi_k)\, b_i , \tag{70}$$

in terms of angular and scalar variables, and reduces the Hamiltonian to its expression with zero θ-angle [22]

$$H^{(2)} = \frac{1}{2}\int d^3x \left[\sum_{i=1}^{3} \pi_i^2 + \sum_{cyclic} \xi_i^2 \frac{\phi_j^2 + \phi_k^2}{(\phi_j^2 - \phi_k^2)^2} + V(\phi,\chi)\right] \tag{71}$$

5.2 Second-order unconstrained Lagrangian

We are now ready to derive the Lagrangian up to second-order in derivatives corresponding to the Hamiltonian (68). Carrying out the inverse Legendre transformation,

$$\dot{\phi}_i = \pi_i - \frac{\theta}{8\pi^2}\beta_i\,, \tag{72}$$

$$\dot{\chi}_a = G_{ab}\Big(p_{\chi_b} - \frac{\theta}{8\pi^2}\sum_{cyclic} M^T_{bi}(\phi_j - \phi_k)\, b_i\Big)\,, \tag{73}$$

with the matrix M given in (54), and the 3×3 matrix G

$$G = M^{-1} k M^{-1^T}\,, \tag{74}$$

similar to the diagonal matrix $k = \mathrm{diag}\|k_1, k_2, k_3\|$ with entries k_i of (69), we arrive at the second-order Lagrangian

$$L^{(2)}(\phi,\chi) = \frac{1}{2}\int d^3x \left[\sum_{i=1}^{3} \dot{\phi}_i^2 + \sum_{i,j=1}^{3} \dot{\chi}_i G^{-1}_{ij}\dot{\chi}_j - V(\phi,\chi)\right] - \theta\int d^3x\; Q^{(2)}(\phi,\chi)\,, \tag{75}$$

with all θ-dependence gathered in the reduced topological charge density

$$Q^{(2)} = \frac{1}{8\pi^2}\sum_{a=1}^{3}\left(\dot{\phi}_a\beta_a + \sum_{cyclic}^{i,j,k} \dot{\chi}_a M^T_{ai}(\phi_j - \phi_k)\, b_i\right)\,. \tag{76}$$

Using the Maurer-Cartan structure equations for the 1-forms ω_i

$$d\omega_a = \Gamma_{a0c} dt \wedge \omega_c + \Gamma_{abc}\omega_b \wedge \omega_c\,, \tag{77}$$

with the space components of Γ given in (59), and the time components correspondingly defined as

$$\Gamma_{a0b} = \left(\dot{R}R^T\right)_{ab}\,, \tag{78}$$

Eq. (76) can be rewritten as

$$Q^{(2)} = dC^{(2)}\,, \tag{79}$$

with the 3-form

$$
\begin{aligned}
C^{(2)} &= \frac{1}{8\pi^2}\sum_{a<b}^{3}(\phi_a-\phi_b)^2\,\Gamma_{a0b}dt\wedge\omega_a\wedge\omega_b - \\
&\quad -\frac{3}{8\pi^2}\sum_{cyclic}^{3}\left[(\phi_a-\phi_b)^2\,\Gamma_{acb}-\frac{2}{3}\varepsilon_{abc}\phi_1\phi_2\phi_3\right]\omega_a\wedge\omega_b\wedge\omega_c\;. \quad (80)
\end{aligned}
$$

This completes our construction of the second-order Lagrangian with all θ-contributions gathered in a total differential (76). The $Q^{(2)}$ in the effective Lagrangian (75) can be represented as the divergence

$$
Q^{(2)} = \partial^\mu K^{(2)}_\mu\,, \tag{81}
$$

of the 4-vector $K^{(2)\mu} = (K^{(2)}_0, K^{(2)}_i)$, with the components

$$
\begin{aligned}
K^{(2)}_0 &= \frac{1}{16\pi^2}\left[(\phi_2-\phi_3)^2\Gamma_{213}+(\phi_3-\phi_1)^2\Gamma_{321}+(\phi_1-\phi_2)^2\Gamma_{132}-2g\phi_1\phi_2\phi_3\right] (82)\\
K^{(2)}_i &= \frac{1}{16\pi^2}\left[R^T_{i1}(\phi_2-\phi_3)^2\Gamma_{203}+R^T_{i2}(\phi_3-\phi_1)^2\Gamma_{301}+R^T_{i3}(\phi_1-\phi_2)^2\Gamma_{102}\right]\;. (83)
\end{aligned}
$$

