СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E2-2000-179

N.Makhaldiani*

NEW HAMILTONIZATION OF THE SCHRÖDINGER EQUATION BY CORRESPONDING NONLINEAR EQUATION FOR THE POTENTIAL

*E-mail: mnv@cv.jinr.ru

2000

1. Introduction

The Hamiltonian mechanics (HM) is in the ground of mathematical description of the physical theories [1]. But HM is in a sense blind, e.g., it does not make difference between two opposites: the ergodic Hamiltonian systems (with just one integral of motion) and integrable Hamiltonian systems (with maximal number of the integrals of motion).

By our proposal [2] Nambu's mechanics (NM) [3, 4] is proper generalization of the HM, which makes difference between dynamical systems with different numbers of integrals of motion explicit.

The Schrödinger equation, [5] is the base of one of the most effective formulations of the quantum theory, [6]. An interesting reformulations of the Schrödinger theory as infinit dimentional generalization of Nambu's theory where given in [7, 8].

In Sec.2, of this paper, we consider the general method of the Hamiltonian extension [9] of the nonlinear partial differential equations which describes dynamical systems with infinit number of degrees of freedom. For one of the nonlinear equations we find that the Hamiltonian companion in the extended system is the Schrödinger equation. We find some solutions of the considered systems, including the solutions corresponding to the conformal quantum mechanics, [10].

In Sec.3, we consider the d-dimensional generalization of the extended quantum theory.

In Sec.4, we find radially symmetric static solutions.

In Sec.5, we consider de Broglie-Bohm formulation of the quantum theory and show universality of the inverse-square potentials.

In Sec.6, we find an extra quadratic integral of motion for the d-dimensional inverse-square potential and give the corresponding Nambu-Poisson formulation.

In Sec.7, we integrate in quadratures the radial part of the d-dimensional inverse-square potential dynamics.

In Sec.8, we consider the supersymmetric extension of the conformal quantum mechanics and find corresponding nonlinear systems.

In Sec.9, we find Nambu-theoretic formulation of the extended quantum theory.

In Sec.10, we present our conclusions and show some perspectives.

2. The method of Hamiltonization of the infinite dimensional systems (partial differential equations) and new Hamiltonization of the Schrödinger equation

The well known (integrable) system from the hydrodynamics, the KdV equation, (see, e.g., [1])

$$V_t = VV_x - V_{xxx}, \tag{1}$$

can be put in the Hamiltonian form.
Indeed, let us take as a Lagrangian

$$L = (V_t - VV_x + V_{xxx})\psi. \tag{2}$$

Corresponding (extended) system of the equations of motion is

$$\begin{aligned} V_t &= VV_x - V_{xxx}, \\ \psi_t &= V\psi_x + \psi_{xxx}, \end{aligned} \tag{3}$$

the momentum is

$$P = \frac{\partial L}{\partial V_t} = \psi, \tag{4}$$

the Hamiltonian is

$$H = (VV_x - V_{xxx})\psi, \tag{5}$$

the (fundamental) bracket is

$$\begin{aligned} &\{V(t,x), \psi(t,y)\} = \delta(x-y), \\ &\{A, B\} = \int dx A(\frac{\overleftarrow{\delta}}{\delta V(t,x)}\frac{\overrightarrow{\delta}}{\delta\psi(t,x)} - \frac{\overleftarrow{\delta}}{\delta\psi(t,x)}\frac{\overrightarrow{\delta}}{\delta V(t,x)})B. \end{aligned} \tag{6}$$

2.1 Now it is easy to see that for some dynamical system described by the following equation

$$iV_t = V_{xx} - \frac{1}{2}V^2, \tag{7}$$

the Hamilton companion system is given by the Schrödinger equation

$$i\psi_t = -\psi_{xx} + V\psi, \tag{8}$$

with units chosen so that $\hbar = 1,\ 2m = 1$ [1].

[1] We can return to the usual form of the Schrödinger equation by the following change of the variables $t \to t/\hbar$ and $x \to x\sqrt{2m}/\hbar$.

Corresponding Lagrangian is

$$L = (iV_t - V_{xx} + \frac{V^2}{2})\psi, \tag{9}$$

Hamiltonian is

$$H = (V_{xx} - \frac{V^2}{2})\psi, \tag{10}$$

the extended system of the equations of motion is

$$\begin{aligned} iV_t &= V_{xx} - \frac{V^2}{2}, \\ i\psi_t &= -\psi_{xx} + V\psi, \end{aligned} \tag{11}$$

where the variable V maybe interpreted as potential function and the variable ψ-as quantum amplitude.

