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ON THE CLASSICAL AND QUANTUM SCATTERING CROSS-SECTIONS ON THE IMPENETRABLE SPHERE

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1. It is known [1-9] that quantum cross section (c.s.) on the impenetrable sphere is twice as much as the classical one. In the calculation of the quantum c.s. one does not take into account the interference of the incident and scattered waves. This is related to the fact that experimental conditions exclude the measurement of this interference. Theoretically one may remore the latter using the wave packets. Here we prefer to deal with the plane waves and with the solutions of the Schroedinger equation corresponding to the definite energy and momenta. In Sec. 2 we reproduce the classical cross sections. In Sec. 3 we analyse the quantum scattering c.s. and conclude that difference between the quantum c.s. corresponding to the Schroedinger equation and classical one disappears if one takes into account the interference effects and defines the incident flow similarly in both cases.

2. The classical scattering c.s. on the impenetrable sphere is calculated in an elementary way. Only those incident particles are subjected to scattering, whose impact parameter b is less or equal to the radius of the impenetrable sphere a . The number of such particles is equal to πa^2 (if the particle density in the incident flow is equal to unity). The scattering angle equals

$$\theta = 2L \int_{\infty}^{r_{min}} \frac{dr}{r^2[2m(E-V)-L^2/r^2]^{1/2}} + \pi . \tag{1}$$

From (1) we extract the dependence of the impact parameter $b(\equiv \frac{L}{\sqrt{2mE}})$ on the θ :

$$b = a \cdot \cos\frac{\theta}{2} ,$$

The classical differential c.s. is given by

$$\sigma(\theta) = \frac{a^2}{4} . \tag{2}$$

At last, the total c.s. is obtained by integrating $\sigma(\theta)$

$$2\pi \int_0^{\pi} \sigma(\theta)\sin\theta\, d\theta = \pi a^2 , \tag{3}$$

$\sigma(\theta)$ defines the distribution of scattered particles on the surface of the infinite radius sphere. The coincidence of (3) with the number of incident particles which undergone the collision with the impenetrable sphere is due to the conservation of the particle number. The same equality is fulfilled if one takes the sphere of the finite radius R (instead of the infinite one). In this case the impact parameter b is related to the angle at the surface of the sphere in the following way:

$$\theta_R = \pi - 2 \arcsin \frac{b}{a} + \arcsin \frac{b}{R}.$$

The angular distribution of the scattered particles on this surface has a sharp shadow for the angles

$$0 \leq \theta_R < \epsilon , \qquad \text{where} \qquad \sin \epsilon = \frac{a}{R} .$$

For $R=a$, that is if the balance of the scattered particles takes place at the surface of the impenetrable sphere, the shadow occupies the whole front semisphere. The prescriptions for obtaining the total c.s. are the same in all cases: The flow of the scattered particles is integrated over the surface of an arbitrary sphere. Integrating the total flow (including the incident one) one obtains zero. This reflects merely a fact that the number of incoming (into the sphere) particles is equal to that of outcoming. Having a total flow one easily obtains the flow of scattered particles by subtracting from the total flow the incident one. The latter is everywhere constant except for the shadow region where it equals zero.

3. Now go to the quantum case. The wave function is

$$\Psi = e^{ikz} - \frac{1}{kr} \Sigma\, (2\ell+1) i^{\ell}\, \frac{j_\ell(ka)}{h_\ell\ (ka)}\, h_\ell\ (kr) P_\ell\ (\cos\theta)$$

$$(h_\ell^{(1,2)}(x) = \sqrt{\frac{\pi x}{2}} H^{(1,2)}_{\ell+1/2}(x), \quad j_\ell(x) = \sqrt{\frac{\pi x}{2}} J_{\ell+1/2}(x)). \tag{4}$$

For large distances

$$\Psi = e^{ikz} + \frac{1}{r} \cdot e^{ikr} \cdot f(\theta), \tag{5}$$

where

$$f(\theta) = -\frac{1}{ik} \Sigma\, (2\ell+1) \cdot \frac{j_\ell(ka)}{h_\ell^{(1)}(ka)} \cdot P_\ell\ (\cos\theta).$$