Thus we have found the unconstrained analog of the Chern-Simons current $K^{(2)}_\mu$, linear in the derivatives. Under the assumption, that the vector part $K^{(2)}_i$ vanishes at spatial infinity, the unconstrained form of the Pontryagin index p_1 can be represented as the difference of the two surface integrals

$$
W_\pm = \int d^3x\; K^{(2)}_0\,(t\to\pm\infty,\vec{x})\;, \tag{84}
$$

which are the winding number functional (12) for the physical field S in terms of main-axis variables (55) at $t\to\pm\infty$ respectively, since $K^{(2)}_0(\phi,\chi)$ of (82) coincides with the full $K_0[S[\phi,\chi]]$ of (13). In the next Section we shall show, how for certain field configurations, it reduces to the Hopf number of the mapping from the 3-sphere $\mathbb{S}^3$ to the unit 2-sphere $\mathbb{S}^2$.

6 Nonlinear σ-type model with Hopf invariant as infinite coupling limit of $SU(2)$ Yang-Mills theory

Following [22], let us consider the behavior of the classical system for the configurations that correspond to the minima of the homogeneous potential

$$
V^{(0)} = g^2\sum_{i\neq j}\phi_i^2(x)\phi_j^2(x)\;, \tag{85}
$$

which is the zeroth-order term of the derivative expansion for the potential term in the Hamiltonian (68). The stationary points of the potential (85) are the three field configurations $\phi_i = \phi_j = 0, \quad \phi_k \ -$ arbitrary $(i \neq j \neq k)$, each of them forming a continuous line, a "valley", of degenerate absolute minima at zero energy. As mentioned in Section 5.1, these configurations correspond to singular orbits of the $SO(3,\mathbb{R})$ group action, whereas, in our consideration above, we have restricted ourselves to the principle orbits $\phi_1 < \phi_2 < \phi_3$. In order to consider the contribution from the singular configuration $\phi_i = \phi_j = 0$, it is in principle necessary to investigate the dynamics on singular orbits using a decomposition of the S field different from the main-axis transformation (47). Instead of this, we shall use here the fact, that the singular orbits can be regarded as the boundary of the principle orbits and find the corresponding dynamics using a certain limiting procedure. [6]

Suppose now that the classical system spontaneously chooses one of the zero energy minima of the potential (85) with two scalar fields vanishing,

$$\phi_1(x) = \phi_2(x) = 0 \ , \quad \text{and} \qquad \phi_3(x) \quad \text{arbitrary} . \tag{86}$$

For the classical vacuum configuration (86), the potential term in the second order Hamiltonian (68) reduces to the expression [22, 23]

$$\begin{aligned} V^{(2)} = \quad & \phi_3^2\left[(\Gamma_{213})^2 + (\Gamma_{223})^2 + (\Gamma_{233})^2 + (\Gamma_{311})^2 + (\Gamma_{321})^2 + (\Gamma_{331})^2 + (\Gamma_{3[12]})^2\right] \\ & + \left[(X_1\phi_3)^2 + (X_2\phi_3)^2\right] + 2\phi_3\left[\Gamma_{331}X_1\phi_3 + \Gamma_{332}X_2\phi_3\right] , \end{aligned} \tag{87}$$

which can be rewritten as [22, 23]

$$V^{(2)} = (\nabla\phi_3)^2 + \phi_3^2\left[(\partial_i \mathbf{n})^2 + (\mathbf{n}\cdot \text{rot } \mathbf{n})^2\right] - (\mathbf{n}\cdot\nabla\phi_3)^2 + ([\mathbf{n}\times \text{rot } \mathbf{n}]\cdot\nabla\phi_3^2) \ , \tag{88}$$

introducing the unit vector

$$n_i(x) := R_{3i}(\chi(x)) . \tag{89}$$

Hence the unconstrained second-order Lagrangian (75) reduces to the nonlinear σ-model type effective Lagrangian

$$\begin{aligned} L^{(2)}_{\text{Eff}} \ = \ & \frac{1}{2}\int d^3x\Big[(\partial_\mu\phi_3)^2 + \phi_3^2(\partial_\mu \mathbf{n})^2 - \phi_3^2(\mathbf{n}\cdot\text{rot } \mathbf{n})^2 + (\mathbf{n}\cdot\nabla\phi_3)^2 \\ & \qquad\qquad - ([\mathbf{n}\times\text{rot } \mathbf{n}]\cdot\nabla\phi_3^2)\Big] - \theta\int d^3x \ Q^{(2)} , \end{aligned} \tag{90}$$

for the unit vector $\mathbf{n}(x)$ coupled to the scalar field $\phi_3(x)$. The $Q^{(2)}$ in the effective Lagrangian (90) can be represented as the divergence

$$Q^{(2)} = \partial^\mu K^{(2)}_\mu \tag{91}$$

[6] Note that for the study of the limit $\phi_i, \phi_j \to 0$ for $(i, j \neq k)$ in the Hamiltonian formalism, the conditions that follow from the dynamical invariance of the singular orbits, have to be taken into account. In particular, it is obvious from the representation (71) of the unconstrained Hamiltonian, that it is necessary to have $\xi_k \to 0$ for some fixed k, in order to obtain a finite contribution of the kinetic term to the effective Hamiltonian in the limit $\phi_i, \phi_j \to 0$ for $(i, j \neq k)$.