2.2 Generally the solution of the equation (7), V is complex valued. Real valued maybe the static solutions, $V_t = 0$,

$$V_{xx} - \frac{1}{2}V^2 = 0. \tag{12}$$

It is easy to find, in the vanishing at the infinity conditions, the following solution

$$V(x) = \frac{12}{(x - x_0)^2}, \tag{13}$$

where x_0 is a real constant. We have also an approximate ("dilute gas") solutions

$$V(x) = \sum_{n=0}^{N} \frac{12}{(x - x_n)^2} \tag{14}$$

where, $x_0 << x_1 << ... << x_N$, are constants.

The equation (8) with the potential (13) is known as the conformal quantum mechanics, [10].

To find the general static solution of the equation (7,12), we transform the equation (12) to the integral form,

$$x - x_0 = \int\limits_{V(x_0)}^{V(x)} \frac{dV}{\sqrt{(\frac{1}{3}V^3 - C_1)}}, \tag{15}$$

where

$$C_1 = \frac{V^3(x_0)}{3} - (V'(x_0))^2. \tag{16}$$

From expression (15) we find the solution

$$V(x) = \wp(\frac{1}{2\sqrt{3}}(x - x_0), 0, g_3), \ g_3 = 12C_1, \tag{17}$$

where, $\wp(x, g_2, g_3)$ is the elliptic function of Weierstrass, (see [11] and Appendix 1). Note that the solution (13) corresponds to the case $g_3 = C_1 = 0$.

2.3 Let us show that the general static solution of the equation (7) coincides with the subclass of the static solutions of the KdV, (1). In fact, for static solutions of KdV, we have

$$V_{xxx} - VV_x = 0 \tag{18}$$

and after one integration we obtain

$$V_{xx} - \frac{V^2}{2} = C_0. \tag{19}$$

Last equation coincides with the equation (12), for $C_0 = 0$.

The static solutions of the equation (8),

$$\psi_{xx} - V\psi = 0, \tag{20}$$

with potentials in the form (17) is

$$\psi(x) = V_x(x) = \wp_x(\frac{1}{2\sqrt{3}}(x - x_0), 0, g_3). \tag{21}$$

In fact. From the equation (12) we have

$$V_{xxx} - VV_x = 0. \tag{22}$$

So the function (21) fulfils the equation (20).

2.4 For stationary solutions

$$\psi(x,t) = e^{-i\omega t}\psi(x), \tag{23}$$

of the equation (8), we have

$$\psi_{xx} + (\omega - V)\psi = 0. \tag{24}$$

This equation reduce to the Lame's equation, [13]

$$\psi_{yy} = (C_1\wp(y) + C_2)\psi, \tag{25}$$

for $C_1 = 12$, $C_2 = -12\omega$ and $y = 2\sqrt{3}(x - x_0)$.

2.5 Note that there are the stationary solutions

$$V(x,t) = e^{-i\Omega t}V(x), \tag{26}$$

of the following equation

$$iV_t = V_{xx} - \frac{1}{2}e^{i\Omega t}V^2, \tag{27}$$

where $V(x)$ fulfils

$$V_{xx} - \frac{V^2}{2} - \Omega V = 0. \tag{28}$$

and is defined by the following integral

$$\int_{V_0}^{V(x)} \frac{dV}{\sqrt{\frac{1}{3}V^3 + \Omega V^2 + C}} = x - x_0. \tag{29}$$

This integral reduce to the Weierstrass elliptic function after the shift of the integration variable, $V \to V - \Omega$.

We have the following soliton-like localized stationary solutions of the equation (27)

$$V(x,t) = -\frac{3\Omega e^{-i\Omega t}}{ch^2(\sqrt{\Omega}/2(x - x_0))}. \tag{30}$$

In the limit $\Omega \to 0$, the solution (30) disappears.

Corresponding Schrödinger equation

$$i\psi_t = -\psi_{xx} - \frac{3\Omega}{ch^2(\sqrt{\Omega}/2(x - x_0))}\psi, \tag{31}$$

is exactly solvable (see e.g [14]).

2.6 Let us consider the following generalization of the equation (7)

$$iV_t = V_{xx} - \frac{1}{n}V^n, \quad n \neq 1. \tag{32}$$

To the static solution (13) corresponds the following solution of the equation (32)

$$V(x) = \frac{a_n}{(x-x_0)^{\alpha_n}}, \quad a_n = \Big(\frac{2n(n+1)}{(n-1)^2}\Big)^{\frac{1}{n-1}}, \quad \alpha_n = \frac{2}{n-1}. \tag{33}$$