The standard reasoning proceeds along the following lines. The total c.s. is defined as

$$\sigma = 2\pi \int |f|^2 \sin\theta\, d\theta = \frac{4\pi}{k^2}\, \Sigma\, (2\ell+1) \cdot \frac{j_\ell^2 (ka)}{j_\ell^2(ka) + h_\ell^2 (ka)} \tag{6}$$

$$(h_\ell(x) = \sqrt{\frac{\pi\kappa}{2}} N_{\ell+1/2}(x)).$$

If $ka \gg 1$ (classical limit), then the quantity

$$\frac{j_\ell^2}{j_\ell^2 + h_\ell^2}$$

is very small for $\ell > ka$; for $\ell < ka$ it rapidly oscillates near the average value 1/2. So (6) approximately equals

$$\frac{2\pi}{k^2} \sum_0^{ka} (2\ell+1) \approx 2 \cdot \pi a^2$$

which is twice as much as classical c.s. (3). The key point of the preceeding reasoning is the identification of (6) with the total c.s.

Now let us analyse the situation in more detail. The particle current is given by

$$j_r = \frac{\hbar}{2mi} (\bar{\Psi} \frac{\partial \Psi}{\partial r} - \Psi \frac{\partial \bar{\Psi}}{\partial r}). \tag{7}$$

On the surface of the infinite sphere one should use for Ψ its asymptotic value (5):

$$j_r = \frac{\hbar k}{m}\cos\theta$$

$$+ \frac{\hbar k}{mR^2}\cdot|f|^2$$

$$+ \frac{\hbar k}{2mR}(1+\cos\theta)\cdot[f\cdot e^{ikR(1-\cos\theta)} + \bar{f}\cdot e^{-ikR(1-\cos\theta)}]$$

$$- \frac{\hbar}{2imR^2}[f\, e^{ikR(1-\cos\theta)} - \bar{f}\, e^{-ikR(1-\cos\theta)}]. \tag{8}$$

The term in the first line of (8) is the flow corresponding to a plane wave. Being integrated over the sphere it gives zero contribution to the total integral flow. The term in the second line is usually referred as the flow of the scattered particles. Being integrated it gives the double value of the classical flow. The term in the third line is the flow arising from the interference of the plane and scattered waves. The integral over it is equal to

$$-\frac{4\pi v}{k}\,\mathrm{Im}\, f(0)$$

and exactly compensates the flow of the previous line. This is just another formulation of the well-known optical theorem. Finally, the integral contribution of the current in the last line of (8) has an order of $1/R$ and vanishes for infinite R. So, the total integrated flow is exactly zero as in the classical case. The problem is how to separate the total current (8) into parts corresponding to the incident and scattered currents. The complications are certainly due to the interference terms. One might try to quess the part of the total current corresponding to the incident flow. Then the remaining part would have presented the flow of the scattered particles. Unfortunately, it is impossible to choose the flow corresponding to the plane wave (1st line of (8)) as a candidate for the incident flow (being integrated this flow gives zero; because of this the integral from the remaining part of flow is also zero). Now define the incident flow exactly as in the classical case.

This means that incident flow given by the first line of (8) is everywhere constant except for the shadow region (i.e., for angles $0 \le \theta \le \arcsin\frac{a}{R}$) where it equals zero. Then the remaining part of the total flow is the scattering flow. The integral from it over the sphere gives for the cross section value πa^2 exactly as in the classical case. We see that once the identical definition of the incident flow is made,the same answer is obtained in both cases.Two questions are arising now.Suppose one sets up the particle counters on the surface of the sphere of the large radius R. What integral flow will be measured by them? The answer is, of course, zero if one measures the total flow (including incident one) and if the counters have infinite resolution. On the other hand if one can separate and subtract the contribution of the incident flow and the counters have finite resolution (so they cannot detect a very rapid angle oscillations of the interference terms), then the measured c.s. will be equal to $2\pi a^2$. One may easily see that finite dimensions of the real counter lead to the same cancellation of the interference terms. For example, suppose that a counter has finite radial dimension ΔR. The averaging of the interference terms over ΔR results in an additional factor $1/k$ in front of these terms. For large values of k the contribution of the interference terms becomes negligible. The second question is whether the separation of the incident current from the total one, which was based on analogy with the classical case, has only the formal sense. The following answer is obvious. If the wavelength is much larger than the radius of the impenetrable sphere a, then the distortion of the incident wave takes place over the large distances and the mentioned above definition of the incident flow loses its sense. If the wave-length is small (this means $\hbar \to 0$ or $E \to \infty$) with respect to a, then the quantum picture is very close to the classical one and the used definition of the incident flow has sense. In order to see this more clearly we calculate the particle flow through the surface of finite radius sphere:

$$j_r = \frac{\hbar k}{m}\cos\theta$$

$$+ \frac{\hbar}{2mR}\, e^{-ikR\cos\theta}\left(-\cos\theta + \frac{1}{ikR}\right)\times$$

$$\times \sum (2\ell+1)\cdot i^{\ell}\,\frac{j_\ell(ka)}{h_\ell^{(1)}(ka)}\cdot h_\ell^{(1)}(kR)\cdot P_\ell(\cos\theta) + \mathrm{c.c.} \tag{9}$$

$$+\frac{\hbar}{2mRi}e^{ikR\cos\theta}\sum(2\ell+1)(-i)^{\ell}\frac{j_{\ell}(ka)}{h_{\ell}^{(2)}(ka)}\dot{h}_{\ell}^{(2)}(kR)\,P_{\ell}(\cos\theta)+c.c.$$

$$+\frac{\hbar}{2miR^{2}}\sum(2\ell+1)(-i)^{\ell}\frac{j_{\ell}(ka)}{h_{\ell}^{(2)}(ka)}h_{\ell}^{(2)}(kR)\cdot P_{\ell}(\cos\theta)\times$$

$$\times\sum(2\ell'+1)(-i)^{\ell'}\frac{j_{\ell'}(ka)}{h_{\ell'}^{(1)}(ka)}h_{\ell'}^{(1)}(kR)\cdot P_{\ell}(\cos\theta)+c.c.$$

As earlier the interference terms after integration cancel completely the flow of the scattered wave, so the total integral flow is again zero. Turning now to (4) one sees that for R close to a and $ka \gg 1$ the wave function Ψ vanishes for small angles. This means that Ψ describes the shadow effect. Setting $R=a$ in (9) one finds that $j_r=0$ locally (i.e., for every point of the sphere). So, for the incident flow one has

$$j_r^{inc} = \begin{cases} \frac{\hbar k}{m}\cos\theta & \frac{\pi}{2}<\theta<\pi \\ 0 & 0<\theta<\frac{\pi}{2} \end{cases}$$

(shadow effect)

The scattered flow is then equal to

$$j_r^{scat} = -j_r^{inc} .$$

For small distances and large k these formulas exactly reproduce the classical ones.

4. Now summarize rezults. In the standard quantum-mechanical treatment the interference of the incident and scattered waves is removed implicitly (using Eq.(6)) or explicitly (using the wave packets) in order to meet the requirements of the experimental situation. The latter means that thin collimated beams of particles are used and the particle counters are located well outside the interference region. The wave packets being a superposition of the Shroedinger equation wave functions have no definite energy and momenta whereas the classical particles have. It is known /10,11/ that wave packets are spreading in time. For example, if at the initial moment the absolute square of the wave function is given by the Gauss distribution with the width a , then at time t one has for the same

quantity:

$$\frac{1}{1+(\frac{t}{\tau})^{2}}\exp\left[-\frac{r^{2}}{a^{2}}\frac{1}{1+(\frac{t}{\tau})^{2}}\right], \qquad \tau=\frac{ma^{2}}{h} ,$$

If for m one takes the electron mass and for a classical electron radius, one may estimate the time constant $\tau \approx 0.7\cdot10^{-25}$ sec. The spreading of such packet takes place on the microscopic distances $c\tau \approx 2\cdot10^{-15}$ cm. The experimental study of the wave packets is just started /12/.

We aimed above to compare quantum and classical c.s. using the plane waves as the incoming particles, the solutions of the Schroedinger equation with definite energy and momenta and not disregarding the interference between incident and scattered waves. It was shown that mentioned c.s. are coinciding if the incident flow is the same in both cases and if the used measuring devices are ideal.

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