of the 4-vector

$$K^{(2)\mu} = \frac{1}{16\pi^2}\phi_3^2\left((\mathbf{n}(x)\cdot \mathrm{rot}\ \mathbf{n}(x)),\ [\mathbf{n}(x)\times \dot{\mathbf{n}}(x)]\right) . \tag{92}$$

If we impose the usual boundary condition that the field $\mathbf{n}$ becomes time-independent at spatial infinity, the contribution from the vector part $K_i^{(2)}$ vanishes and the unconstrained form of the Pontryagin topological index p_1 can be represented as the difference

$$p_1 = n_+ - n_- \tag{93}$$

of the surface integrals

$$n_\pm = \frac{1}{16\pi^2}\int d^3x\,(\mathbf{V}_\pm(\vec{x})\cdot \mathrm{rot}\ \mathbf{V}_\pm(\vec{x})) \tag{94}$$

of the fields

$$\mathbf{V}_\pm(\vec{x}) := \lim_{t\to\pm\infty}\ \phi_3(x)\mathbf{n} . \tag{95}$$

We shall show now that the surface integrals (94) are Hopf invariants in the representation of Whitehead [29].

Under the Hopf mapping of a 3-sphere to a 2-sphere having unit radius, $N:\ \mathbb{S}^3\to\mathbb{S}^2$, the preimage of a point on $\mathbb{S}^2$ is a closed loop. The number Q_H of times, the loops corresponding to two distinct points on $\mathbb{S}^2$ are linked to each other, is the so-called Hopf invariant. According to Whitehead [29], this linking number can be represented by the integral

$$Q_H = \frac{1}{32\pi^2}\int_{S^3} w^1\wedge w^2\,, \tag{96}$$

with the so-called Hopf 2-form curvature $w^2 = H_{ij}dx^i\wedge dx^j$ given in terms of the map N as

$$H_{ij} = \varepsilon_{abc}N_a\,(\partial_i N_b)\,(\partial_j N_c)\ , \tag{97}$$

and the 1-form w^1 related to it via $w^2 = dw^1$. Since the curvature H_{ij} is divergence-free,

$$\varepsilon_{ijk}\partial_i H_{jk} = 0\ , \tag{98}$$

it can be represented as the rotation

$$H_{ij} = \partial_i\mathcal{A}_j - \partial_j\mathcal{A}_i\ , \tag{99}$$

in terms of some vector field $\mathcal{A}_i$ $(i = 1,2,3)$ defined over the whole of $\mathbb{S}^3$. Thus the Hopf invariant takes the form

$$Q_H = \frac{1}{16\pi^2}\int d^3x\,(\mathcal{A}\cdot \mathrm{rot}\ \mathcal{A})\ . \tag{100}$$

Therefore, the surface integrals (94) are just Hopf invariants in the Whitehead representation (100) and the unconstrained form of the topological term $Q^{(2)}$ is an

3-dimensional Abelian Chern-Simons term [4] with "potential" V_i and the corresponding "magnetic field" rot**V**.

The topological term in the original $SU(2)$ Yang-Mills theory reduces in our effective non-linear σ-model not to a winding number, but the linking number Q_H of the field lines. The importance of the linking number for the stability of the solitons has been emphasised in [39], studing solitonic solutions of the $O(3)$ Faddeev-Skyrme σ-model. Furthermore we point out, that the stabilising term in the Faddeev-Skyrme model, used in [40] as an quantum effective theory for the infrared sector of Yang-Mills theory, is the square of the Hopf curvature in the form (97), quadratic in derivatives, whereas in our effective theory the Hopf curvature appears in the Whitehead form (99), linear in derivatives. A Hopf invariant as topological characteristic of low energy gluon field configurations has been introduced [41] also in the context of the Faddeev-Niemi (FN) effective theory [40]. In difference to our case it is quadratic in derivatives and is obtained from the 3-dimensional non-Abelian Chern-Simons action. A similar such relation was obtained also in [42] using the representation of gauge fields in terms of the complex two-component $\mathbb{CP}^1$ variables.