To the Schrödinger equation (8) corresponds the following Hamiltonian partner of the equation (32)

$$i\psi_t = -\psi_{xx} + V^{n-1}\psi. \tag{34}$$

This equation for the static solution (33) reduce to the following conformal quantum mechanics

$$i\psi_t = -\psi_{xx} + \frac{2n(n+1)}{(n-1)^2}\frac{1}{x^2}\psi. \tag{35}$$

2.7 Note that there are several approaches to the quantum mechanics with such a singular potential as (13) (see e.g [15] and references therein). Let us show a connection between harmonic oscilator and inverse-square potential problems. If we take two independent solutions, u and v, of the harmonic oscilator equation of motion

$$\ddot{x} + \omega^2 x = 0. \tag{36}$$

Then it is easy to show that the variable $\rho = \sqrt{u^2+v^2}$ fulfils the following equation

$$\ddot{\rho} + \omega^2\rho - \frac{g^2}{\rho^3} = 0, \tag{37}$$

where $g = u\dot{v} - v\dot{u}$ is a constant of motion. So we have a connection between problems corresponding to the Hamiltonians

$$H = \frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}x^2 \tag{38}$$

and

$$H = \frac{1}{2}\dot{x}^2 + \frac{\omega^2}{2}x^2 + \frac{g^2}{2x^2}. \tag{39}$$

2.8. Static solutions of the heat equation with polynomial nonlinearites

Let us consider the following nonlinear heat equation

$$V_t = V_{xx} - \frac{1}{2}P_n'(V), \tag{40}$$

where $P_n = a_n V^n + a_{n-1} V^{n-1} + ... + a_0$.

The static solutions of this equation are given by hyperelliptic functions (see Appendix 2.)

$$V(x) = \wp_n(x, c_{n-2}, ..., c_0). \tag{41}$$

3. Any dimensional generalization and many particle interpretation

Let us consider n-dimansional case. The system (11) takes the following form

$$\begin{aligned} iV_t &= \Delta V - \frac{V^2}{2}, \\ i\psi_t &= -\Delta\psi + V\psi. \end{aligned} \tag{42}$$

Corresponding Lagrangian, Hamiltonian and Poisson brackets are obvious modifications of the expressions (9),(10),(6).

Let us take the simplest generalization, two dimentional case, $n = 2$ and make its two particle interpretation.

It is easy to see that

$$V(x_1, x_2) = \frac{12}{(x_1 - x_2)^2} + \frac{12}{(x_2 - x_1)^2} = \frac{24}{(x_1 - x_2)^2} \tag{43}$$

is the static solution of the following equation

$$iV_t = (\Delta_1 + \Delta_2)V(x_1, x_2, t) - \frac{V^2}{2}. \tag{44}$$

Corresponding Schrödinger equation

$$i\psi_t = -(\Delta_1 + \Delta_2)\psi + \frac{24}{(x_1 - x_2)^2}\psi, \tag{45}$$

describes the two particle quantum system. In n-dimensional case of two particle system x_1 and x_2 become an n-dimensional vectors. For N-particle case with different masses m_i, the nonlinear equation becomes

$$iV_t = (\Delta_1/m_1 + \Delta_2/m_2 + ... + \Delta_N/m_N)V(x_1, x_2, ..., x_N, t) - \frac{V^2}{2}, \tag{46}$$

with corresponding static solution

$$V(x_1, x_2, ..., x_N) = \sum_{i \neq j}^{N} \frac{12 m_{ij} p_{ij}}{(x_i - x_j)^2} = \sum_{i<j}^{N} \frac{24 m_{ij} p_{ij}}{(x_i - x_j)^2}, \tag{47}$$

where

$$m_{ij} = \frac{m_i + m_j}{m_i m_j} \tag{48}$$

and p_{ij} are projectors,

$$p_{ij} = p_{ji}, \quad p_{ij} p_{kl} = p_{ij} \delta_{ik} \delta_{jl}. \tag{49}$$

4. Radially symmetric static solutions and a new mechanism of computation of coupling constants

Now we find static, vanishing at the infinity, radially symmetric solutions of the system (42). The equation for V is

$$V_{rr} + \frac{n-1}{r} V_r - \frac{1}{2} V^2 = 0. \tag{50}$$

It is easy to find the following solution

$$V = \frac{4(4-n)}{r^2}. \tag{51}$$

From the point of view of the Schrödinger equation, (8) the solution (51) corresponds to attraction (repulsion) for space dimensions $n > 4$ ($n < 4$).

Let us make comparision with the scalar potential of the point source given by the following equation

$$\Delta V = g \delta^n(x). \tag{52}$$

The solution of this equation is (see, e.g. [16])

$$V = -\frac{g}{(n-2)\Omega_n} \frac{1}{r^{n-2}}, \; \Omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}. \tag{53}$$