7 Conclusions and remarks

We have generalized the Hamiltonian reduction of $SU(2)$ Yang-Mills gauge theory to the case of nonvanishing θ-angle, and shown that there is agreement between reduced and original constrained equations of motions. We have employed an improved derivative expansion to the non-local kinetic term in the obtained unconstrained Hamiltonian and investigated it in long-wavelength approximation. The corresponding second order Lagrangian has been constructed, with all θ-dependence gathered in a 4-divergence of a current, linear in the derivatives, which is the unconstrained analog of the original Chern-Simons current. Close to the minimum of the classical potential the obtained long-wavelength Lagrangian reduces to a classical effective theory with an Abelian Chern-Simons term originating from the Pontryagin topological functional. The obtained reduced topological term is not a winding number, but the linking number of the field lines.

Such a "metamorphosis" of non-Abelian topological invariants to Abelian fields with nonvanishing helicity sheds some light on the relation between the derived classical effective theory of unit vector field $\mathbf{n}(x)$ and FN quantum effective theory. The stabilising term in the FN model is the square of the Hopf curvature in the form (97), quadratic in derivatives, whereas in our effective theory the Hopf curvature appears in the Whitehead form (99), linear in derivatives. Furthermore we point out, that it has been emphasized in [43, 44], that the FN action has the same symmetry breaking properties, $SU(2) \to U(1)$, as the nonlinear σ-model, and thus two unwanted Goldstone bosons should appear. In order to overcome this problem, therefore, explicit symmetry breaking terms have been introduced in the lattice study of the FN action [44]. Our classical effective theory already contains such terms, for example

$(\mathbf{n}\cdot\nabla\phi_3)^2$ in (??), that, in the corresponding quantum effective theory, can break the symmetry explicitly and thus avoid the appearance of Goldstone bosons.

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Appendix A: Conventions and notations

In this Appendix, we collect several notations and definitions for $SU(2)$ Yang-Mills theory used in the text following [4].

The classical Yang-Mills action of the $su(2)$-valued connection 1-form A in 4-dimensional Minkowski space-time with a metric $\eta = \mathrm{diag}\|1,-1,-1,-1\|$ reads

$$I = -\frac{1}{g^2}\int \mathrm{tr}\, F\wedge {}^*F - \frac{\theta}{8\pi^2 g^2}\int \mathrm{tr}\, F\wedge F\,, \tag{101}$$

with the curvature 2-form

$$F = dA + A\wedge A \tag{102}$$

and its Hodge dual ${}^*F$. The trace in (101) is calculated in the antihermitian $su(2)$ algebra basis $\tau^a = \sigma^a/2\,i$ with Pauli matrices σ^a, $a = 1,2,3$, satisfying $[\tau_a, \tau_b] = \varepsilon_{abc}\,\tau_c$, and $\mathrm{tr}\,(\tau_a\tau_b) = -\frac{1}{2}\,\delta_{ab}$.

In the coordinate basis the components of the connection 1-form A are

$$A = g\,\tau^a\, A^a_\mu\, dx^\mu\,, \tag{103}$$

and the components of the curvature 2-form F are

$$F = \frac{1}{2}\, g\,\tau^a\, F^a_{\mu\nu}\, dx^\mu \wedge dx^\nu\,, \tag{104}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\varepsilon^{abc} A^b_\mu A^c_\nu\,. \tag{105}$$

Its dual ${}^*F$ are given as

$${}^*F = \frac{1}{2}\, g\,\tau^a\, {}^*F^a_{\mu\nu}\, dx^\mu \wedge dx^\nu\,, \tag{106}$$

$${}^*F^a_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma} F^{a\,\rho\sigma}\,, \tag{107}$$

with totally antisymmetric Levi-Civita pseudotensor $\varepsilon_{\mu\nu\rho\sigma}$ using the convention

$$\varepsilon^{0123} = -\,\varepsilon_{0123} = 1. \tag{108}$$

The θ-angle enters the classical action as coefficient in front of the Pontryagin index density

$$Q = -\,\frac{1}{8\pi^2}\,\mathrm{tr}\,F \wedge F\,. \tag{109}$$

The Pontryagin index density is a closed form $d\,Q = 0$ and thus locally exact

$$Q = d\,C\,, \tag{110}$$

with the Chern 3-form

$$C = -\frac{1}{8\pi^2}\,\mathrm{tr}\left(A \wedge dA + \frac{2}{3}\,A \wedge A \wedge A\right)\,. \tag{111}$$

The corresponding Chern-Simons current K^μ is a dual of the 3-form C,

$$K^\mu = (1/3!)\,\varepsilon^{\mu\nu\rho\sigma}\,C_{\nu\rho\sigma} = -\,\frac{1}{16\pi^2}\varepsilon^{\mu\alpha\beta\gamma}\mathrm{tr}\left(F_{\alpha\beta}\,A_\gamma - \frac{2}{3}\,A_\alpha A_\beta A_\gamma\right)\,. \tag{112}$$

with the notations $A_\mu := g\,\tau^a\,A^a_\mu$ and $F_{\mu\nu} := g\,\tau^a\,F^a_{\mu\nu}$. The chromomagnetic field is given as

$$B^a_i = \frac{1}{2}\,\varepsilon_{ijk}\,F^a_{jk} = \varepsilon_{ijk}\,\left(\partial_j A_{ak} + \frac{g}{2}\,\varepsilon_{abc}\,A_{bj}\,A_{ck}\right)\,, \tag{113}$$

and the covariant derivative in the adjoint representation as

$$(D_i(A))_{ac} = \delta_{ac}\partial_i + g\,\varepsilon_{abc}A_{bi}\,. \tag{114}$$

Finally, we frequently use the matrix notations

$$A_{ai} := A^a_i\,, \qquad B_{ai} := B^a_i\,. \tag{115}$$

Appendix B: Unconstrained Lagrangian in $1/g$ approximation

In this Appendix it is shown that straightforward application of expansion of the nonlocal part P_a of the kinetic term in the unconstrained Hamiltonian to zeroth-order discussed in Section 4.1, leads to the appearance of θ-dependence of the reduced system on the classical level. Expressing the Hamiltonian (41), in terms of the main-axis variables, defined in Section 5, and performing an inverse Legendre transformation, one obtains the Lagrangian density

$$\begin{aligned}\mathcal{L}^{(2)}(\phi,\chi) &= \frac{1}{2}\left(\sum_{i=1}^{3}\dot{\phi}_i^2 + \sum_{i,j=1}^{3}\dot{\chi}_i G^{-1}_{ij}\dot{\chi}_j - V(\phi,\chi)\right) - \frac{1}{2}\left(\frac{\theta}{8\pi^2}\right)^2 \sum_{cyclic}\frac{\Delta_i^2}{\phi_j^2+\phi_k^2} \\ &\quad -\frac{\theta}{8\pi^2}\sum_{a=1}^{3}\left(\dot{\phi}_a\beta_a + \sum_{cyclic}\dot{\chi}_a M^T_{ai}(\phi_j-\phi_k)\left(b_i + \frac{(\phi_j-\phi_k)}{\phi_j^2+\phi_k^2}\Delta_i\right)\right)\end{aligned} \tag{116}$$

denoting the difference

$$\Delta_i = \frac{1}{2}(\phi_j - \phi_k)b_i - (\phi_j + \phi_k)\, R_{is}B_s^{(-)} \,, \tag{117}$$

with b_i of (65) and $B_i^{(-)}$ of (66), or explicitly,

$$\Delta_i = -\left[X_i(\phi_j\phi_k) + (\Gamma_{ijj} + \Gamma_{ikk})\phi_j\phi_k - \phi_i(\phi_j\Gamma_{ikk} + \phi_k\Gamma_{ijj})\right] \,. \tag{118}$$

It easy to convince oneselves that the term proportional to θ^2 is not a surface term. Indeed, considering for simplicity configurations of spatially constant angular variables χ_i and $\phi_1 = \phi_2 = \phi_3 =: \phi$,it reduces to

$$-\left(\frac{\theta}{8\pi^2}\right)^2 \sum_{i=1}^{3} \partial_i\phi\ \partial_i\phi \,, \tag{119}$$

which is not a 4-divergence. For $\Delta_i = 0$ the Lagrangian density (116) reduces to (75), obtained from the improved Hamiltonian (46), free of the divergence problem.

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[23] Erratum to [22]: The last line of (3.42) should be replaced by $\left(\Gamma^1{}_{23}\,\phi_3 + \Gamma^1{}_{32}\,\phi_2 + \Gamma^1{}_{[23]}\,\phi_1 - g\,\phi_2\,\phi_3\right)^2$. Correspondingly, in formula (4.6) the term $\left(\Gamma^3{}_{[12]}\,\phi_3\right)^2$ should be added. Finally, in formulae (4.8), (4.10) and (4.11) the term $\left(\vec{n}\cdot\vec{\partial}\times\vec{n}\right)^2$ should be included. The correct formulae are given in the present work in (62),(87), and (88) correspondingly.

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