According to the contemporary theoretical conceptions, on small scales (or for early Univerce) space(-time) geometry is a coherent superposition of different classical geometries, with different metric structures, topologies, dimensions (see, e.g. [17]). Let us take the system (11) as fundamental model describing the Univerce at this scales. Than according to the solution (51), for the components of geometry with $n > 4$ the matter has tendency to concentrate and take less dimension. For components with $n < 4$, the matter repulse and try to rise the space dimension. For $n = 4$, the matter is free. So in the model described by the system (11) with solution (51), the matter choice the dimension $n = 4$. On a bigger scales, we can qualitatively describe the same picture by the equation (52) and the solution (53) in dimension $n = 4$ with different values of the point charge g,

$$g = 16\pi^2(n-4), \qquad n = 4 + \frac{g}{16\pi^2} = 4 - 2\varepsilon, \tag{54}$$

where $\varepsilon = \sqrt{4\pi\alpha}/32\pi^2 = 10^{-3}$ for quantum electrodynamics ($\alpha = 1/137$) and $\varepsilon = 0.04$ for nucleon- pion strong interection model ($\alpha_s = 14.7$), [18].

This equation gives a simpl(ified) example of the calculation of the charges of elementary particles and/or corresponding fractal dimension of the space [2].

To be more realistic, we can consider relativistic modification of this model, but the static considerations remain unchanged.

4.1 Let us consider d-dimensional generalization of the equation (32)

$$iV_t = \Delta V - \frac{1}{n}V^n, \quad n \neq 1. \tag{55}$$

Radially symmetric, static solution of this equation is

$$V(r) = \frac{a}{r^\alpha}, \tag{56}$$

where

$$\alpha = \frac{2}{n-1}, \quad a^{n-1} = \alpha n(\alpha + d - 2) = \frac{2n(2-(d-2)(n-1))}{(n-1)^2}. \tag{57}$$

Note that, for $n = 2$, $a < 0$, for $d > 4$ and $a > 0$, for $d < 4$.

For $n \geq 2$, a is real for

$$d < 2 + \frac{2}{n-1}. \tag{58}$$

So, we have real solutions only for $d \leq 2$. Corresponding Schrödinger equation

$$i\psi_t = -\Delta\psi + V^{n-1}\psi, \tag{59}$$

has the following potential

$$V_n = V^{n-1} = \frac{2n(2-(d-2)(n-1))}{(n-1)^2}\frac{1}{r^2}, \tag{60}$$

which is attractive for $n \geq 3$ and $d \geq 3$. So we conclude, that the minimal nonlinearity, the $n = 2$ case, is the most "attractive" one.

5. de Broglie-Bohm formulation of the quantum theory and inverce-square potential as universal quantum potential

In text-books on quantum mechanics (see e.g. [19]), quantum and classical theories are connected by representation of the wave function in the form

$$\psi = \Re exp(i\frac{S}{\hbar}). \tag{61}$$

[2] See [17] for general method of calculations of fractal dimension of space in different models of particle physics and field and string theores.

Inserting this expression into the Schrödinger equation, we obtine

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V = \frac{\hbar^2}{2m}\frac{\Delta\Re}{\Re}$$
$$\frac{\partial \Re^2}{\partial t} + \nabla(\Re^2 \frac{\nabla S}{m}) = 0. \tag{62}$$

It was found by E. Madelung, [20] that Schrödinger equation can be recast as a hydrodynamic equations. The key step in the de Broglie-Bohm formulation of the quantum theory, [21, 22, 23] is to regard

$$v = \frac{\nabla S}{m} \tag{63}$$

as a particle velocity. The first equation of the system (62) is then the Hamolton-Jacobi equation and the second equation is the continuity equation relating the particle probability density $\rho = \Re^2$ to the current density $j = \rho v$. Note that if we neglect the right hand side therm of the equation (62), we obtain exectly the classical Hamilton- Jacobi theory. With nonzero right hand side (Quantum potential) we have full quantum theory.

In the previous sections of this work, we have seen, that the inverse-square potentials appears naturaly in our extended quantum theory, (11,42). Now we will give some qualitative arguments in favour of inverse-square potentials in the de Broigle-Bohm formulation of quantum theory, [22, 23]. The vorticity of the flow

$$\nabla \times v = \nabla \times (\nabla S)/m, \tag{64}$$

is zero as long as S is nonsingular. In this case, the probability field is irrotational. At points where $\Re = 0$, i.e., wave function nodes, S may be singular. At such points vorticity may be nonzero. So at the wave function nodes, may be (quantum) vortices, ([24]).

Let us suppose that at the node of the wave function, $\Re \sim r^\alpha$, $\alpha(\vec{n}) \geq 0$, than the quantum potential, (62) has the following form, (70)

$$V_q = -\frac{\hbar^2}{2m}\frac{\Delta\Re}{\Re} = \frac{g_q(\vec{n})}{r^2}. \tag{65}$$

From the system (62) (for static configurations) we have

$$\nabla S \sim \frac{1}{\Re^2}, \quad V = \frac{g_c}{r^{4\alpha}} + \frac{g_q}{r^2}, \tag{66}$$

where

$$g_c < 0, \quad g_q = (\alpha(\alpha + d - 2) + \frac{\Delta_n r^\alpha}{r^\alpha})\frac{\hbar^2}{2m}, \tag{67}$$

$$\Delta = \frac{d^2}{dr^2} + \frac{d-1}{r}\frac{d}{dr} + \frac{\Delta_n}{r^2}. \tag{68}$$

So, for $\alpha = 1/2$, quantum potential renormalize the classical one ($g = g_c + g_q$) and can improve its, e.g. "fall of the particle to the center" [14], properties. For $\alpha < 1/2$ ($\alpha > 1/2$), quantum (classical?) potential dominates for smoll scales.

If we have a nonanalytical behaviour $\Re \sim exp(-(r_0/r)^\alpha)$, $\alpha > 0$, then [3]

$$V = g_c exp(4(r_0/r)^\alpha) + \frac{g(\vec{n})}{r^{2(1+\alpha)}}. \tag{69}$$

So, nonanalytical case corresponds to the "fall of the particle to the center" potentials.

6. Extra quadratic integral of motion for the d-dimensional inverse-square potential and Nambu-Poisson formulation

In spherical coordinates, $\vec{r} = r\,\vec{n}$, $\mid \vec{n} \mid = 1$, the inverse-square potential is

$$V(r) = \frac{g(\vec{n})}{r^2}. \tag{70}$$

The Hamiltonian is

$$H_1 = \frac{m}{2}(\dot{r}^2 + r^2\dot{n}^2) + \frac{g(\vec{n})}{r^2}. \tag{71}$$

Corresponding Lagrangian is

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{n}^2) - \frac{g(\vec{n})}{r^2} - \lambda(\vec{n}^2 - 1), \tag{72}$$

where λ is the Lagrange multiplier.

The equations of motion are

$$\begin{aligned} m\ddot{r} - mr\dot{n}^2 - 2\frac{g(\vec{n})}{r^3} &= 0, \\ m(r^2\dot{n}_i)^{\cdot} + \frac{\partial g}{\partial n_i}\frac{1}{r^2} + 2\lambda n_i &= 0. \end{aligned} \tag{73}$$

If we multiply the last equation by $r^2\dot{n}_i$ and take a sum over i, we will have

$$\frac{d}{dt}(\frac{m}{2}r^4\dot{n}^2 + g(\vec{n})) = 0. \tag{74}$$

So

$$H_2 = \frac{m}{2}r^4\dot{n}^2 + g(\vec{n}) = r^2(H_1 - \frac{m}{2}\dot{r}^2), \tag{75}$$

[3] This note appears after conversation with B. Magradze.

is the new integral of motion.

This integral for the case of dimensions of space $d = 2$ and $d = 3$ were considered in [24]. We obtained this integral by universal way for general d.

Now having two integrals of motion we can put the equations of motion (73) in the following (second level) Nambu-Poisson form, [2]

$$\begin{aligned} \dot{\vec{r}} &= \{\vec{r}, H_1, H_2\} \\ &= \omega_{ijk}(x) \frac{\partial \vec{r}}{\partial x_i} \frac{\partial H_1}{\partial x_j} \frac{\partial H_2}{\partial x_k}, \end{aligned} \tag{76}$$

where the structure functions ω_{ijk} can be determined by comparision of the equations (76) and (73).

7. Explicit integration of the radial part of the d-dimensional and complet integration of $d = 2$ dimensional inverse-square potential dynamics

The integral H_2, (75) depends explicitly just on the variable r. So we can find the dynamics of the variable by one quadrature. Indeed, from the expression (75) we find:

$$1.\ H_1 = 0,\quad a)\ H_2 = 0,\quad r = r_0 = const, \tag{77}$$

$$b)\ H_2 < 0,\quad \dot{r}_0^2 = \pm\sqrt{\frac{8|H_2|}{m}},\quad r^2 = r_0^2 \pm \sqrt{\frac{8|H_2|}{m}}t; \tag{78}$$

$$\begin{aligned} &2.\ H_1 \neq 0,\ a)\ H_2 = 0,\quad \dot{r} = \pm\sqrt{\frac{2H_1}{m}},\quad r = r_0 \pm \sqrt{\frac{2H_1}{m}}t; \\ &b)\ H_2 \neq 0,\quad r^2 = \frac{H_2}{H_1} + \frac{2H_1}{m}(t - t_0)^2. \end{aligned} \tag{79}$$

Now we put the solution for $r(t)$ into the 2-dimentional expression of H_1,

$$H_1 = \frac{m}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{g(\phi)}{r^2} \tag{80}$$

and after some calculations we obtain

$$\int_{\phi_0}^{\phi} \frac{d\phi}{\sqrt{H_2 - g(\phi)}} = \pm\sqrt{\frac{2}{m}} \int_{t_0}^{t} \frac{dt}{r^2}. \tag{81}$$

This completes the integration of the two dimensional inverse-square potential in quadratures (see also [24]).

8. Supersymmetric extension

A minimal realization of the algebra of supersymmetry

$$\begin{aligned}
&\{Q, Q^+\} = H, \\
&\{Q, Q\} = \{Q^+, Q^+\} = 0,
\end{aligned} \tag{82}$$

is given by a point particle in one dimension, [25]

$$\begin{aligned}
&Q = a(-iP + W_x), \\
&Q^+ = a^+(iP + W_x),
\end{aligned} \tag{83}$$

where $P = -i\partial/\partial x$, the superpotential $W(x)$ is any function of x and the spinor operators a and a^+ obey the anticommuting relations

$$\begin{aligned}
&\{a, a^+\} = 1, \\
&a^2 = (a^+)^2 = 0.
\end{aligned} \tag{84}$$

The operator

$$B = [a^+, a], \tag{85}$$

is the generator of the U(1) transformations

$$\begin{aligned}
\psi^- &\to \psi^-_\alpha = e^{i\alpha B}\psi^- = e^{-i\alpha}\psi^-, \\
\psi^+ &\to \psi^+_\alpha = e^{i\alpha B}\psi^+ = e^{i\alpha}\psi^+.
\end{aligned} \tag{86}$$

There is the following representation of the operators a, a^+ and B by the Pauli spin matrices

$$\begin{aligned}
a &= \frac{\sigma_1 - i\sigma_2}{2}, \\
a^+ &= \frac{\sigma_1 + i\sigma_2}{2}, \\
B &= \sigma_3.
\end{aligned} \tag{87}$$

From formulas (82) and (83) than we have

$$H = P^2 + W_x^2 + BW_{xx}, \tag{88}$$

$$\begin{aligned}
H\psi^- = \;& H_-\psi^- = (P^2 + W_x^2 - W_{xx})\psi^-, \\
H\psi^+ = \;& H_+\psi^+ = (P^2 + W_x^2 + W_{xx})\psi^+.
\end{aligned} \tag{89}$$

Now we can identify the potential of H_- with the solutien (13)

$$V_- = W_x^2 - W_{xx} = \frac{12}{x^2}. \tag{90}$$

Then we find

$$W_- = \frac{3}{2} ln x^2. \tag{91}$$

If we take instead of the potential V_- of the Hamiltonian H_- the potential V_+ of the Hamiltonian H_+, than we will find

$$W_+ = 2 ln x^2. \tag{92}$$

The superpartner of the potential (90, 91) is

$$V_+ = W_x^2 + W_{xx} = \frac{6}{x^2}, \tag{93}$$

which is the static solution of the "superpartner" of the equation (7)

$$iU_t = U_{xx} - U^2. \tag{94}$$

An interesting quation is how the supersymmetry is realized on the system

$$\begin{aligned} iV_t &= V_{xx} - \frac{1}{2}V^2, \\ iU_t &= U_{xx} - U^2. \end{aligned} \tag{95}$$

Note that the supersymmetric generalization of the conformal quantum mechanics, [10] were considered in [26, 27].

9. The Nambu-theoretic (re)formulation of the extended Schrödinger quantum theory

The variational formulation of the extended Schrödinger quantum theory, (42) we can construct by the following Lagrangian

$$L = (iV_t - \Delta V + \frac{1}{2}V^2)\psi. \tag{96}$$

The momentum variables are

$$\begin{aligned} P_v &= \frac{\partial L}{\partial V_t} = i\psi, \\ P_\psi &= 0. \end{aligned} \tag{97}$$

As a Hamiltonians of the Nambu-theoretic formulation we take the following integrals of motion

$$\begin{aligned} H_1 &= (\Delta V - \frac{1}{2}V^2)\psi, \\ H_2 &= P_v - i\psi, \end{aligned}$$

$$H_3 = P_\psi. \tag{98}$$

We invent unifying vector notation, $\phi = (\phi_1, \phi_2, \phi_3, \phi_4) = (\psi, P_\psi, V, P_v)$. Then it may be verified that the equations of the extended quantum theory can be put in the following Nambu-theoretic form

$$\begin{aligned}\phi_t(x) &= \{\phi(x), H_1, H_2, H_3\} \\ &= i\int \frac{\partial(\phi(x), H_1, H_2, H_3)}{\partial(\phi_1(y), \phi_2(y), \phi_3(y), \phi_4(y))} dy,\end{aligned} \tag{99}$$

where the bracket is defined as

$$\{A, B, C, D\} = i\varepsilon_{ijkl}\int \frac{\delta A}{\delta\phi_i(y)}\frac{\delta B}{\delta\phi_j(y)}\frac{\delta C}{\delta\phi_k(y)}\frac{\delta D}{\delta\phi_l(y)} dy. \tag{100}$$

Note that, from the point of view of the Nambu-theoretic representation (99) we have the following (total) class of equivalent Hamiltonians

$$H = H_1 + \lambda_2 H_2 + \lambda_3 H_3, \tag{101}$$

where, λ_2 and λ_3 are arbitrary functions of t. In components, the system of equations (99) is

$$\phi_{1t} = i(\Delta\phi_1 - \phi_3\phi_1) \Rightarrow i\psi_t = -\Delta\psi + V\psi, \tag{102}$$

$$\begin{aligned}&\phi_{2t} = 0 \Rightarrow \phi_2 = H_3 = const = 0, \\ &\phi_{3t} = -i(\Delta\phi_3 - \frac{1}{2}\phi_3^2) \Rightarrow iV_t = \Delta V - \frac{1}{2}V^2, \\ &\phi_{4t} = -(\Delta\phi_1 - \phi_3\phi_1).\end{aligned} \tag{103}$$

$$H_2 = const = 0 \Rightarrow \phi_4 = i\phi_1, \tag{104}$$

Due to the line (104), the equation (103) reduce to the equation (102).

It is easy to make the highest level Nambu-Poisson formulation, [28, 2, 29]. In our case the third level (with three Hamiltonians) is highest. Then we make corresponding reductions on the low level (Nambu-)Poisson form(s). In our case we have the following reductions

$$\begin{aligned}\phi_t(x) &= \{\phi(x), H_1, H_2, H_3\} \\ &= \{\phi(x), H_1, H_2\} = \{\phi(x), H_2, H_3\} = \{\phi(x), H_3, H_1\} \\ &= \{\phi(x), H_1,\} = \{\phi(x), H_2\} = \{\phi(x), H_3\},\end{aligned} \tag{105}$$

where, e.g.

$$\begin{aligned}&\phi_{it}(x) = \{\phi(x), H_1\} = f_{ij}\frac{\partial H_1}{\partial\phi_j(x)}, \\ &f_{ij} = \varepsilon_{ij12} - i\varepsilon_{ij24}.\end{aligned} \tag{106}$$

As matrix, the structure tensor f has the following form

$$f = (f_{ij}) = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

After strike out the second row and column, corresponding to the eigenvector

$$X_{1i} = \frac{\partial H_3}{\partial \phi_i}, \tag{107}$$

we will have 3x3 antisymmetric matrix, with one zero eigenvalue corresponding to the eigenvector

$$X_{2i} = \frac{\partial H_2}{\partial \phi_i}. \tag{108}$$

In the subspace ortogonal to the vectors (107) and (108), our Poisson dynamics reduce to the symplectic one.

Note that the standard Schrödinger quantum theory were presented in the form (99) in [7]. The Weyl-Wigner-Moyal formulation of the quantum theory were presented in the Nambu-theoretic form in [8].

10. Conclusions and perspectives

For inverse-square potential problems in three (and more) dimensional case, we need third (and more) integral(s) of motion. This integral(s) can not depends only on r and $\dot{r}$. For some given (testing) form of the integral(s) $H_3 = H(r, \dot{r}, n, \dot{n})$, we have the following condition

$$\dot{H} = H_r\dot{r} + H_{\dot{r}}\ddot{r} + H_n\dot{n} + H_{\dot{n}}\ddot{n} = 0, \tag{109}$$

which is the partial differential equation for $g(n)$, (70). Nontrivial solutions of this equation give us thee (and higher) dimensional integrable systems.

Another interesting question is the connection of the algebraic integrability (in radicals) of the general polynomial equations and analytic properties of the (hyper)elliptic functions, (see appendix 2).

Concerning to the relation of our Hamiltonian extended quantum theory and supersymmetry, considered in Sec.8 in the simplest case of superconformal quantum mechanics, it is interesting to investigate the connection between general solutions (and equations) corresponding to the following diagram

$$V_- \Rightarrow W_- \Rightarrow W_+ \Rightarrow V_+. \tag{110}$$

The work on the applications of the formalism of this paper for several dynamical systems is in progress [30].

Appendix 1 The (famous) Weierstrass elliptic function

$$V(x) = \wp(x, g_2, g_3) \tag{111}$$

can be defined from the integral

$$\int_{V(x)}^{\infty} \frac{dV}{\sqrt{4V^3 - g_2 V - g_3}} = x. \tag{112}$$

Expending under the integral in powers of g_2 and g_3, we find x as a series of $V(x)$. Inverting that series, we find, (see,e.g. [12])

$$\wp(x, g_2, g_3) = \frac{1}{x^2} + \frac{g_2}{20}x^2 + \frac{g_3}{28}x^4 + ... \tag{113}$$

Appendix 2 Let us introduce the following generalization of the Weierstrass function

$$V_n(x) = \wp_n(x, C_{n-2}, C_{n-3}, ..., C_0), \tag{114}$$

which is defined by the following integral

$$\int_{V_n(x)}^{\infty} \frac{dV}{\sqrt{P_n(V)}} = x, \tag{115}$$

where

$$P_n(V) = \frac{4}{(n-2)^2}V^n + C_{n-2}V^{n-2} + ... + C_0. \tag{116}$$

Note that, by simple shift of the variable, $U = V + a$, we can always eliminate the next to the leading therm of the general polynomial

$$\begin{aligned} P_n(U) &= a_n U^n + a_{n-1}U^{n-1} + ... + a_0, \\ P_n(V + a) &= a_n V^n + (n a_n a + a_{n-1})V^{n-1} + ... + C_0, \end{aligned} \tag{117}$$

if we take

$$a = -\frac{a_{n-1}}{n a_n}. \tag{118}$$

So the expression (115) we can be consider as the standard form of the general polynomial case.

As in the previous appendix, we obtain the following series (re)presentation

$$V_n(x) = \wp_n(x, C_{n-2}, ..., C_0) = \frac{1}{x^{2/(n-2)}} - \frac{(n-2)^2}{4(n+2)}C_{n-2}x^{2/(n-2)} + ... \tag{119}$$

This expression defines simple analytic function just in the case of the elliptic functions, $n = 3$ and $n = 4$. The case of $n = 1$ and $n = 2$ correspond to elementary functions.

References

[1] L.D. Faddeev and L.A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer, Berlin, 1987.

[2] D. Baleanu, N. Makhaldiani, *Communications of the JINR*, Dubna **E2-98-348** 1998, solv-int/9903002, *Roumanian J. Phys.* **44** N9-10 (1999).

[3] Y. Nambu, *Phys.Rev. D* **7** 2405 (1973).

[4] L.A. Takhtajan, *Comm.Math.Phys.* **160** 295 (1994).

[5] F. Berezin, M. Shubin, *Schrödinger equation*, Kluwer, Dordrecht, 1991.

[6] Anthony Sudbery, *Quantum mechanics and the particles of nature*, Cambridge University Press, 1986.

[7] I. Cohen, *International J. Theor. Phys.* **12** 69 (1975).

[8] Iwo Bialynicki-Birula and P.J. Morrison, *Phys.Lett. A* **158** 453 (1991).

[9] N. Makhaldiani, O. Voskresenskaya, *Communications of the JINR*, Dubna **E2-97-418** 1997.

[10] V. de Alfaro, S. Fubini and G. Furlan, *Nuovo Cimento* **34A** 569 (1976)

[11] Serge Lang, *Elliptic Functions*, Addison-Wesley, London, 1973.

[12] H. Bateman and A. Erdelyi, *Higer Transendental Functions*, Vol.**3**, New York, 1955.

[13] E. Kamke, *Differentialgleichungen*, Leipzig, 1959.

[14] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics*, Vol.**3** of *Course of Theoretical Physics*, 3rd ed. Pergamon Press, Oxford, 1977.

[15] A.T. Filippov, *Physics of Elementary Particles and Atomic Nuclei*, Vol. **10** Part **3** Atomizdat, Moscow, 1979.

[16] N.V. Makhaldiani, *Approximate methods of the field theory and their applications in physics of high energy, condensed matter, plasma and hydrodynamics*, Dubna, 1980.

[17] N.V. Makhaldiani, *Communications of the JINR*, Dubna **P2-87-306** 1987.

[18] N.N. Bogoliubov and D.V. Shirkov, *Introduction to the Theory of Quantized Fields*, New York, 1959.

[19] David Bohm, *Quantum Theory*, New York, 1952.

[20] E. Madelung, *Z.Phys.* **40** 322 (1926).

[21] L. de Broglie, *Nonlinear wave mechanics, a causal interpretation*, Elsevier, 1960.

[22] D. Bohm, *Phys. Rev.* **85** 166 (1952).

[23] D. Bohm and B.J. Hiley, *The Undivided Universe*, Routlege and Chapman & Hall, London, 1993.

[24] Hua Wu and D.W.L. Sprung, *Phys. Rev. A* **49** (1994) 4305.

[25] E. Witten, *Nucl. Phys. B* **188** 513 (1981).

[26] V. Akulov, A. Pashnev, *Theor. Math. Phys.* **56** 862 (1983).

[27] S. Fubini and E. Rabinovici, *Nucl. Phys. B* **245** 17 (1984).

[28] N. Makhaldiani, *Communications of the JINR*, Dubna **E2-97-407** 1997, solv-int/9804002.

[29] N. Makhaldiani, *Communications of the JINR*, Dubna **E2-99-337** 1999.

[30] N. Makhaldiani, in preparation.

Received by Publishing Department
on August 1, 2000.